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CHARACTERIZATIONS OF p-SPACES

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1. **Introduction.** The concept of p-space is quite recent. It was introduced by Arhangel'skii [2]. The definition of p-space given in [2] involves compactification of the space. In view of the interesting properties of p-spaces obtained in [2], Alexadroff [1] suggested a problem of finding a direct intrinsic definition (without appeal to compactification). The main aim of this note is to answer the above problem.

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2. **Preliminary.** We require the following definitions:

DEFINITION 2.1. A completely regular space X is called a *p-space* iff there is a countable family $\{\mathbf{V}_i\}_{i=1}^{\infty}$ of open covers of X in any one (hence in all) of its Hausdorff compactifications such that $\bigcap_{i=1}^{\infty} \mathrm{St}(x, \mathbf{V}_i) \subset X$, for all $x \in X$.

DEFINITION 2.2. Let $\{A_s \mid s \in S\}$ be a family of subsets of a set X and $\{\mathbf{V}_i\}_{i=1}^{\infty}$ be a countable family of covers of X. Then, we say that the family $\{A_s \mid s \in S\}$ has sets which are *base point strictly small* relative to $\{\mathbf{V}_i\}_{i=1}^{\infty}$ iff there exists $x_0 \in X$ such that for each i, there is $s_i \in S$ and $V^i \in \mathbf{V}_i$ for which no $x_0 \in V^i$ and $A_{s_i} \subset V^i$.

Unless otherwise specified, we use the terminology of Engelking [3].

3. **Characterizations of p-spaces.**

THEOREM 3.1. *A completely regular space X is a p-space iff there exists a countable family $\{\mathbf{V}_i\}_{i=1}^{\infty}$ of open covers of X such that for every family of closed sets $\{F_s \mid s \in S\}$ which has the finite intersection property and contains sets which are base point strictly small relative to $\{\mathbf{V}_i\}_{i=1}^{\infty}$ the inequality $\bigcap(F_s \mid s \in S) \neq \varnothing$ holds.*

Proof. Let us suppose that there exists in X a countable family $\{\mathbf{V}_i\}_{i=1}^{\infty}$ of open covers of X which has the required property. Let $\mathbf{V}_i = \{V_s^i \mid s \in S_i\}$ for $i = 1, 2, \ldots,$ and let W_s^i denote an open set in βX (the Stone Čech compactification of X) such that $W_s^i \cap X = V_s^i$ for $s \in S_i$ and $i = 1, 2, \ldots$. Evidently, $\{\mathbf{W}_i\}_{i=1}^{\infty}$ where $\mathbf{W}_i = \{W_s^i \mid s \in S\}$ is a countable family of open covers of X in βX for each i. We now show that $\bigcap_{i=1}^{\infty} \mathrm{St}(x, \mathbf{W}_i) \subset X$ for all $x \in X$.

Let $y \in \bigcap_{i=1}^{\infty} \mathrm{St}(x, \mathbf{W}_i)$, and let $\mathbf{B}(y)$ be the family of all its neighborhoods in βX. The family $\{(\mathrm{cl}_{\beta X} B) \cap X \mid B \in \mathbf{B}(y)\}$ consists of closed subsets of the space X and has the finite intersection property. Also for each i there exists s_i such that x, y is in

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$W^i_{s_i}$. By the regularity of βX there is $B \in \mathbf{B}(y)$ depending on i such that $y \in B$ and $\mathrm{cl}_{\beta X} B \subset W^i_{s_i}$. This implies that the family $\{(\mathrm{cl}_{\beta X} B) \cap X \mid B \in \mathbf{B}(y)\}$ contains sets which are base point strictly small relative to $\{\mathbf{V}_i\}_{i=1}^{\infty}$, the base point being x. Therefore by the hypothesis

$$\bigcap (X \cap (\mathrm{cl}_{\beta X} B) \mid B \in \mathbf{B}(y)) = X \cap (\bigcap (\mathrm{cl}_{\beta X} B \mid B \in \mathbf{B}(y))) \neq \varnothing .$$

But $\bigcap (\mathrm{cl}_{\beta X} B \mid B \in \mathbf{B}(y)) = y$, hence $y \in X$. Since y is an arbitrary member of $\bigcap_{i=1}^{\infty} \mathrm{St}(y, \mathbf{V}_i)$. Consequently, $\bigcap_{i=1}^{\infty} \mathrm{St}(x, \mathbf{V}_i) \subset X$.

Conversely, let us assume that X is a p-space, i.e. there exists a countable family $\{\mathbf{V}_i\}_{i=1}^{\infty}$ of open covers of X in βX such that for each $x \in X$ we have $\bigcap_{i=1}^{\infty} \mathrm{St}(x, \mathbf{V}_i) \subset X$. For each $x \in X$ and $i = 1, 2, \ldots$, let W^i_x be an open set in βX such that $x \in W^i_x \subset \mathrm{cl}_{\beta X} W^i_x \subset V$ for some $V \in \mathbf{V}_i$. We shall show that the countable family $\{\mathbf{U}_i\}_{i=1}^{\infty}$ of open covers of the space X, where $\mathbf{U}_i = \{X \cap W^i_x \mid x \in X\}$ has the required property.

Let $\{F_s \mid s \in S\}$ be a family of closed subsets of X which has the finite intersection property and contains sets which are base point strictly small relative to $\{\mathbf{U}_i\}_{i=1}^{\infty}$. The family $\{\mathrm{cl}_{\beta X} F_s \mid s \in S\}$ has the finite intersection property and consists of closed subsets of βX. Therefore, by the compactness of βX, $\bigcap (\mathrm{cl}_{\beta X} F_s \mid s \in S) \neq \varnothing$. Suppose $x \in \bigcap (\mathrm{cl}_{\beta X} F_s \mid s \in S)$. Since $F_s = X \cap (\mathrm{cl}_{\beta X} F_s)$, in order that $x \in \bigcap (F_s \mid s \in S)$, it is enough to show that $x \in X$.

Because $\{F_s \mid s \in S\}$ has sets which are base point strictly small relative to $\{\mathbf{U}_i\}_{i=1}^{\infty}$, there exists $x_0 \in X$ such that for each i, one can choose $s_i \in S$ and $U^i \in \mathbf{U}_i$ such that $F_{s_i} \subset U^i$ and $x_0 \in U^i$. Since

$$x \in \mathrm{cl}_{\beta X} F_{s_i} \subset \mathrm{cl}_{\beta X} U^i \subset \mathrm{cl}_{\beta X} W^i_{x_1} \subset \mathrm{St}(x_0, \mathbf{V}_i),$$

it follows that $x \in \mathrm{St}(x_0, \mathbf{V}_i)$ for all i; but, by the hypothesis $\bigcap_{i=1}^{\infty} \mathrm{St}(x_0, \mathbf{V}_i) \subset X$. Consequently, $x \in X$. Hence the theorem is proved.

We can formulate the following result, which is similar in flavor to various results of Tamano [4]:

THEOREM 3.2. *Let X be a completely regular space and βX be the Stone Čech compactification of X. Then X is a p-space iff there exists a sequence $\{G_i\}_{i=1}^{\infty}$ of open sets in $X \times \beta X$ such that $\Delta_x \subset \bigcap_{i=1}^{\infty} G_i \subset X \times X$, where $\Delta_x = \{(x, x) \mid x \in X\}$.*

We leave the proof to the reader.

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