Ergod. Th. & Dynam. Sys. (1985), **5**, 445–447
Printed in Great Britain

k-fold mixing lifts to weakly mixing isometric extensions

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(*Received* 12 *August* 1984)

Abstract. If $\hat{T}$ is a weakly mixing isometric extension of a finite measure preserving, k-fold mixing map T, then $\hat{T}$ must also be k-fold mixing.

We here complete a collection of results each of which reads 'If T is $[\cdots]$ and $\hat{T}$ is a weakly mixing isometric extension of T, then $\hat{T}$ is $[\cdots]$,' where $[\cdots]$ can be [weakly mixing], [k-fold mixing], [K] or [Bernoulli]. The first is trivial. Each of the others has a distinctly different proof (see [**1**] for K and [**2**] for Bernoulli). Here we will prove the k-fold mixing case.

Let $(T, X, \mathscr{F}, \mu)$ be an ergodic finite measure preserving transformation of a non-atomic Lebesgue probability space. Let Y be a compact metric space with a transitive group G of isometries. By a *G-cocycle* $f(x, n)$ over T we mean a measurable map $f: \Omega \times \mathbb{Z} \to G$ so that

$$f(x, n_1 + n_2) = f(T^{n_1}(x), n_2) \circ f(x, n_1)$$

i.e.

$$f(x, n) = \prod_{i=0}^{n-1} f(T^i(x), 1).$$

We will abbreviate $f(\omega, 1) = f(\omega)$, the generating function of the cocycle.

As G is transitive on Y, Y is isometric to G/H, H an isotropy subgroup of some point y_0, and Haar measure on G projects to a G-invariant normalized measure ν on Y.

On the probability space $(X \times Y, \mathscr{F} \times \mathscr{G}, \mu \times \nu)$ we can define a measure preserving action $\hat{T}$ by

$$\hat{T}^n(x, y) = (T^n(x), f(x, n)(y)).$$

We call this the '*f-extension*' of T, and generically an 'isometric extension' of T.

THEOREM 1. *If T is k-fold mixing, i.e. for any measurable sets $A_0, A_1, \ldots, A_{k-1}$,*

$$\lim_{n_1, n_{i+1} - n_i \to \infty} \mu(A_0 \cap T^{n_1}(A_1) \cdots \cap T^{n_{k-1}}(A_{k-1})) = \mu(A_0)\mu(A_1) \cdots \mu(A_k),$$

and $\hat{T}$ is a weakly mixing isometric extension of T, then $\hat{T}$ is also k-fold mixing.

Proof. It is enough to verify for functions

$$g_i(x, y) = \chi_{A_i}(x)\bar{g}_i(y),$$

where χ_{A_i} is the characteristic function of $A_i \in \mathscr{F}$ and $\bar{g}_i$ is continuous and ≤ 1, that

$$\lim_{n_1, n_{i+1}-n_i \to \infty} \left(\int g_0(x, y) g_1(\hat{T}^{n_1}(x, y)) \cdots g_{k-1}(\hat{T}^{n_{k-1}}(x, y)) \, d\mu \times \nu \right) = \prod_{i=0}^{k-1} \int g_i(x, y) \, d\mu \times \nu,$$

as such functions generate an L^1 dense algebra.

Assume $\bar{g}_1, \ldots, \bar{g}_{k-1}$ are fixed continuous functions, and $\delta(\varepsilon)$ a uniform modulus of continuity for all $(k-1)$ of them.

As $\hat{T}$ is weakly mixing, the k-fold product $\hat{T} \times \hat{T} \times \cdots \times \hat{T}$ acting on $((X \times Y)^k, (\mathscr{F} \times \mathscr{G})^k, (\mu \times \nu)^k)$ is ergodic. Thus for $(\mu \times \nu)^k$-a.e. point $((x_0, y_0), \ldots, (x_{k-1}, y_{k-1}))$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \left(\prod_{j=0}^{k-1} g_j(\hat{T}^i(x_j, y_j)) \right) = \prod_{i=0}^{k-1} \int g_i(x, y) \, d\mu \times \nu.$$

Fix an $\varepsilon > 0$ and select $\bar{y}_1, \ldots, \bar{y}_s$, a $\delta(\varepsilon/2^{k+3})$ dense subset of Y, and partition Y into sets B_i, $y_i \in B_i$, of diameter less than $\delta(\varepsilon/2^{k+3})$. Let $0 < \alpha = \min(\nu(B_i))$.

Thus for all points $(x_0, \ldots, x_{k-1}) \in X^k$ and $(y_0, \ldots, y_{k-1})$, $(y'_0, \ldots, y'_{k-1}) \in Y^k$ if y_k and y'_k are in the same $B_{i(k)}$,

$$\frac{1}{N} \sum_{i=0}^{N-1} \left(\prod_{j=0}^{k-1} g_j(\hat{T}^i(x_j, y'_j)) \right) = \frac{1}{N} \sum_{i=0}^{N-1} \left(\prod_{j=0}^{k-1} g_j(\hat{T}^i(x_j, y_j)) \right) \pm \frac{\varepsilon}{8}.$$

Select N so large that for $(\mu \times \nu)^k$ all but $\varepsilon \alpha^k/4$ of the points $((x_0, y_0), \ldots, (x_{k-1}, y_{k-1}))$,

$$\frac{1}{N} \sum_{i=0}^{N-1} \left(\prod_{j=0}^{k-1} g_j(\hat{T}^i(x_j, y_j)) \right) = \prod_{i=0}^{k-1} \int g_i(x, y) \, d\mu \times \nu \pm \frac{\varepsilon}{8}.$$

It now follows that for μ^k all but $\varepsilon/4$ of the points $(x_0, \ldots, x_{k-1})$, for *all* $(y_0, y_1, \ldots, y_{k-1})$,

$$\frac{1}{N} \sum_{i=0}^{N-1} \left(\prod_{j=0}^{k-1} g_j(\hat{T}^i(x_j, y_j)) \right) = \prod_{j=0}^{k-1} \int g_j(x, y) \, d\mu \times \nu \pm \frac{\varepsilon}{4},$$

as the existence of one point $(y_0, \ldots, y_{k-1})$ not satisfying this error bound implies a set of measure at least α^k not satisfying the earlier error bound for a given $(x_0, \ldots, x_{k-1})$. Partition X into subsets $C_1, \ldots, C_p$ so that if $x_1, x_2 \in C_j$ and $n = 1, \ldots, N$ then

$$|f(x_1, n) - f(x_2, n)| < \delta(\varepsilon/2^{k+2}),$$

and

each A_i is a union of C_j's.

Thus if $x_j, x'_j \in C_{k(j)}$ for $j = 0, \ldots, k-1$ then

$$\frac{1}{N} \sum_{i=0}^{N-1} \left(\prod_{j=0}^{k-1} g_j(\hat{T}^i(x_j, y_j)) \right) = \frac{1}{N} \sum_{i=0}^{N-1} \left(\prod_{j=0}^{k-1} g_j(\hat{T}^i(x'_j, y_j)) \right) \pm \frac{\varepsilon}{4}.$$

https://doi.org/10.1017/S0143385700003060 Published online by Cambridge University Press

Now as T is k-fold mixing, we can select M so large that if n_1, $n_{i+1}-n_i>M$, then for any $C_{j(0)},\ldots,C_{j(k-1)}$,

$$\mu(C_{j(0)}\cap T^{-n_1}(C_{j(1)})\cap\cdots(T^{-n_{k-1}}(C_{j(k-1)}))=\prod_{i=0}^{k-1}\mu(C_{i(j)})\left(1\pm\frac{\varepsilon}{4}\right).$$

Fix such a choice of $n_1,\ldots,n_{k-1}$ and construct an invertible measure preserving map $\phi:(X,\mathscr{F},\mu)\to(X^k,\mathscr{F}^k,\mu^k)$ so that for all but $\varepsilon/4$ of the $x\in X$, if $(x,T^{n_1}(x),\ldots,T^{n_{k-1}}(x))\in C_{i(0)}\times C_{i(1)}\times\cdots\times C_{i(k-1)}$ then $\phi(x)\in(C_{i(0)}\times C_{i(1)}\times\cdots\times C_{i(k-1)})$, $(\phi(x)=(\phi(x)_1,\phi(x)_2,\ldots,\phi(x)_k))$.

Now

$$\int\prod_{j=0}^{k-1}(g_j(\hat{T}^{n_j}(x,y)))\,d\mu\times\nu=\int\frac{1}{N}\sum_{i=0}^{N-1}\left(\prod_{j=0}^{k-1}g_j(\hat{T}^{n_j+i}(x,y))\right)d\mu\times\nu$$

$$=\int\frac{1}{N}\sum_{i=0}^{N-1}\left(\prod_{j=0}^{k-1}g_j(\hat{T}^{i}(\phi(x)_j,f(x,n_j)(y)))\right)d\mu\times\nu\pm\frac{\varepsilon}{4}.$$

But for μ^k all but $\varepsilon/4$ of the $\phi(x)$, for all y,

$$\frac{1}{N}\sum_{i=0}^{N-1}\prod_{j=0}^{k-1}g_j(\hat{T}^{j}(\phi(x)_j,f(x,n_j)(y))=\prod_{j=0}^{k-1}\int g_j(x,y)\,d\mu\times\nu\pm\frac{\varepsilon}{2}.$$

Hence if n_1, $n_{i+1}-n_i>M$,

$$\int\prod_{j=0}^{k-1}(g_j(\hat{T}^{n_j}(x,y)))\,d\mu\times\nu=\prod_{j=0}^{k-1}\int g_j(x,y)\,d\mu\times\nu\pm\varepsilon,$$

completing the result.

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