Canad. Math. Bull. Vol. 39 (2), 1996 pp. 129–137

ON THE LINEAR INVARIANCE OF LINDELÖF NUMBERS

JAN BAARS AND HELMA GLADDINES

ABSTRACT. Let X and Y be Tychonov spaces and suppose there exists a continuous linear bijection from $C_p(X)$ to $C_p(Y)$. In this paper we develop a method that enables us to compare the Lindelöf number of Y with the Lindelöf number of some dense subset Z of X. As a corollary we get that if for perfect spaces X and Y, $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic, then the Lindelöf numbers of X and Y are equal. Another result in this paper is the following. Let X and Y be any two linearly ordered perfect Tychonov spaces such that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. Let $\mathcal{P}$ be a topological property that is closed hereditary, closed under taking countable unions and closed under taking continuous images. Then X has property $\mathcal{P}$ if and only if Y has. As examples of such properties we consider certain cardinal functions.

1. **Introduction.** Let X and Y be Tychonov spaces. By $C(X)$, we denote the set of all real-valued continuous functions on X. We endow $C(X)$ with the topology of pointwise convergence and we denote that by $C_p(X)$. The function space $C_p(x)$ is a topological vector space which is a dense subspace of $\mathbb{R}^X$ with the product topology. The topological and linear structure of $C_p(X)$ have widely been investigated. Arkhangel'skiĭ's papers [1] and [3] and his book [4] contain a survey of results.

Our main interest is in the question of which topological properties are ℓ-invariant. We say that a topological property $\mathcal{P}$ is *ℓ-invariant* if for all Tychonov spaces X and Y such that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic we have X has property $\mathcal{P}$ if and only if Y has. Two function spaces $C_p(X)$ and $C_p(Y)$ are *linearly homeomorphic* if there exists a homeomorphism between them which is also linear. Among properties that are ℓ-invariant are, for example, compactness, σ-compactness and pseudocompactness (*cf.* [2]). Many other properties have been proved to be so as well (see for example [1] and [6]).

In [1], Arkhangel'skiĭ announced the result of Veličko that Lindelöfness is an ℓ-invariant property. It seems that his proof cannot be generalized for higher Lindelöf numbers. In [5], it is shown that for the class of paracompact first countable spaces, the Lindelöf number is ℓ-invariant. In the next section we will develop a method that allows us to decide that the Lindelöf number for another class of spaces is ℓ-invariant. We will show that the Lindelöf number is ℓ-invariant for the class of perfect Tychonov spaces. In fact we prove a much stronger result: Suppose $C_p(X)$ and $C_p(Y)$ are linearly

This paper was written while the first author was a postdoc at the Thomas Stieltjes Institute for Mathematics.
Received by the editors April 4, 1994.
AMS subject classification: Primary 54C35, 57N17.
Key words and phrases: Function Spaces, Lindelöf Numbers.

homeomorphic. Let κ be the Lindelöf number of Y and assume that in Y each closed subset can be written as the intersection of κ-many open subsets of Y. Then the Lindelöf number of X is less than or equal to κ.

In the last section of this paper we prove the following result. Let X and Y be any two linearly ordered perfect Tychonov spaces such that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. Let $\mathcal{P}$ be a topological property that is closed hereditary, closed under taking countable unions and closed under taking continuous images. Then X has property $\mathcal{P}$ if and only if Y has. Of course there are many topological properties that satisfy these conditions. Section 4 will mention some specific cardinal functions that apply.

2. **The main tools.** Let X and Y be Tychonov spaces, let $\phi\colon C_p(X) \longrightarrow C_p(Y)$ be a continuous linear map and let $y \in Y$ be fixed. The map $\psi_y\colon C_p(X) \longrightarrow \mathbb{R}$ defined by $\psi_y(f) = \phi(f)(y)$ is continuous and linear. So $\psi_y \in L(X)$, the dual of $C_p(X)$. Since the evaluation mappings $\xi_x(x \in X)$ defined by $\xi_x(f) = f(x)$ for $f \in C_p(X)$ form a Hamel basis for $L(X)$ (*cf.* [4] or [6]), there are for $\psi_y \neq 0$, $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}\setminus\{0\}$ such that $\psi_y = \sum_{i=1}^n \lambda_i \xi_{x_i}$. We define the *support* of y in X to be the finite set $\{x_1, \ldots, x_n\} \subseteq X$. If $\psi_y = 0$, the *support* of y is defined to be the empty set (notice that whenever ϕ is onto, $\psi_y \neq 0$ for every $y \in Y$). For $A \subseteq Y$ we denote $\bigcup_{y\in A} \operatorname{supp}(y)$ by $\operatorname{supp} A$.

LEMMA 2.1 ([6]). *Let X and Y be Tychonov spaces and let $\phi\colon C_p(X) \longrightarrow C_p(Y)$ be a continuous linear map. Then for $y \in Y$,*

(1) for every $z \in \operatorname{supp}(y)$, there is $\lambda_z^y \in \mathbb{R}$ such that $\phi(f)(y) = \sum_{z\in\operatorname{supp}(y)} \lambda_z^y f(z)$, for every $f \in C_p(X)$,

(2) if $f, g \in C_p(X)$ coincide on $\operatorname{supp}(y)$, then $\phi(f)(y) = \phi(g)(y)$.

LEMMA 2.2 ([6]). *Let X and Y be Tychonov spaces and let $\phi\colon C_p(X) \longrightarrow C_p(Y)$ be a continuous linear injective map. Then $\operatorname{supp} Y$ is dense in X. If ϕ is a linear homeomorphism, then $\operatorname{supp} Y = X$.*

LEMMA 2.3 ([6]). *Let X and Y be Tychonov spaces and let $\phi\colon C_p(X) \longrightarrow C_p(Y)$ be a continuous linear map. Then the set valued map $\operatorname{supp}\colon Y \longrightarrow \mathcal{P}(X)$ is LSC. That is, for every open subset O of X, the set $\{y \in Y : \operatorname{supp}(y) \cap O \neq \emptyset\}$ is open in Y.*

Let X and Y be Tychonov spaces and let $\phi\colon C_p(X) \longrightarrow C_p(Y)$ be a continuous linear map. By $\operatorname{card} A$, we denote the cardinality of a set A. For every $n \in \mathbb{N}$ we define $Y_n = \{y \in Y : \operatorname{card}\{\operatorname{supp}(y)\} \leq n\}$. Clearly $Y_n \subseteq Y_{n+1}$ for all $n \in \mathbb{N}$. Since $\operatorname{supp}(y)$ is finite for all $y \in Y$, $Y = \bigcup_{n=1}^\infty Y_n$. Note that if ϕ is onto, then Y_1 is the set of elements of Y that have exactly one element in their support.

Let $X_n = \operatorname{supp} Y_n$ for all $n \in \mathbb{N}$. Then $\operatorname{supp} Y = \bigcup_{n=1}^\infty X_n$ and $X_n \subseteq X_{n+1}$ for all $n \in \mathbb{N}$.

LEMMA 2.4. *The set Y_n is a closed subset of Y for all $n \in \mathbb{N}$.*

PROOF. Let $y \in Y \setminus Y_n$. Then $\operatorname{card}\{\operatorname{supp}(y)\} > n$. Let $\operatorname{supp}(y) = \{x_1, \ldots, x_m\}$, where $m > n$ and $x_i \neq x_j$ if $i \neq j$. Find open subsets O_i $(1 \leq i \leq m)$ in X such that $x_i \in O_i$ and $O_i \cap O_j = \emptyset$ if $i \neq j$. Let $U_i = \{y \in Y : \operatorname{supp}(y) \cap O_i \neq \emptyset\}$ $(1 \leq i \leq m)$. By Lemma 2.3,

https://doi.org/10.4153/CMB-1996-017-0 Published online by Cambridge University Press

U_i is open in Y. Note that $y \in U_i$. Let $U = \bigcap_{i=1}^{m} U_i$. Then U is open in Y and $y \in U$. If $z \in U$, then $\mathrm{supp}(z) \cap O_i \neq \emptyset$ for all $1 \leq i \leq m$. Hence $\mathrm{card}\{\mathrm{supp}(z)\} \geq m$. We conclude that $U \cap Y_n = \emptyset$. ∎

Suppose that ϕ is onto. In that case, the map $f_1 : Y_1 \longrightarrow X_1$ defined by $f_1(y) = x$, where $\mathrm{supp}(y) = \{x\}$, is well defined.

LEMMA 2.5. *The map f_1 is continuous.*

PROOF. Let $y \in Y_1$ and let $\mathrm{supp}(y) = \{x\}$. Take an arbitrary open O in X such that $x \in O$. By Lemma 2.3, the set $U = \{y \in Y : \mathrm{supp}(y) \cap O \neq \emptyset\}$ is open in Y. Note that $y \in U$. It easily follows that $f_1(U \cap Y_1) \subseteq O$. ∎

Let $n > 1$ and let A be an arbitrary subset of $Y_n \setminus Y_{n-1}$. We define an equivalence relation $\sim$ on A as follows: for $y_1, y_2 \in A$ we have $y_1 \sim y_2$ if and only if $\mathrm{supp}(y_1) = \mathrm{supp}(y_2)$.

Let $A/\sim$ denote the quotient space and let $q_A : A \longrightarrow A/\sim$ denote the corresponding quotient map. Define

$$B_A = \big\{(x_1, \ldots, x_n) \in X^n : \exists y \in A,\ \mathrm{supp}(y) = \{x_1, \ldots, x_n\}\big\}$$

and define $p_A : B_A \longrightarrow A/\sim$ by $p_A(x_1, \ldots, x_n) = q_A(y)$ where $y \in A$ is such that $\mathrm{supp}(y) = \{x_1, \ldots, x_n\}$. The map p_A is well defined. Note that if $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ is a permutation, then $p_A(x_1, \ldots, x_n) = p_A(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ and that the fibers of points are finite.

LEMMA 2.6. *The map p_A satisfies the following conditions:*

(a) p_A is closed.

(b) p_A is open.

(c) p_A is locally injective.

PROOF. For (a) let F be an arbitrary closed subset of B_A and suppose $y \in A$ is such that $q_A(y) \in \overline{p_A[F]} \setminus p_A[F]$. Since $y \in Y_n \setminus Y_{n-1}$, $\mathrm{card}\{\mathrm{supp}(y)\} = n$. So $\mathrm{supp}(y) = \{x_1, \ldots, x_n\}$ where $x_i \neq x_j$ for $i \neq j$. For every permutation $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ it follows that $p_A(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = q_A(y)$ so $(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \notin F$. Hence we can find open sets $U_1, \ldots, U_n$ in X such that $x_i \in U_i$ for every $1 \leq i \leq n$, $U_i \cap U_j = \emptyset$ for $i \neq j$ and for every permutation $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$, $U_{\sigma(1)} \times \cdots \times U_{\sigma(n)} \cap F = \emptyset$. Let $O_i = \{z \in Y : \mathrm{supp}(z) \cap U_i \neq \emptyset\}$ for every $1 \leq i \leq n$. By Lemma 2.3, O_i is open in Y and $y \in O_i$. So $O = \bigcap_{i=1}^{n} O_i$ is open in Y and $y \in O$.

CLAIM. $q_A^{-1}\big[q_A[O]\big] = O$.

Let $z \in q_A^{-1}\big[q_A[O]\big]$. Then there is $w \in O$ such that $q_A(z) = q_A(w)$. Hence $\mathrm{supp}(z) = \mathrm{supp}(w)$. Since $\mathrm{supp}(w) \cap U_i \neq \emptyset$ for every $1 \leq i \leq n$ we get the same for $\mathrm{supp}(z)$. This gives $z \in O$ which proves the claim.

The claim gives us that $q_A[O]$ is open in $A/\sim$. Since $q_A(y) \in q_A[O]$, $q_A[O] \cap p_A[F] \neq \emptyset$. Let $z \in O$ and $(z_1, \ldots, z_n) \in F$ be such that $q_A(z) = p_A(z_1, \ldots, z_n)$. Hence $\mathrm{supp}(z) = \{z_1, \ldots, z_n\}$. Since $\mathrm{supp}(z) \cap U_i \neq \emptyset$ for every $1 \leq i \leq n$, $z_i \neq z_j$ and $U_i \cap U_j = \emptyset$ for every $i \neq j$, there exists a permutation $\sigma\colon \{1,\ldots,n\} \longrightarrow \{1,\ldots,n\}$ such that $(z_1,\ldots,z_n) \in U_{\sigma(1)} \times \cdots \times U_{\sigma(n)}$. But then $(z_1,\ldots,z_n) \notin F$. This completes the proof of part (a) of this lemma.

For (b) let O be an arbitrary open subset of B_A. Put $U = p_A^{-1}\big[p_A[O]\big]$. Then U is also an open subset of B_A, because if $(x_1,\ldots,x_n) \in U$, then there exists a permutation $\sigma\colon \{1,\ldots,n\} \longrightarrow \{1,\ldots,n\}$ such that $(x_{\sigma(1)},\ldots,x_{\sigma(n)}) \in O$. We can find open sets $U_1,\ldots,U_n$ in X such that $x_i \in U_i$ for every $1 \leq i \leq n$, $U_i \cap U_j = \emptyset$ for $i \neq j$ and $(U_{\sigma(1)} \times \cdots \times U_{\sigma(n)}) \cap B_A \subseteq O$. Then $(U_1 \times \cdots \times U_n) \cap B_A \subseteq U$. It is clear that $p_A[U] = p_A[O]$; hence it suffices to prove that $p_A[U]$ is an open subset of $A/\!\sim$. By part (a), $(A/\!\sim) \setminus p_A[B_A \setminus U]$ is open in $A/\!\sim$. To finish the proof of part (b), observe that $(A/\!\sim) \setminus p_A[B_A \setminus U] = p_A[U]$.

For (c) let $(x_1,\ldots,x_n) \in B_A$ and let O be open in B_A such that $(x_1,\ldots,x_n) \in O$. We can find open sets $U_1,\ldots,U_n$ in X such that $x_i \in U_i$ for every $1 \leq i \leq n$, $U_i \cap U_j = \emptyset$ for $i \neq j$ and $(U_1 \times \cdots \times U_n) \cap B_A \subseteq O$. We claim that $p_A|_{(U_1\times\cdots\times U_n)\cap B_A}$ is injective. Suppose $p_A(z_1,\ldots,z_n) = p_A(w_1,\ldots,w_n)$, where $(z_1,\ldots,z_n)$ and $(w_1,\ldots,w_n)$ are elements of $U_1 \times \cdots \times U_n \cap B_A$. Then there exists a permutation $\sigma\colon \{1,\ldots,n\} \longrightarrow \{1,\ldots,n\}$ such that $(z_1,\ldots,z_n) = (w_{\sigma(1)},\ldots,w_{\sigma(n)})$. But since the sets U_i are pairwise disjoint we must have that σ is the identity. ∎

We emphasize that the map p_A is not continuous in general. We do not use property (b) and (c) of Lemma 2.6 in this paper, but we think these properties are interesting in themselves.

Let π_A denote the restriction of B_A of the projection $\pi\colon X^n \longrightarrow X$ onto the first coordinate.

LEMMA 2.7. *If $A = Y_n \setminus Y_{n-1}$, then $X_n \setminus X_{n-1} \subseteq \pi_A[B_A] \subseteq X_n$.*

PROOF. The set $\pi_A[B_A]$ is obviously contained in X_n. Let $x \in X_n \setminus X_{n-1}$. Then $x \in \operatorname{supp}(y)$ for some $y \in Y_n \setminus Y_{n-1}$. Let $\operatorname{supp}(y) = \{x_1,\ldots,x_n\}$, where $x_1 = x$. Then $(x_1,\ldots,x_n) \in p_A^{-1}\big(q_A(y)\big)$ and $x = \pi_A(x_1,\ldots,x_n)$. Hence $x \in \pi_A\big[p_A^{-1}\big(q_A(y)\big)\big]$. ∎

3. **Results on Lindelöf numbers.** We will use the notation of the previous section without explicit reference. The following well known lemma will be used in our first results. Recall that the Lindelöf number $l(X)$ of a topological space X is defined to be the smallest cardinal $\kappa \geq \aleph_0$ such that each open cover of X has a subcover of cardinality less than or equal to κ.

LEMMA 3.1. *Let X and Y be topological spaces and let f be a closed map of X onto Y such that $f^{-1}(y)$ is compact for every $y \in Y$. Then*

(a) X is σ-compact if Y is,

(b) $l(X) \leq l(Y)$.

A continuous map having the properties of this lemma is usually called a *perfect map* (*cf.* [7]). We want to emphasize that when we apply Lemma 3.1 in this paper we usually don't deal with a continuous map.

Recall that a topological space X is perfect if every closed subset of X is a G_δ in X.

THEOREM 3.2. *Let X and Y be Tychonov spaces and let $\phi: C_p(X) \longrightarrow C_p(Y)$ be a continuous linear bijection. Suppose that Y is a perfect σ-compact space. Then X contains a dense σ-compact subset.*

PROOF. Since ϕ is injective it suffices to show, by Lemma 2.2, that supp(Y) is σ-compact. We will do that by showing that each X_n is σ-compact. Lemma 2.4 gives us that Y_1 is σ-compact. Then, since ϕ is onto, Lemma 2.5 implies that X_1 is σ-compact. Let $n > 1$ and suppose we showed that X_{n-1} is σ-compact. Since Y is perfect, Y_{n-1} is a G_δ subset of Y. Hence $Y_n \setminus Y_{n-1}$ is an F_σ-subset of Y so that $Y_n \setminus Y_{n-1}$ is σ-compact. Let $A = Y_n \setminus Y_{n-1}$. Then $A/\sim$ is also σ-compact. By Lemma 3.1 (a), B_A is σ-compact; hence $\pi_A(B_A)$ is a σ-compact subspace of X. Then Lemma 2.7 shows that $X_n = X_{n-1} \cup \pi_A[B_A]$ is σ-compact. ■

In [2], Arkhangel'skiĭ showed that σ-compactness (and also compactness) is an ℓ-invariant property.

For a cardinal function f we denote $\sup\{f(A) : A \subseteq X\}$ by $hf(X)$.

THEOREM 3.3. *Let X and Y be Tychonov spaces and let $\phi: C_p(X) \longrightarrow C_p(Y)$ be a continuous linear bijection. Then X contains a dense subset Z such that $l(Z) \leq hl(Y)$.*

PROOF. We follow the same strategy as in the previous theorem by showing that for each $n \in \mathbb{N}$, $l(X_n) \leq hl(Y)$. It then easily follows that $l(\text{supp } Y) \leq hl(Y)$. Lemma 2.5 shows us that $l(X_1) \leq l(Y_1) \leq hl(Y)$. Let $n > 1$ and suppose we showed that $l(X_{n-1}) \leq hl(Y)$. Let $A = Y_n \setminus Y_{n-1}$. Then $l(A/\sim) \leq l(A) \leq hl(Y)$. Lemma 3.1 (b) then gives $l(B_A) \leq hl(Y)$; hence $l(\pi_A(B_A)) \leq hl(Y)$. Since $X_n = X_{n-1} \cup \pi_A[B_A]$ it follows that $l(X_n) \leq hl(Y)$. ■

COROLLARY 3.4. *Let X and Y be Tychonov spaces and let $\phi: C_p(X) \longrightarrow C_p(Y)$ be a continuous linear bijection. If Y is hereditary Lindelöf, then X contains a dense Lindelöf subset.*

Let X be a topological space and let $\kappa \geq \aleph_0$ be a cardinal. A subset A of X is of type G_κ if A is the intersection of κ many open subsets of X. Then $hl(X) \leq \kappa$ if and only if $l(X) \leq \kappa$ and in X every closed subset is of type G_κ. Note that X is perfect if and only if each closed subset of X is of type $G_\delta (= G_{\aleph_0})$.

COROLLARY 3.5. *Let X and Y be Tychonov spaces and let $\phi: C_p(X) \longrightarrow C_p(Y)$ be a continuous linear bijection. Let $\kappa = l(Y)$ and suppose in Y every closed subset is of type G_κ. Then X contains a dense subset Z such that $l(Z) \leq \kappa$.*

COROLLARY 3.6. *Let X and Y be Tychonov spaces and let $\phi: C_p(X) \longrightarrow C_p(Y)$ be a continuous linear bijection. If Y is a perfect Lindelöf space, then X contains a dense Lindelöf subset.*

If Z is a dense subset of a paracompact space X, then $l(X) = l(Z)$ (*cf.* [7]). This observation gives us four other corollaries.

COROLLARY 3.7. *Let X and Y be Tychonov spaces and let $\phi\colon C_p(X) \longrightarrow C_p(Y)$ be a continuous linear bijection. If X is paracompact, then $l(X) \leq hl(Y)$.*

COROLLARY 3.8. *Let X and Y be Tychonov spaces and let $\phi\colon C_p(X) \longrightarrow C_p(Y)$ be a continuous linear bijection. If Y is hereditary Lindelöf and X is paracompact, then X is Lindelöf.*

COROLLARY 3.9. *Let X and Y be Tychonov spaces and let $\phi\colon C_p(X) \longrightarrow C_p(Y)$ be a continuous linear bijection. Let $\kappa = l(Y)$ and suppose in Y every closed subset is of type G_κ. If X is paracompact, then $l(X) \leq \kappa$.*

COROLLARY 3.10. *Let X and Y be Tychonov spaces and let $\phi\colon C_p(X) \longrightarrow C_p(Y)$ be a continuous linear bijection. If Y is a perfect Lindelöf space and X is paracompact, then X is Lindelöf.*

THEOREM 3.11. *Let X and Y be Tychonov spaces such that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. Then $l(X) \leq hl(Y)$ and $l(Y) \leq hl(X)$.*

PROOF. Let $\phi\colon C_p(X) \longrightarrow C_p(Y)$ be a linear homeomorphism. In the proof of Theorem 3.3 we showed that $l(\text{supp } Y) \leq hl(Y)$. Lemma 2.2 gives us $l(X) \leq hl(Y)$. ∎

COROLLARY 3.12. *Let X and Y be Tychonov spaces such that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. Let $\kappa = \max\{l(X), l(Y)\}$ and suppose that in X and Y every closed subset is of type G_κ. Then $l(X) = l(Y)$.*

COROLLARY 3.13. *Let X and Y be perfect Tychonov spaces such that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. Then $l(X) = l(Y)$.*

In [1], Arkhangel'skiĭ announced the following theorem proved by Veličko: Lindelöfness is an ℓ-invariant property. His proof doesn't seem to generalize to higher Lindelöf numbers. Corollaries 3.12 and 3.13 show that the Lindelöf number is ℓ-invariant in at least a class of spaces that contains all perfect Tychonov spaces. In [5], it was shown that the Lindelöf number is ℓ-invariant for the class of first countable paracompact spaces.

4. **Other results.** In this section we consider a class of spaces for which topological properties that are closed hereditary, closed under countable unions and continuous images, are ℓ-invariant. The class of spaces under consideration is the class of all linearly ordered perfect Tychonov spaces. There are many topological properties that are closed hereditary, closed under countable unions and continuous images: for example, the properties we considered in the previous section. Those are not of special interest for the above class of spaces since they are now known to be ℓ-invariant in a larger class of spaces, the class of all perfect Tychonov spaces. There are however properties for which it is unknown if they are ℓ-invariant for all perfect Tychonov spaces, but for which we can show that they are for the class of all linearly ordered perfect Tychonov spaces. Of course this leaves some open questions. We will mainly be interested in certain cardinal functions, that is, the spread, the extent, the cellularity, the hereditary density and the

hereditary Lindelöf number. Note that all these cardinal functions do not increase under taking closed subsets, countable unions and continuous images. Of course there are many other cardinal functions that satisfy our assumptions: for example the width, the depth and the height of a topological space. For a survey of cardinal functions we refer to [8].

Again we use the notation of Section 2 without explicit reference. In addition if X is a Tychonov space, if $n \in \mathbb{N}$ and if $1 \leq i \leq n$, then $\pi_i : X^n \longrightarrow X$ denotes the projection on the i-th factor. Let $\mathcal{A}$ be a subclass of the class of all Tychonov spaces. A Tychonov space Y is *related* to $\mathcal{A}$ if

for all $X \in \mathcal{A}$, for all continuous linear bijections $\phi: C_p(X) \longrightarrow C_p(Y)$ and for all $n > 1$, if $A = Y_n \setminus Y_{n-1}$, then there exists a continuous map $r_A: A \longrightarrow B_A$ such that $X_n \setminus X_{n-1} \subseteq \bigcup_{i=1}^{n} (\pi_i \circ r_A)[A]$.

LEMMA 4.1. *Let X and Y be Tychonov spaces and let $\phi: C_p(X) \longrightarrow C_p(Y)$ be a continuous linear bijection. Suppose that Y is related to some class $\mathcal{A}$ for which $X \in \mathcal{A}$. Let $\mathcal{P}$ be a topological property that is hereditary, closed under taking countable unions and continuous images. If Y has property $\mathcal{P}$, then X contains a dense subset that has property $\mathcal{P}$.*

PROOF. We will show that supp Y has property $\mathcal{P}$. Since $\mathcal{P}$ is closed under taking countable unions it suffices to show that for each $n \in \mathbb{N}, X_n$ has property $\mathcal{P}$. Since $\mathcal{P}$ is closed under taking continuous images and since Y_1 has property $\mathcal{P}$, Lemma 2.5 gives us that X_1 has property $\mathcal{P}$. For $n > 1$, suppose that X_{n-1} has property $\mathcal{P}$. Let $A = Y_n \setminus Y_{n-1}$ and let $r_A: A \longrightarrow B_A$ be a continuous map such that $X_n \setminus X_{n-1} \subseteq \bigcup_{i=1}^{n} (\pi_i \circ r_A)[A]$. Since $\mathcal{P}$ is hereditary and closed under taking continuous images, it follows that for all $1 \leq i \leq n, (\pi_1 \circ r_A)[A]$ has property $\mathcal{P}$. Since $\mathcal{P}$ is also closed under taking countable unions, we are done. ∎

COROLLARY 4.2. *Let $\mathcal{A}$ be a class of Tychonov spaces. Let X and Y be elements of $\mathcal{A}$ that are both related to $\mathcal{A}$ and such that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. Let $\mathcal{P}$ be a topological property as in Lemma 4.1. Then X has property $\mathcal{P}$ if and only if Y has.*

For perfect Tychonov spaces, Lemma 4.1 and Corollary 4.2 also hold for topological properties that are closed hereditary instead of hereditary. In general we have

LEMMA 4.3. *Let X and Y be Tychonov spaces and let $\phi: C_p(X) \longrightarrow C_p(Y)$ be a continuous linear bijection. Suppose that Y is related to some class $\mathcal{A}$ for which $X \in \mathcal{A}$. Let $\kappa \geq \aleph_0$ be a cardinal and suppose that in Y each closed subset is of type G_κ. Let $\mathcal{P}$ be a topological property that is closed hereditary, closed under taking κ-many unions and continuous images. If Y has property $\mathcal{P}$, then X contains a dense subset that has property $\mathcal{P}$.*

PROOF. Note that by the assumptions on $\mathcal{P}$, all layers $Y_n \setminus Y_{n-1}$ have property $\mathcal{P}$. Hence we can proceed as in the proof of Lemma 4.1. ∎

https://doi.org/10.4153/CMB-1996-017-0 Published online by Cambridge University Press

COROLLARY 4.4. *Let $\mathcal{A}$ be a class of Tychonov spaces. Let X and Y be elements of $\mathcal{A}$ that are both related to $\mathcal{A}$ and such that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. Let $\kappa \geq \aleph_0$ be a cardinal and suppose that in X and Y each closed subset is of type G_κ. Let $\mathcal{P}$ be a topological property as in Lemma 4.3. Then X has property $\mathcal{P}$ if and only if Y has.*

Corollary 4.4 applies to the class of all linearly ordered Tychonov spaces.

LEMMA 4.5. *Tychonov spaces are related to the class of all linearly ordered Tychonov spaces.*

PROOF. Let $(X,<)$ be a linearly ordered Tychonov space and let Y be an arbitrary Tychonov space. Let $\phi: C_p(X) \longrightarrow C_p(Y)$ be a continuous linear bijection, let $n > 1$ and let $A = Y_n \setminus Y_{n-1}$. Define $h_1: A \longrightarrow X$ by $h_1(y) =$ the first element of supp(y). We claim that h_1 is continuous. Let $y \in A$ and let $x_1 < \cdots < x_n$ in X be such that $\operatorname{supp}(y) = \{x_1, \ldots, x_n\}$. So $h_1(y) = x_1$. Let U be an arbitrary open neighborhood of x_1. Find disjoint open intervals $U_1, \ldots, U_n$ in X such that $U_1 \subseteq U$ and such that for each $1 \leq i \leq n$, $x_i \in U_i$. Let $O = \bigcap_{i=1}^n \{z \in A : \operatorname{supp}(y) \cap U_i \neq \emptyset\}$. Then O is an open neighborhood of y such that $h_1[0] \subseteq U$. So actually h_1 is a continuous selection of the LSC map supp. If we define $h_2, \ldots, h_n$ by $h_i(y) = x_i$, *i.e.*, $h_i(y)$ is the i-th element of supp(y), then we have in the same way that $h_2, \ldots, h_n$ are continuous, and moreover that for each $y \in A$, $\operatorname{supp}(y) = \{h_1(y), \ldots, h_n(y)\}$. Define $r_A: A \longrightarrow B_A$ by $r_A(y) = \big(h_1(y), \ldots, h_n(y)\big)$. Then clearly r_A is a well defined continuous mapping. We have to show that $X_n \setminus X_{n-1} \subseteq \bigcup_{i=1}^n (\pi_i \circ r_A)[A]$. For that, let $x \in X_n \setminus X_{n-1}$. By Lemma 2.7, there is $(x_1, \ldots, x_n) \in B_A$ such that $\pi_A\big((x_1, \ldots, x_n)\big) = x$. Hence $x = x_1$. Let $y \in A$ be such that $\operatorname{supp}(y) = \{x_1, \ldots, x_n\}$. There exists a permutation $\sigma: \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ such that for each $1 \leq i \leq n$, $h_{\sigma(i)}(y) = x_i$. So $x \in (\pi_{\sigma(1)} \circ r_A)(y)$. ∎

LEMMA 4.6. *Let X be a linearly ordered Tychonov space and let Y be an arbitrary Tychonov space. Let $\phi: C_p(X) \longrightarrow C_p(Y)$ be a continuous linear bijection. Let $\mathcal{P}$ be a topological property as in Lemma 4.1. If Y has property $\mathcal{P}$, then X contains a dense subset that has property $\mathcal{P}$.*

PROOF. Apply Lemmas 4.1 and 4.5. ∎

COROLLARY 4.7. *Let X and Y be linearly ordered Tychonov spaces such that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. Let $\mathcal{P}$ be a topological property that is hereditary, closed under taking countable unions and continuous images. Then X has property $\mathcal{P}$ if and only if Y has property $\mathcal{P}$.*

LEMMA 4.8. *Let X be a linearly ordered Tychonov space and let Y be an arbitrary Tychonov space in which each closed subset is of type G_κ, where $\kappa \geq \aleph_0$ is a cardinal. Let $\phi: C_p(X) \longrightarrow C_p(Y)$ be a continuous linear bijection. Let $\mathcal{P}$ be a topological property as in Lemma 4.3. If Y has property $\mathcal{P}$, then X contains a dense subset that has property $\mathcal{P}$.*

PROOF. Apply Lemmas 4.3 and 4.5. ∎

COROLLARY 4.9. *Let X and Y be linearly ordered Tychonov spaces such that $C_p(X)$ and $C_p(Y)$ and linearly homeomorphic. Let $\kappa \geq \aleph_0$ be a cardinal and suppose that in X and Y each closed subset is of type $G\kappa$. Let $\mathcal{P}$ be a topological property that is closed hereditary, closed under taking κ-many unions and continuous images. Then X has property $\mathcal{P}$ if and only if Y has property $\mathcal{P}$.*

COROLLARY 4.10. *Let X and Y be linearly ordered perfect Tychonov spaces such that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. Then the spread, the extent, the cellularity, the hereditary Lindelöf number and the hereditary density number of X and Y are equal.*

In [10], Tkachuk already showed that cellularity is ℓ-invariant for the class of all Tychonov spaces, so this corollary does not give us anything new on cellularity. We finish this paper by posing some open questions

QUESTION 4.11. Are the spread, the hereditary Lindelöf number and the hereditary density number ℓ-invariant properties for all Tychonov spaces? For all perfect Tychonov spaces ?

In [9], Okunev gave an example of two Tychonov spaces X and Y such that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic, but with unequal extents for X and Y. Both spaces are pseudocompact, but one of them is not normal. This example and Corollary 4.10 imply the following question

QUESTION 4.12. For which class of spaces is the extent of ℓ-invariant property? For the class of normal spaces? For all perfectly normal spaces?

REFERENCES

1. A. V. Arkhangel'skiĭ, *C_p-theory*, In: Recent Progress in General Topology, (eds. J. van Mill and M. Hušek), North Holland (1992), 1–56.
2. ———, *On linear homeomorphisms of function spaces*, Soviet Math. Dokl. **25**(1982), 852–855.
3. ———, *Problems in C_p-theory*, In: Open Problems in Topology, (eds. J. van Mill and G. M. Reed), North Holland (1990), 601–616.
4. ———, *Topological function spaces*, Kluwer Academic Publishers, Dordrecht, 1992.
5. J. Baars, *Function spaces on first countable paracompact spaces*, Bull. Pol. Acad. Sci. Math. **42**(1994), 29–35.
6. J. Baars and J. de Groot, *On Topological and Linear Equivalence of certain Function Spaces*, CWI-tract **86**, Centre for Mathematics and Computer Science, Amsterdam.
7. R. Engelking, *General Topology*, Helderman Verlag, Berlin, 1989.
8. I. Juhász, *Cardinal Functions II, Ten Years Later*, MC Tract 123. Mathematical Centre, Amsterdam.
9. O. G. Okunev, *A method for constructing examples of M-equivalent spaces*, Top. and its Appl. **36**(1990), 157–171.
10. V. V. Tkachuk, *Duality with respect to the C_p-functor and cardinal invariants of Souslin type*, Mat. Zametki **37**(1985), 441–450.

Faculteit der Wiskunde en Informatica
Vrije Universiteit
De Boelelaan 1081a
1081 HV Amsterdam
The Netherlands

Current address:
Jan Baars
Oosteinderweg 11
1432 AC Aalsmeer
The Netherlands

Current address:
Helma Gladdines
Het Hoogt 123
1025 GT Amsterdam
The Netherlands