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PERIODIC WAVES IN A RUNNING STREAM

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1. **Introduction.** In this paper, we discuss questions of the existence and calculation of periodic, steady flows over periodic streambeds. There are some surprises.

Problems such as this, of flows in running streams, are free-surface problems, and part of the difficulty is that the domain occupied by the fluid is not completely known *a priori*. That is, one is given a streambed, usually assumed to be described by a function $-B: R^2 \to R^1$, and one looks for the upper surface $H: R^2 \to R^1$ so that the fluid flows in the domain[1]

$$G = \{(X, Y, Z) \in R^3 : -B(X, Z) < Y < H(X, Z)\}. \tag{1.1}$$

A velocity potential is a function $\Phi: \bar{G} \to R^1$ such that the velocity of the fluid at any point is the gradient of Φ at that point. (See, e.g., [8, 14, 17].) Such a potential necessarily satisfies Laplace's equation

$$\Phi_{XX} + \Phi_{YY} + \Phi_{ZZ} = 0 \text{ in } G. \tag{1.2}$$

Also, if the flow is to be steady, there can be no component of the velocity normal to the boundary of G. Since the fluid velocity is the gradient of Φ, its normal component is the normal component of the gradient of Φ—i.e., the normal derivative of Φ. Thus, for steady flows, the normal derivative of Φ must be zero on the boundary of G, which gives

$$\Phi_Y + B_X\Phi_X + B_Z\Phi_Z = 0 \quad \text{when} \quad Y = -B(X, Z), \tag{1.3}$$

and

$$\Phi_Y = H_X\Phi_X + H_Z\Phi_Z \quad \text{when} \quad Y = H(X, Z). \tag{1.4}$$

Since H is not known *a priori*, we expect a second condition on the free surface. This condition is usually that the pressure is constant (equal to atmospheric pressure) there, but, more generally, one can allow for the effects

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(1) It is common (cf. [17]) in problems of water waves to choose a coordinate system with the Y-axis vertical and positive upward. This allows for the reduction to two dimensions by simply omitting the variable Z.

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of surface tension. In that case, the condition becomes that the pressure on the free surface differs from atomospheric pressure by a term proportional to the mean curvature of the surface [8, 9]. This reduces to the condition of constant pressure on the free surface when the constant of proportionality is zero. When surface tension is included, Bernoulli's equation [8, 14, 17] can be used to show that the condition on the pressure becomes

$$(1.5) \qquad gH-\frac{T_1}{\rho}D_2\cdot\frac{D_2H}{\sqrt{1+|D_2H|^2}}+\tfrac{1}{2}|D_3\Phi|^2=\text{constant} \quad \text{when} \quad Y=H(X,Z).$$

Here, g denotes the acceleration due to gravity, ρ is the density of the fluid, and $T_1 \geq 0$ is the surface tension. All three of these objects are assumed constant. D_2 denotes the two- and D_3 the three-dimensional gradient. As we remarked, (1.5) reduces to the constant pressure condition when $T_1=0$. Notice that the problem remains nonlinear, even when $T_1=0$, because of the term $\frac{1}{2}|D_3\Phi|^2$ occurring in (1.5).

The basic problem to be solved is (1.2)-(1.5), with $-B:R^2\to R^1$ given and the functions $H:R^2\to R^1$ and $\Phi:\bar{G}\to R^1$ to be found. But there is one more constant to be specified, the mean speed, U, of the flow. If we think of the problem stated as modelling the flow in a laboratory flume, U can be adjusted by using a more or less powerful pump.

What follows is a brief outline of some known results when B is periodic. For a more detailed history, see [18, 19]. All but one of the known results that I am aware of are two-dimensional. This means only that B, H and Φ are independent of Z. Until further notice, then, we consider the two-dimensional problem. Also, most of the known results assume no surface tension, so that $T_1=0$.

Even in this simplest case of two-dimensional flows without surface tension, it was indicated as early as 1886 that interesting things might be expected to happen. In 1886, Kelvin (see [8, p. 409]) looked at a simple *linearized* version of the problem, assuming B to be a simple sinusoid. He then showed that even the qualitative features of the flow may be expected to depend strongly on the mean speed U. In fact, Kelvin showed that, in his linear model, H is a sinusoid when B is, and that, when U is large enough, the top follows the bottom, the maxima of the top lying directly over the maxima of the bottom and the minima over the minima. However, he also showed that, at a certain speed U_1, the flow inverts, so that, for $U<U_1$, the maxima of the top lie over the minima of the bottom and the minima over the maxima! This phenomenon was recently verified experimentally [16].

The first rigorous results for the full, nonlinear problem are due to Gerber [1]. He considered two-dimensional flow in the absence of surface tension. Assuming B periodic and, like a sinusoid, having one maximum and one minimum per period, Gerber showed that, if U is large enough, a periodic flow exists with the free surface also having one maximum and one minimum per period and following the bottom in the sense defined in the preceding paragraph. Gerber's proof uses Leray-Schauder theory and is strictly nonconstructive.

In 1957, Moiseev [10] went further, making the same hypotheses as Gerber, that the flow is two-dimensional, without surface tension, and that the bottom has exactly two extrema per period. He also assumed that the variation of B is small enough, so that B does not deviate too far from being flat. In this case, Moiseev showed that there is a periodic flow, except for a sequence $\{U_n\}$ of mean speeds. He showed further that the only limit point of the sequence $\{U_n\}$ is zero, that the speeds U_n are bifurcation points, and that the top follows the bottom when $U > U_1$, but that it inverts when[2] $U < U_1$. Moiseev uses the implicit function theorem and Krasnosel'skii's [6] bifurcation theorem. Accordingly, Moiseev's arguments are, in principle, at least, constructive, although one would have to put some time into the details.

Still making the hypotheses that the flow is two-dimensional and without surface tension, Krasovskii [7] then showed that he can avoid Gerber's and Moiseev's hypothesis that the bottom has two extrema per period. Krasovskii also does not assume, as Moiseev did, that the bottom is nearly flat. Assuming only that the bottom is periodic and that U is large enough, Krasovskii showed that there is a periodic solution to the problem. He also showed that he was able to reproduce Gerber's result that the top follows the bottom when U is large, if B has only one maximum and one minimum per period. Krasovskii's methods involve the theory of monotone operators and cannot be called constructive.

Also assuming the bottom nearly flat, Hewgill, Reeder & Shinbrot [3] recently rederived Moiseev's existence results by a different method and eliminated the hypothesis that B has only two extrema per period. The paper [3] is the first I know of that makes no use of the theory of functions of a complex variable. Accordingly, it is the first with the possibility of generalization to three dimensions[3]. However, there were still difficulties with the generalization, and it was only provided later. Hewgill, Reeder & Shinbrot invoked the hypothesis that B is nearly flat by assuming it to have the form

$$B = d_0(1 + \varepsilon b), \tag{1.6}$$

where d_0 is the mean depth and ε is a (small) parameter. Let $\{U_n\}$ be the

(2) Moiseev also claims that the top follows the bottom when $U_{2k-1} < U < U_{2k}$ and inverts when $U_{2k} < U < U_{2k-1}$. Since such behavior is not a feature of Kelvin's linear model, since it is not supported by experimental evidence [16], and since, finally, Moiseev's paper does not contain details of his proof, I view these further conclusions as doubtful, or, at best, unproved.

(3) In different contexts, and to solve different problems, similar methods were used by Joseph [5] and by Sattinger [13].

sequence of bifurcation speeds introduced by Moiseev. Hewgill, Reeder & Shinbrot then showed that, when $U \neq U_n$, the solution exists and is an *analytic* function of ε in a neighborhood of $\varepsilon = 0$. Accordingly, the solution can be found to any desired degree of accuracy by simply expanding H and Φ in a series of powers of ε and equating coefficients of like powers. (Actually, the process involves one additional step: for some details, see [3].) We call a solution *regular* if it is analytic in ε in a neighborhood of $\varepsilon = 0$. In the later sections, we discuss only regular solutions, since it is they which can be calculated explicitly. Incidentally, by showing the solution to be regular, Hewgill, Reeder & Shinbrot also proved Kelvin's calculations to be valid for small enough ε (and $U \neq U_n$).

The results of Moiseev and of Hewgill, Reeder & Shinbrot raise a curious question. They showed that periodic flows exist for all small enough ε provided $U \notin \{U_n\}$. They also showed that $\{U_n\}$ decreases to zero. But why should *zero* be a limit of singular velocities of any kind? After all, experience shows that, as the mean speed goes to zero, the flow simply grows more tranquil, eventually coming to a complete stop. This question was answered implicitly by the introduction of surface tension. In a series of papers by Beckert [2], Hilbig [4], and Zeidler [20] (see Zeidler [19, chap. 10] for a complete account of these results), these authors showed that, in the presence of surface tension, the problem has a solution except for a sequence $\{U_n(T_1)\}$, where $\{U_n(0)\}$ is Moiseev's sequence $\{U_n\}$ and has zero as its only limit point. When $T_1 > 0$, on the other hand, ∞ is the only limit point of $\{U_n(T_1)\}$, and $\{U_n(T_1)\}$ is bounded away from zero. It follows, using the methods of [3] (see [15]), that regular flow exists in an interval $0 \leqq U < U'$, and that this flow goes to zero as $U \downarrow 0$.

The first—and, to this date, the only[4]—three-dimensional results are due to Shinbrot [15]. In [15], I discussed two- and three-dimensional periodic flows over periodic bottoms, with and without surface tension. The work [15] is the subject of the rest of this paper.

2. **The basic theorems.** In this section, we state the main results proved in [15]. They are then interpreted in two and in three dimensions in §§3 and 4, respectively.

2.1. *The coordinate system.* We always use a rectangular coordinate system defined in the following way. The Y-axis is vertical and points up. The plane $Y = 0$ is taken as the free surface when there is no flow (i.e., when $U = 0$). The X-axis is chosen in the direction of the mean flow. (In two dimensions, this statement is redundant; in three dimensions, it means that the X- and Z-axes are chosen so that the mean value of $\partial\Phi/\partial Z$ is zero.) In this coordinate system, we suppose the bottom to be described by a formula $Y = -B(X, Z) < 0$, and we

(4) See, however, [12] and, especially, [11], where different, but related problems in water waves are solved.

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define d_0 to be the mean value of B. Then, d_0 is the mean depth when there is no flow. We always assume that B has the form (1.6):

$$B = d_0(1+\varepsilon b). \tag{2.1}$$

Here, ε is a parameter, and b has mean value zero.

2.2. *Dimensionless parameters.* None of the results discussed in the introduction were originally stated in terms of U or T_1, although they can be restated in those terms. Rather, as is usual (and usually most fruitful) in physical problems, the results are proved using dimensionless parameters that are measures of U and T_1. The usual parameters are the Froude number F and the Bond number τ, defined by

$$F = \frac{U^2}{gd_0} \quad \text{and} \quad \tau = \frac{T_1}{\rho g d_0^2}. \tag{2.2}$$

It is easy to see that F and τ are dimensionless. Also, it turns out that, when B is periodic and has the form (2.1), the flow is generally completely determined by the three parameters F, τ, and ε.

2.3. *The sets* $\mathcal{F}_\tau$. As indicated in the introduction, one can prove existence of periodic solutions of the problem (1.1)–(1.5) except when the Froude number lies in a certain set which depends on τ. The set is different in two and in three dimensions, but, in order to state the main results without reference to dimension, we give the set the same name in both two and three dimensions.

In two dimensions, let B have period Ld_0. Then, when $\tau \geqq 0$ is fixed, the set of Froude numbers that cause difficulty is defined by

$$\mathcal{F}_\tau = \left\{ \left(\frac{L}{2\pi\ell} + \frac{2\pi\ell}{L}\tau \right) \tanh \frac{2\pi\ell}{L} : \ell = 1, 2, \ldots \right\}. \tag{2.3$_2$}$$

In three dimensions, the definition of $\mathcal{F}_\tau$ is an extension of the set defined in (2.3)$_2$. Here, we assume B doubly periodic, so that it is periodic in two directions with two periods. The X-axis is fixed, as in §2.1, by the direction of the mean flow. We suppose B is periodic in two different directions, making angles α and β with the X-axis. Without loss in generality, we may assume $-\pi/2 < \alpha, \beta \leqq \pi/2$. We suppose also that B has period Ld_0 in the direction α and Md_0 in the direction β. For integers ℓ and m, let

$$\begin{aligned} \lambda_{\ell m} &= \frac{\dfrac{2\pi\ell}{L}\sin\beta - \dfrac{2\pi m}{M}\sin\alpha}{\sin(\beta-\alpha)}, \\ \mu_{\ell m} &= \frac{\dfrac{2\pi\ell}{L}\cos\beta - \dfrac{2\pi m}{M}\cos\alpha}{\sin(\alpha-\beta)}, \\ \rho_{\ell m}^2 &= \lambda_{\ell m}^2 + \mu_{\ell m}^2, \qquad \rho_{\ell m} \geqq 0. \end{aligned} \tag{2.4}$$

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We can now define the set $\mathscr{F}_\tau$ in three dimensions. It is given by

$$(2.3)_3 \qquad \mathscr{F}_\tau = \left\{ \frac{1+\tau\rho_{\ell m}^2}{\lambda_{\ell m}^2} \rho_{\ell m} \tanh \rho_{\ell m} : \ell, m = \pm 1, \pm 2, \ldots \right\}.$$

Notice that $(2.3)_2$ is formally obtained from $(2.3)_3$ by taking $\alpha = 0$, $\beta = \pi/2$, and $M = \infty$, as might be expected.

2.4. *Definition of a solution.* In the statements of the theorems that follow, no precise smoothness conditions are given, either on B or the solution. Let us say here that $B \in C^{3+\alpha}(R^2)$ certainly suffices for the results, although one can do with quite a bit less. (See [3] and [15].) Also, although the precise conditions on the solution are not given, we can remark that all functions appearing in (1.1)–(1.5) are at least continuous. (Again, for details of this, see [15].)

Once d_0 is given, the numbers α, β, L and M define a fundamental parallelogram of periodicity of B. We say that B is *periodic*-$(\alpha, Ld_0; \beta, Md_0)$, these four numbers defining the periodicity of B. By a *solution of* (1.2)–(1.5), we mean a pair (H, Φ) of sufficiently smooth functions satisfying (1.2)–(1.5), where G is the domain defined by (1.1). A *periodic solution* is a solution for which the functions $(X, Z) \mapsto H(X, Z)$ and

$$(2.5) \qquad (X, Z) \mapsto -UX + \Phi(X, yB(X, Z) + (1+y)H(X, Z), Z)$$

are periodic $-(\alpha, Ld_0; \beta, Md_0)$ for every y, $-1 < y < 0$.

A remark about the complicated function (2.5) is, perhaps, in order. First, note that Φ itself cannot be periodic, because $\partial\Phi/\partial X$ must have mean value U. This is the reason for the term $-UX$ appearing in (2.5). Also, it would make no sense to say that $(X, Z) \mapsto -UX + \Phi(X, Y, Z)$ is periodic for fixed Y, since this function is not, in general, even defined on all of R^2 for all Y. $y \mapsto yB + (1+y)H$ is merely the simplest homotopy connecting $-B$ to H such that (2.5) is defined on all of R^2.

Notice that, by a periodic solution, we mean a solution having the same periodicity properties (i.e., the same values of α, β, L and M) as B.

Finally, recall that a solution is *regular* if it is analytic in ε in a neighborhood of $\varepsilon = 0$.

2.5. *The main results.* We now have the definitions required to state the main results of [15]. The only ambiguity is in the definition of $\mathscr{F}_\tau$; I remind the reader that $\mathscr{F}_\tau$ is defined by $(2.3)_2$ in two dimensions and by $(2.3)_3$ in three dimensions. The first result, then, is

THEOREM 2.5.1. *Let B have the form* (2.1), *where b is periodic and sufficiently smooth. If $\tau > 0$, then* (1.2)–(1.5) *has a unique, periodic solution for all small enough $|\varepsilon|$ if only $F \in [0, \infty) \setminus \mathscr{F}_\tau$.*

Of course, this is a mere existence statement. However, we have, as indicated in the introduction.

THEOREM 2.5.2. *The solutions of theorem* 2.5.1 *are regular. Moreover,* $H = 0(\varepsilon)$ *and* $\Phi = UX + 0(\varepsilon)$. *If we write* $H = \varepsilon H_1 + 0(\varepsilon^2)$ *and* $\Phi = UX + \varepsilon\Phi_1 + 0(\varepsilon^2)$, *then* (H_1, Φ_1) *is the solution of the usual* [16] *linear water waves problem associated with* (1.1)–(1.5).

The last sentence here is the justification for Kelvin's conclusion, discussed in the introduction, of inversion of the two-dimensional flow when F passes through $F_1 = \max \mathcal{F}_\tau$. The argument here is valid for $\tau > 0$ (see [15]), while Kelvin's argument applies also when $\tau = 0$. However, we have the following result, which is the only difference (with regard to existence) between two- and three-dimensional flows.

THEOREM 2.5.3. *In two dimensions, theorems* 2.5.1 *and* 2.5.2 *remain valid also when* $\tau = 0$.

The difficulty with the case $\tau = 0$ in three dimensions is discussed in §4. It contains one of the surprises mentioned in the first paragraph of §1.

It remains to ask what happens when $F \in \mathcal{F}_\tau$. In this case, we have

THEOREM 2.5.4. *Let* α, L, β, M *and* d_0 *be given. Let* $\tau \geqq 0$. *Then, there exists a periodic-*$(\alpha, Ld_0; \beta, Md_0)$ *function* $b \in C^\infty(R^2)$ *such that* (1.2)–(1.5) *has no regular, periodic solution for any* $F \in \mathcal{F}_\tau$.

This shows that for Froude numbers in $\mathcal{F}_\tau$ there is the opportunity for the situation to be genuinely nasty. It is probably true, at least, that isolated points of $\mathcal{F}_\tau$ are bifurcation points and that there is a periodic solution, analytic in some power of ε, although this is still an open question. More serious is the fact that points of $\mathcal{F}_\tau$ are not always isolated, as we shall see. Whether there is any solution at all in such cases is doubtful. Again, though, the matter is open.

The proofs of theorems 2.5–1–4 can be found in [15]. We leave this for the interested reader.

3. **Two-dimensional flows.** Since, in all cases, there is a regular, periodic solution of our problem whenever $F \notin \mathcal{F}_\tau$, it remains to describe in some detail the sets $\mathcal{F}_\tau$. We begin with two dimensions; the description of $\mathcal{F}_\tau$ in that case, taken together with the results of §2, give just the theorems of Moiseev [10], Hewgill, Reeder & Shinbrot [3], Hilbig [4] and Zeidler [20].

3.1. *The case* $\tau = 0$. When $\tau = 0$, $(2,3)_2$ gives

$$\mathcal{F}_0 = \left\{ \frac{L}{2\pi\ell} \tanh \frac{2\pi\ell}{L} : \ell = 1, 2, \ldots \right\}. \tag{3.1}$$

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Thus, $\mathcal{F}_0$ is a sequence converging to zero. Theorem 2.5.3 gives, therefore,

THEOREM 3.1. (Moiseev [10], Hewgill, Reeder & Shinbrot [3]). *Let b be periodic, sufficiently smooth, and independent of Z. Then, the two-dimensional problem without surface tension has a unique, periodic solution for all small enough ε, except for a sequence $\{F_\ell\}$ of Froude numbers whose only limit point is zero. Moreover, the problem always has a solution when the Froude number* $F \geqq 1$.

The only thing that does not follow immediately from (3.1) and theorem 2.5.3 is the last sentence of theorem 3.1. However, $\max \mathcal{F}_\tau = F_1 = (L/2\pi)\tanh(2\pi/L) < 1$. Since the problem has a solution whenever $F \notin \mathcal{F}_\tau$, and $\mathcal{F}_\tau$ contains no $F \geqq 1$ (for any L), the last sentence follows.

3.2. *The case* $\tau > 0$. When $\tau > 0$, $(2.3)_2$ gives immediately that $\mathcal{F}_\tau$ is a sequence whose only limit point is infinity. Most interesting is the fact that

$$\min \mathcal{F}_\tau \geqq \min_{\rho>0} \left(\frac{1+\tau\rho^2}{\rho} \tanh \rho\right) > 0. \tag{3.2}$$

These facts, together with theorem 2.5.3 give the following result.

THEOREM 3.2 (Hilbig [4], Zeidler [19]). *Let b be periodic, sufficiently smooth, and independent of Z. Then, if* $\tau > 0$, *the two-dimensional problem has a unique, periodic solution for all small enough ε, except for a sequence* $\{F_\ell(\tau)\}$ *of Froude numbers whose only limit point is ∞. The minimum of* $\{F_\ell(\tau)\}$ *is positive when* $\tau > 0$, *so the problem has a solution for all small enough Froude numbers.*

The last sentences of theorems 3.1 and 3.2 indicate that widely different results can be expected in the two cases when surface tension is present and when it is not. When surface tension is taken into account, the flow is perfectly regular in an interval $[0, F')$ of Froude numbers. This is what our intuition would lead us to expect. On the other hand, when surface tension is neglected, zero is a limit point of Froude numbers at which bifurcation takes place [10]. It is surprising, but apparently true, that what makes our intuition correct about fluids at small Froude numbers is the presence of surface tension.

4. **Three-dimensional flows.** It is impossible, of course, to make a flow perfectly two-dimensional in reality. Thus, the three-dimensional case holds most of the physical interest. What is most significant is the difference between the two- and the three-dimensional results when $\tau = 0$. One can probably conclude that most earlier work on flows in a running stream—work in two dimensions with no surface tension—has little or no descriptive significance for actual flows.

4.1. *The case* $\tau = 0$. Because we only have theorems 2.5.1 and 2.5.2 available in three dimensions, and not theorem 2.5.3, we can say nothing positive about existence in three dimensions when $\tau = 0$. However, we have theorem 2.5.4 available to prove non-existence when $F \in \mathcal{F}_\tau$, even when $\tau = 0$.

In three dimensions, $\mathcal{F}_\tau$ is defined by $(2.3)_3$. When $\tau = 0$, we have

LEMMA 4.1.1. *In three dimensions, the set* $\mathcal{F}_0$ *is dense in* $(0, \infty)$.

The proof of lemma 4.1.1 is cumbersome and unenlightening, so we refer the reader to the proof given in [15].

When taken together with theorem 2.5.4, lemma 4.1.1 gives

THEOREM 4.1.2 (Shinbrot [15]). *Let* α, L, β, M *and* d_0 *be given, and let* $\tau = 0$. *Then, there exists a periodic-*$(\alpha, Ld_0, \beta, Md_0)$ *function* $b \in C^\infty(R^2)$ *such that* (1.2)–(1.5) *has no regular, periodic solution for any Froude number in a* dense *subset of* $(0, \infty)$.

This means that the two-dimensional solutions of theorem 3.1 can in no sense be limits of three-dimensional solutions without surface tension. Since, furthermore, waves closely approximating two-dimensional ones can be obtained in practice [16], this probably means that the entire body of literature on two-dimensional waves without surface tension must be reappraised. Of course, before this reappraisal can be undertaken seriously, more work must be done on flows that are not regular. One thing is certain, however: modern theories of bifurcation are not equipped to handle situations where the possible bifurcation points are dense.

4.2. *The case* $\tau > 0$. When $\tau > 0$, on the other hand, the two- and the three-dimensional results are in many ways the same. In particular, (2.4) shows that $|\lambda_{\ell m}| \leqq \rho_{\ell m}$, so that we obtain once again the lower bound (3.2) for $\mathcal{F}_\tau$. Also, since $\rho_{\ell m} \to \infty$ when either $|\ell|$ or $|m| \to \infty$, while $|\lambda_{\ell m}| \leqq \rho_{\ell m}$, it follows easily that

$$\frac{1 + \tau\rho_{\ell m}^2}{\lambda_{\ell m}^2} \rho_{\ell m} \tanh \rho_{\ell m} \to \infty \text{ as either } |\ell| \text{ or } |m| \to \infty.$$

Therefore, we have the following analog of theorem 3.2.

THEOREM 4.2. (Shinbrot [15]). *Let* b *be doubly periodic and sufficiently smooth. Then, if* $\tau > 0$, *the three-dimensional problem has a unique, periodic solution for all small enough* ε, *except for a sequence* $\{F_{\ell m}(\tau)\}$ *of Froude numbers whose only limit point is infinity. The minimum of* $\{F_{\ell m}(\tau)\}$ *is positive when* $\tau > 0$, *so the problem has a unique, regular solution for all small enough Froude numbers.*

This result and theorem 4.1.2 show that, in three dimensions, at least, the problem only has physical significance when $\tau > 0$.

References

1. Robert Gerber, Sur les solutions exactes des équations du mouvement avec surface libre d'un liquide pesant. *J. de Math.* 34 (1955) 185–299.

2. Herbert Beckert, Existenzbeweise in der Theorie permanenter Schwerewellen. *Arch. Rat. Mech. Anal.* 9 (1962) 379–394.

3. Denton E. Hewgill, John Reeder, and Marvin Shinbrot, Some exact solutions of the nonlinear problem of water waves. To appear in *Pacific J. Math.*

4. H. Hilbig, Existenzbeweis für Potentialströmungen längs eines Kanals mit welliger Kanalsohle unter Einfluss der Schwerkraft und der Oberflächenspannung. *Arch. Rat. Mech. Anal.* 51 (1973) 295–303.

5. Daniel D. Joseph, Domain perturbations: the higher order theory of infinitesimal water waves. *Arch. Rat. Mech. Anal.* 51 (1973) 295–303.

6. M. A. Krasnosel'skii, *Topological Methods in the Theory of Nonlinear Integral Equations.* Macmillan, New York, 1964.

7. Yu. P. Krasovskii, The theory of steady state waves of large amplitude. Translated in: *Soviet Physics Dokl.* 5 (1960) 62–65.

8. Sir Horace Lamb, *Hydrodynamics, 6th ed.* Cambridge, at the University Press, 1932.

9. James Clerk Maxwell, Capillary action. *Encyclopedia Britannica, 11th ed.* Cambridge, 1911.

10. N. N. Moiseev, On the flow of a heavy fluid over a wavy bottom. *Prikl. Mat. Mekh.* (Russian). 21 (1957) 15–20.

11. John Reeder and Marvin Shinbrot, Three-dimensional, nonlinear wave interaction in water of constant depth. To appear.

12. John Reeder and Marvin Shinbrot, The initial value problem for surface waves under gravity, II and III. *Indiana Univ. Math. J.* 25 (1976) 1049–1071, and *J. Math. Anal. and Appl.* 67 (1979) 340–391.

13. D. Sattinger, On the free surface of a viscous fluid motion. *Proc. Roy. Soc. London A* 349 (1976) 183–204.

14. Marvin Shinbrot, *Lectures on Fluid Mechanics.* Gordon and breach, New York, London, Paris, 1973.

15. Marvin Shinbrot, Water waves over periodic bottoms in three dimensions. To appear in *J. of the Inst. of Maths. and Applications.*

16. Troy Shinbrot, *On salient phenomena of stationary waves in water.* Undergraduate thesis, Reed College, Oregon, 1978.

17. J. J. Stoker, *Water Waves.* Interscience Publishers, New York, 1957.

18. John V. Wehausen and Edmond V. Laitone, Surface waves. In: *Handbuch der Physik*, S. Flügge, ed., v. IX, pp. 446–778. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1960.

19. Eberhard Zeidler, *Beiträge zur Theorie und Praxis Freier Randwertaufgaben.* Akademie-Verlag, Berlin, 1971.

20. Eberhard Zeidler, Topologischer Existenzbeweis für Kapillar-Schwerewellen. *Math. Nachr.* 49 (1971) 85–99.

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