FINITE UNITARY REFLECTION GROUPS

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Introduction. Any finite group of linear transformations on n variables leaves invariant a positive definite Hermitian form, and can therefore be expressed, after a suitable change of variables, as a group of unitary transformations (**5**, p. 257). Such a group may be thought of as a group of congruent transformations, keeping the origin fixed, in a unitary space U_n of n dimensions, in which the points are specified by complex vectors with n components, and the distance between two points is the norm of the difference between their corresponding vectors.

In the real case we have a group of orthogonal transformations in Euclidean space E_n. Among such groups the *groups generated by reflections* have been the object of considerable study (**6**; **9**; **10**). The concept of a reflection has recently been extended to unitary space by Shephard (**24**). A reflection in unitary space is a congruent transformation of finite period that leaves invariant every point of a certain prime, and it is characterised by the property that all but one of the characteristic roots of the matrix of transformation are equal to unity. The remaining root, if the reflection is of period m, is a primitive mth root of unity, and the reflection is then said to be m-fold. Shephard, in the paper just quoted, has considered a particular class of unitary groups generated by reflections which possess properties closely analogous to those of the real orthogonal groups considered by Coxeter.

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This paper is divided into two parts. In the first part we determine all finite *unitary groups generated by reflections*. To save constant repetition we shall abbreviate the italicised words to 'u.g.g.r.', the qualification 'finite' being understood throughout. Since any finite group of unitary transformations is either irreducible or completely reducible **(5**, p. 263**)**, it suffices to determine the irreducible groups; in fact, if a u.g.g.r. is reducible, it is necessarily the direct product of its irreducible components, each of which is itself a u.g.g.r. This is accomplished in §§1–4. The method employed depends on the fact that, with each u.g.g.r. $\mathfrak{G}$ in U_n we can associate a collineation group $\mathfrak{G}'$ in projective space of $n - 1$ dimensions which is generated by homologies, and that conversely, from each such group $\mathfrak{G}'$ a finite number (generally only one) of u.g.g.r. can be derived. We are thus able to draw on the considerable literature which exists on collineation groups generated by homologies. In fact all the collineation groups we require are well known except the imprimitive ones, and these, as we shall see, can be determined without difficulty.

In the second part of the paper we place on record some curious properties of these groups, which are enunciated in the form of theorems in §5. These extend and generalise the results obtained by Coxeter **(10)** for the special type of u.g.g.r. which consists of real orthogonal transformations. The next five sections of the paper are devoted entirely to the proofs of these theorems. The last section gives abstract definitions for those groups in U_n which are generated by n reflections.

I. DETERMINATION OF ALL THE FINITE IRREDUCIBLE UNITARY GROUPS GENERATED BY REFLECTIONS

1. The associated collineation group $\mathfrak{G}'$. Let $\mathfrak{G}$ be a u.g.g.r. in U_n. The matrices which correspond to the operations of $\mathfrak{G}$ can be regarded as collineation matrices in projective space S_{n-1} of $n - 1$ dimensions, and the corresponding collineations form a group $\mathfrak{G}'$ which is isomorphic to the quotient group of $\mathfrak{G}$ by the cyclic subgroup $\mathfrak{Z}$ which consists of the elements of $\mathfrak{G}$ represented by scalar matrices. To the m-fold reflections of $\mathfrak{G}$ correspond collineations of finite period m leaving fixed all points of a prime of S_{n-1}. Such collineations are known as *homologies*, and thus $\mathfrak{G}'$ is generated by homologies.

Conversely, suppose $\mathfrak{G}'$ is a collineation group in S_{n-1} generated by homologies. The matrix of any such homology has a set of characteristic roots of the form $(\lambda, \lambda, \ldots, \lambda, \mu)$ with $\lambda \neq \mu$, $\lambda\mu \neq 0$. If $n > 2$ this matrix can be normalised in a unique way (by multiplying by λ^{-1}) so that it has all but one of its characteristic roots equal to unity, while if $n = 2$ this can be done in two ways (the other multiplier being μ^{-1}); and such a normalised matrix is equivalent to a reflection. Thus from any group $\mathfrak{G}'$ generated by homologies we can construct, in at most a finite number of ways, a group $\mathfrak{G}$ generated by reflections. More precisely, this can be done in a unique way provided that $n > 2$ and that the cyclic subgroups generated by the homologies in $\mathfrak{G}'$ are all conjugate in $\mathfrak{G}'$. As we shall soon see, these subgroups are conjugate when $n > 2$ except in a very few cases. But when

$n = 2$ a variety of cases arise which need separate discussion. When $n = 1$, $\mathfrak{G}$ is evidently a cyclic group generated by a reflection, and $\mathfrak{G}'$ is trivial.

2. Imprimitive u.g.g.r. in U_n. The groups $\mathfrak{G}$ and $\mathfrak{G}'$ are clearly both primitive or both imprimitive. In this section we shall determine the irreducible u.g.g.r. in U_n which are imprimitive.

For any imprimitive group $\mathfrak{G}$ it must be possible to arrange the transformation variables in sets which are either unchanged or permuted among themselves by every operation of $\mathfrak{G}$. If $\mathfrak{G}$ is irreducible, these sets of variables must be permuted transitively, so that each set contains the same number, n_1 say, of variables. The number of sets is then k where $kn_1 = n$. If we consider the corresponding collineation group $\mathfrak{G}'$ in S_{n-1}, there are then k linear spaces of dimension $n_1 - 1$, whose join is S_{n-1}, which are permuted among themselves by the collineations of $\mathfrak{G}'$. Now $\mathfrak{G}'$ is generated by homologies, and in any homology, a space of dimension $r > 0$ which is not invariant under the homology meets its transform in a space of dimension $r - 1$. It follows that, for an irreducible imprimitive group generated by homologies in S_{n-1}, we must have $n_1 = 1$, $k = n$, and therefore there are n distinct isolated points which are permuted by the collineations of $\mathfrak{G}'$. These n points define a simplex (which we naturally take as the simplex of reference) and the group $\mathfrak{G}'$ then consists of monomial substitutions of the form:

2.1 $$T:\quad x_i' = a_i x_{\sigma(i)}$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation σ of $(1, 2, \ldots, n)$. If all the characteristic roots of this, save one, are equal to unity, the transformation must be of one or other of the forms

2.2 $$x_i' = \theta x_j, \quad x_j' = \theta^{-1} x_i, \quad x_k' = x_k \qquad (k \neq i, j),$$

or

2.3 $$x_i' = \phi x_i, \quad x_j' = x_j \qquad (j \neq i),$$

where ϕ is a root of unity. Since $\mathfrak{G}$ is a u.g.g.r., it is generated by transformations of these two types, and since $\mathfrak{G}$ is irreducible, transformations of the former type certainly occur.

The correspondence $T \rightarrow \sigma$ where T is the typical transformation 2.1 of $\mathfrak{G}$, determines a homomorphism Θ of $\mathfrak{G}$ into the symmetric group $\mathfrak{S}_n$ of permutations of the suffixes of the variables. Let $\mathfrak{H}$ be the image of $\mathfrak{G}$ under Θ. Then, since $\mathfrak{G}$ is irreducible, $\mathfrak{H}$ is a transitive subgroup of $\mathfrak{S}_n$. It must be generated by the permutations corresponding to reflections in $\mathfrak{G}$. Now 2.2 is mapped onto the transposition (ij) of $\mathfrak{S}_n$, while 2.3 is mapped onto the identity. It follows that $\mathfrak{H}$ is generated by transpositions, and since it is transitive, $\mathfrak{H}$ must coincide with $\mathfrak{S}_n$, and Θ is a homomorphism of $\mathfrak{G}$ *onto* $\mathfrak{S}_n$.

Consider the subgroup $\mathfrak{G}^*$ of $\mathfrak{G}$ generated by the reflections 2.2. Since $\mathfrak{G}^*$, like $\mathfrak{G}$, is mapped homomorphically onto $\mathfrak{S}_n$ by Θ, the subgroups of $\mathfrak{G}^*$ which keep $n - 2$ variables fixed are all conjugate in $\mathfrak{G}^*$. The linear group induced on the remaining variables, since it leaves the simplex of reference invariant, must be dihedral, of order $2m$ say. It follows that $\mathfrak{G}^*$ is generated by transformations 2.2 where θ is a primitve mth root of unity, and $\mathfrak{G}^*$ is the group, of order $m^{n-1}n!$, of transformations of the form

2.4 $$x_i' = \theta^{\nu_i} x_{\sigma(i)}$$

where

2.5 $$\sum \nu_i \equiv 0 \pmod{m}.$$

If, in addition, $\mathfrak{G}$ contains a transformation of the type 2.3, this must transform the subgroup $\mathfrak{G}^*$ into itself. By considering the effect of 2.3 on the transformations of $\mathfrak{G}^*$ keeping fixed all the variables except x_i and x_j, it follows that ϕ is a power of θ. Thus if 2.3 is a q-fold reflection, so that ϕ is a primitive qth root of unity, q must divide m. Conversely, if $m = pq$, the set of transformations 2.4, where θ is a primitive mth root of unity and

2.6 $$\sum \nu_i \equiv 0 \pmod{p}$$

determine a u.g.g.r. of order $qm^{n-1}n!$ containing, in addition to the 2-fold reflections 2.2, q-fold reflections of type 2.3. This group may be denoted by $G(m, p, n)$. It is defined for all integral m, n ($m > 0$, $n > 1$) and any divisor p of m. But, if $m = 1$, the group leaves fixed the prime $\sum x_i = 0$, and hence is reducible. We therefore suppose that $m > 1$.

$G(m, p, n)$ is the symmetry group of a complex polytope which may be denoted by $\frac{1}{p}\gamma_n^m$ in analogy with the notation of (**24**, p. 378). It has $qm^{n-1} = m^n/p$ vertices:

$$(\theta^{\nu_1}, \theta^{\nu_2}, \ldots, \theta^{\nu_n}), \qquad \sum \nu_i \equiv 0 \pmod{p}.$$

Thus $G(m, 1, n)$ is the symmetry group of γ_n^m, and $G(m, m, n)$ is $[1\ 1; n-2]^m$ ($n > 2$) in the notation of (**24**). $G(m, m, 2)$ is the dihedral group of order $2m$, being the symmetry group of the polygon $\frac{1}{m}\gamma_2^m = \{m\}$, the real regular m-gon (**24**, p. 378).

We have now proved that the only irreducible imprimitive u.g.g.r. are the groups $G(m, p, n)$.

3. Primitive u.g.g.r. in U_n ($n \neq 2$). We now consider the primitive groups. When $n = 1$ it is clear that the only possibility is a cyclic group generated by a single reflection. When $n = 2$ we are led to a number of complications peculiar to this dimension which it will be convenient to discuss separately in the next section. Here we shall consider only the cases in which n exceeds two; these cases are characterised by the fact that the matrix of a homology in S_{n-1} can be normalized *in a unique way* to give a reflection matrix in U_n.

There is one simple general case in which $\mathfrak{G}$ is the symmetric group of order $(n+1)!$, the complete symmetry group of the regular simplex in E_n. The collineation group $\mathfrak{G}'$ in this case is simply isomorphic to $\mathfrak{G}$.

Apart from this, there are only a finite number of irreducible primitive groups generated by homologies. These groups are all known, and many of them have been extensively studied. We shall list them here, together with the u.g.g.r. to which they give rise. A suffix will be used to denote the order of the group.

When $n = 3$, the groups $\mathfrak{G}'$ generated by homologies are of orders 60, 168, 216 or 360 (**18**, ch. XII):

$\mathfrak{G}'_{60}$ is isomorphic with the alternating group on five symbols, and the corresponding group $\mathfrak{G}$, of order 120, is [3, 5], the symmetry group of the regular icosahedron.

$\mathfrak{G}'_{168}$ is Klein's simple group (**14**) and leads to a u.g.g.r. of order 336 containing 21 2-fold reflections.

$\mathfrak{G}'_{216}$ is the *Hessian group* which leaves invariant the configuration of inflections of a cubic curve. This group is generated by homologies of period 3, belonging to 12 cyclic subgroups, and in addition contains nine homologies of period 2 which generate an imprimitive subgroup. A u.g.g.r. corresponding to this collineation group must contain 3-fold reflections, and may, in addition, contain 2-fold reflections corresponding to the homologies of period 2. There are in fact two such groups; one, of order 648, contains only 3-fold reflections and is the symmetry group of the complex regular polytope *3(24)3(24)3* (in the notation of Shephard (**23**)) and the other, of order 1296, containing both 3-fold and 2-fold reflections, is the symmetry group of the regular complex polytope *3(24)3(18)2* (or its reciprocal).

$\mathfrak{G}'_{360}$ is the collineation group first found by Valentiner (**28**) and shown by Wiman (**29**) to be isomorphic with the alternating group of degree six. It leads to a u.g.g.r. of order 2160 generated by 45 2-fold reflections.

The primitive collineation groups in four variables have been given by Blichfeldt (**3**), Bagnera (**1**) and Mitchell (**20**). Those which are generated by homologies are of orders 576, 1920, 7200, 11520 and 25920:

$\mathfrak{G}'_{576}$ is the collineation group leaving fixed a pair of associated sets of desmic tetrahedra, and the corresponding u.g.g.r. is Euclidean of order 1152, namely [3, 4, 3], the symmetry group of the regular 24-cell (**9**).

$\mathfrak{G}'_{7200}$ also gives rise to a Euclidean group, namely [3, 3, 5], of order 14400, the symmetry group of regular 120-cell (**9**).

$\mathfrak{G}'_{11520}$ is the group leaving invariant Klein's 60_{15} figure, and $\mathfrak{G}'_{1920}$ is a subgroup of this leaving fixed a set of five of the 15 fundamental tetrahedra. To these correspond u.g.g.r. of orders 46080 and 7680 respectively, namely the symmetry groups of the complex polytopes $(\frac{1}{2}\gamma_3{}^4)^{+1}$ and $(\frac{1}{4}\gamma_3{}^4)^{+1}$. The latter is the group $[2\ 1;1]^4$ in Shephard's notation (**24**, p. 373).

$\mathfrak{G}'_{25920}$ is unique among the quaternary groups in that all its homologies are of period three, and it gives rise to a u.g.g.r. of order 155,520, the symmetry group of the complex regular polytope *3(24)3(24)3(24)3* (**23**).

The primitive groups generated by homologies in more than four variables were determined by Mitchell (**21**). There are five of them, which fall into two sets. The first set comprises groups in S_5, S_6 and S_7 respectively of orders 72.6!, 4.9! and 96.10! to which correspond Euclidean groups of orders 72.6!, 8.9! and

192.10! which are the symmetry groups of the polytopes 2_{21}, 3_{21} and 4_{21} respectively **(6)**. The remaining two groups, of orders 36.6! and 18.9! in S_4 and S_5 **(2; 12; 25)** correspond to u.g.g.r. (in U_5 and U_6 respectively) of orders 72.6! and 108.9!, they are the groups $[2\ 1;\ 2]^3$ and $[2\ 1;\ 3]^3$ in the notation of Shephard **(24**, p. 373**)**.

4. Primitive u.g.g.r. in U_2. The enumeration of the primitive u.g.g.r. in U_2 is complicated by the fact that any collineation (other than the identity) of finite period m can be regarded as a homology, and its matrix can be normalised in two ways so as to give a reflection matrix. As far as the collineation groups are concerned the matter is simple enough: the only primitive collineation groups in S_1 are the tetrahedral, octahedral and icosahedral groups. To each of these, however, correspond several u.g.g.r. in U_2.

Let $\mathfrak{G}$ be one of these u.g.g.r. and let $\mathfrak{G}'$ be the corresponding collineation group. Let Z be a generator of the cyclic subgroup $\mathfrak{Z}$ of $\mathfrak{G}$ defined in §1; then $\mathfrak{Z}$ is an invariant subgroup of $\mathfrak{G}$ and $\mathfrak{G}' \cong \mathfrak{G}/\mathfrak{Z}$. Thus if k denotes the period of Z, k operations of $\mathfrak{G}$ correspond to any collineation S' of $\mathfrak{G}'$, and if S is any one of these, the whole set is given by SZ^r $(r = 0, 1, \ldots, k-1)$.

Let S be an m-fold reflection, with matrix $\mathbf{S}$, so that its characteristic roots are $(1, \theta_m)$ where θ_m is a primitive mth root of unity, and let $\mathbf{Z}$ be the matrix corresponding to Z. Then, if θ_k is the primitive kth root of unity such that $\mathbf{Z} = \theta_k \mathbf{I}$, the characteristic roots of $\mathbf{SZ}^r$ are $(\theta_k^{\,r}, \theta_m\theta_k^{\,r})$. If this is a reflection then either $\theta_k^{\,r} = 1$ (i.e., $r = 0$) or $\theta_m\theta_k^{\,r} = 1$. This latter case occurs if and only if θ_m is a power of θ_k, i.e., if m is a factor of k. Hence, *if S' is an operation of period m in $\mathfrak{G}'$ which corresponds to an m-fold reflection in $\mathfrak{G}$, there are either one or two such reflections corresponding to S' according as k is not, or is, multiple of m, where k is the order of the subgroup $\mathfrak{Z}$ of $\mathfrak{G}$ represented by scalar matrices.*

If T' is an operation of $\mathfrak{G}'$ conjugate to S', then among the operations of $\mathfrak{G}$ which correspond to T' is one conjugate to S. Hence if one operation of a conjugate set in $\mathfrak{G}'$ has one (or two) corresponding reflections in $\mathfrak{G}$, then the same is true of every operation of the set.

The operation S^{-1} of $\mathfrak{G}$ clearly corresponds to $(S')^{-1}$ in $\mathfrak{G}'$. Suppose now that S' and its inverse are conjugate in $\mathfrak{G}'$, and that $m > 2$ (the theory that follows being trivial if $m = 2$). Then, among the operations $S^{-1}Z^r$ is one conjugate to S in $\mathfrak{G}$. This is a reflection (since it is conjugate to the reflection S) and is distinct from S^{-1}, since the characteristic roots of $\mathbf{S}^{-1}$ are $(1, \theta_m^{-1})$ and are different from those of $\mathbf{S}$. Consequently

If S' is an operation of period m $(m > 2)$ in $\mathfrak{G}'$ conjugate to its inverse, which corresponds to an m-fold reflection S, then there are two reflections in $\mathfrak{G}$ which correspond to S' and so (by the last result proved) m is a factor of k.

The method of constructing all u.g.g.r. corresponding to $\mathfrak{G}'$ is then as follows. We start from the abstract definition of $\mathfrak{G}'$ in the well-known form

$$s^2 = t^3 = (st)^p = 1$$

($p = 3, 4, 5$ according as $\mathfrak{G}'$ is tetrahedral, octahedral or icosahedral). We then take a representation $\mathfrak{G}_1$ of $\mathfrak{G}$ by unitary matrices, for instance, Klein's representation by unimodular matrices (**15**, ch. 2). If $\mathbf{S}_1$ and $\mathbf{T}_1$ are a pair of matrices corresponding to s and t, then $\mathbf{S}_1^2$, $\mathbf{T}_1^3$ and $(\mathbf{S}_1\mathbf{T}_1)^p$ are all scalar matrices. Any u.g.g.r. corresponding to $\mathfrak{G}'$ will then have, as corresponding matrices to s and t, scalar multiples of $\mathbf{S}_1$ and $\mathbf{T}_1$. These multiples are to be chosen in such a way that $\mathfrak{G}$ contains reflection matrices and is generated by them. The sets of possible multipliers are finite in number and can be determined without difficulty. We give explicit results below, but suppress the details of the work involved. It should be noted that here (as elsewhere) we do not regard the group $\mathfrak{G}$ as distinct from its conjugate $\overline{\mathfrak{G}}$ (in which every matrix is replaced by its complex conjugate).

In order to save space, we shall arrange the results in tabular form. For each collineation group $\mathfrak{G}'$ we give explicit forms for Klein's generators $\mathbf{S}_1$, $\mathbf{T}_1$ and $\mathbf{S}_1\mathbf{T}_1$ of the corresponding linear group, together with the values of $\mathbf{S}_1^2$, $\mathbf{T}_1^3$, $(\mathbf{S}_1\mathbf{T}_1)^p$ and the characteristic roots of matrices corresponding to a generator of each cyclic subgroup of $\mathfrak{G}'$. The corresponding u.g.g.r. will then be generated by matrices $\mathbf{S} = \lambda\mathbf{S}_1$, $\mathbf{T} = \mu\mathbf{T}_1$ (where λ, μ are suitably chosen roots of unity) and will have an abstract definition of the form

$$S^2 = Z^{k_1},\quad T^3 = Z^{k_2},\quad (ST)^p = Z^{k_3},\quad ZS = SZ,\quad ZT = TZ,\quad Z^k = 1.$$

The table giving the details of the u.g.g.r. corresponding to each collineation group lists, in its first eight columns, the serial number of the group in the complete list in Table VII, the order of the group, and the values of λ, μ, k_1, k_2, k_3, k. Then follow columns giving the number r_q of q-fold reflections in $\mathfrak{G}$, for various values of q. Typical reflections may easily be identified by considering the characteristic roots of the products of $\mathbf{S}$, $\mathbf{T}$, $(\mathbf{ST})^j$ with powers of $\mathbf{Z}$. The multipliers λ, μ have been chosen so that Z is always represented by the matrix $\exp(2\pi i/k)\,\mathbf{I}$.

The groups obtained are of two kinds according as they can be generated by two reflections, or require three. When $\mathfrak{G}$ can be generated by two reflections it is the symmetry group of a complex regular polygon, whose symbol (in the notation of (**23**)) is given in the table, together with explicit expressions for a pair of generating reflections. These are not necessarily the simplest pair, but have been chosen in such a way as to verify a property of these groups given later (see 5.4).

Groups derived from the tetrahedral group.

$$\mathbf{S}_1 = \begin{pmatrix} i & \\ & -i \end{pmatrix},\ \mathbf{T}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \epsilon & \epsilon^3 \\ \epsilon & \epsilon^7 \end{pmatrix},\ \mathbf{S}_1\mathbf{T}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \epsilon^3 & \epsilon^5 \\ \epsilon^7 & \epsilon^5 \end{pmatrix}\quad (\epsilon = \exp(2\pi i/8)).$$

$$\mathbf{S}_1^2 = \mathbf{T}_1^3 = -\mathbf{I},\ (\mathbf{S}_1\mathbf{T}_1)^3 = \mathbf{I}.$$

The characteristic roots of $\mathbf{S}_1$, $\mathbf{T}_1$, $\mathbf{S}_1\mathbf{T}_1$ are $(i, -i)$, $(-\omega, -\omega^2)$, (ω, ω^2) respectively.

The four corresponding groups $\mathfrak{G}$ are shown in Table I.

TABLE I

No.	Order	λ	μ	k_1	k_2	k_3	k	r_2	r_3	Polygon	Pair of generating reflections
4	24	-1	$-\omega$	1	2	2	2		8	*3(24)3*	ST, T
5	72	$-\omega$	$-\omega$	1	6	6	6		16	*3(72)3*	T, $(ST)^{-1}$
6	48	i	$-\omega$	4	4	1	4	6	8	*3(48)2*	SZ^2, T
7	144	$i\omega$	$-\omega$	8	12	3	12	6	16		

Groups derived from the octahedral group.

$$\mathbf{S}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix}, \quad \mathbf{T}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \epsilon & \epsilon \\ \epsilon^3 & \epsilon^7 \end{pmatrix}, \quad \mathbf{S}_1\mathbf{T}_1 = \begin{pmatrix} \epsilon^3 & \\ & \epsilon^5 \end{pmatrix} \quad (\epsilon = \exp(2\pi i/8)).$$

$$\mathbf{S}_1^2 = \mathbf{T}_1^3 = (\mathbf{S}_1\mathbf{T}_1)^4 = -\mathbf{I}.$$

The characteristic roots of $\mathbf{S}_1$, $\mathbf{T}_1$, $\mathbf{S}_1\mathbf{T}_1$, $(\mathbf{S}_1\mathbf{T}_1)^2$ are $(i, -i)$, $(-\omega, -\omega^2)$, (ϵ^3, ϵ^5), $(i, -i)$ respectively.

The eight corresponding groups $\mathfrak{G}$ are shown in Table II.

TABLE II

No.	Order	λ	μ	k_1	k_2	k_3	k	r_2	r_3	r_4	Polygon	Pair of generating reflections
8	96	ϵ^3	1	1	2	4	4	6		12	*4(96)4*	TS, ST
9	192	i	ϵ	8	7	8	8	18		12	*4(192)2*	S, ST
10	288	$\epsilon^7\omega^2$	$-\omega$	7	12	12	12	6	16	12	*4(288)3*	ST, $(TZ^4)^{-1}$
11	576	i	$\epsilon\omega$	24	21	8	24	18	16	12		
12	48	i	1	2	1	1	2	12				
13	96	i	i	4	1	2	4	18				
14	144	i	$-\omega$	6	6	5	6	12	16		*3(144)2*	SZ^3, T
15	288	i	$i\omega$	12	3	10	12	18	16			

It should be noticed that the two groups (8) and (13), of the same order 96 are not isomorphic, for the squares of the operations of period eight (e.g. S) in the former group belong to the central, while the squares of the operations of period eight (e.g. ST) in the latter do not. Similarly the groups (10) and (15), each of order 288, are distinct abstract groups (being isomorphic with the direct products of the two former groups with a cyclic group of order three).

Groups derived from the icosahedral group.

$$\mathbf{S}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} \eta^4 - \eta & \eta^2 - \eta^3 \\ \eta^2 - \eta^3 & \eta - \eta^4 \end{pmatrix}, \quad \mathbf{T}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} \eta^2 - \eta^4 & \eta^4 - 1 \\ 1 - \eta & \eta^3 - \eta \end{pmatrix}, \quad \mathbf{S}_1\mathbf{T}_1 = \begin{pmatrix} -\eta^3 & \\ & -\eta^2 \end{pmatrix}$$

$$(\eta = \exp(2\pi i/5)).$$

$$\mathbf{S}_1^2 = -\mathbf{I}, \; \mathbf{T}_1^3 = \mathbf{I}, \; (\mathbf{S}_1\mathbf{T}_1)^5 = -\mathbf{I}.$$

The characteristic roots of $\mathbf{S}_1$, $\mathbf{T}_1$, $\mathbf{S}_1\mathbf{T}_1$ are $(i, -i)$, (ω, ω^2), $(-\eta^3, -\eta^2)$ respectively.

The seven corresponding groups $\mathfrak{G}$ are shown in Table III.

TABLE III

No.	Order	λ	μ	k_1	k_2	k_3	k	r_2	r_3	r_5	Polygon	Pair of generating reflections
16	600	$-\eta^3$	1	7	10	10	10			48	*5(600)5*	$(ST)^{-1}, (TS)^{-1}$
17	1200	i	$i\eta^3$	20	11	20	20	30		48	*5(1200)2*	$(ST)^{-1}, (SZ^{10})^{-1}$
18	1800	$-\omega\eta^3$	ω^2	11	30	30	30		40	48	*5(1800)3*	$TZ^{20}, (ST)^{-1}$
19	3600	$i\omega$	$i\eta^3$	40	33	40	60	30	40	48		
20	360	1	ω^2	3	6	5	6		40		*3(360)3*	$ST^{-1}S^{-1}, T^{-1}$
21	720	i	ω^2	12	12	1	12	30	40		*3(720)2*	T^{-1}, S^{-1}
22	240	i	1	4	4	3	4	30				

This completes our determination of all the irreducible u.g.g.r. A complete list appears in Table VII on page 301.

II. SOME PROPERTIES OF FINITE UNITARY GROUPS GENERATED BY REFLECTIONS

5. Statement of results to be proved. The next five sections (§§6–10) of this paper are devoted to establishing the following propositions:

5.1 *Let* $\mathfrak{G}$ *be a finite group, of order* g, *of unitary transformations on* n *variables. Then the following properties are equivalent*:

(a) $\mathfrak{G}$ *is a u.g.g.r.*

(b) $\mathfrak{G}$ *possesses a set of* n *algebraically independent invariant forms* $I_1, I_2, \ldots, I_n$ *of degrees* $m_1 + 1, m_2 + 1, \ldots, m_n + 1$, *such that*

$$\prod_{i=1}^{n} (m_i + 1) = g.$$

(c) *There exists a set of* n *algebraically independent invariant forms of* $\mathfrak{G}$ *such that every invariant form of* $\mathfrak{G}$ *is expressible as a polynomial in the forms of the set.* (Such a set of forms will be called a *basic* set.)

The integers $m_1, m_2, \ldots, m_n$, which we shall call the *exponents* of the group, are determined uniquely. The degrees of the forms of any basic set satisfy the

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condition (b), and any set of n algebraically independent forms which satisfies (b) is a basic set. The number of reflections in $\mathfrak{G}$ is Σm_i.

It will be convenient in what follows to suppose that

$$m_1 \leqslant m_2 \leqslant \ldots \leqslant m_n.$$

5.2 *If $\mathfrak{G}$ is a u.g.g.r. in U_n, then the Jacobian of the n forms of a basic set with respect to the variables $x_1, x_2, \ldots, x_n$ factorises into the product of the reflecting primes, the prime corresponding to a p-fold reflection counting with multiplicity $p - 1$.*

5.3 *If $\mathfrak{G}$ is a u.g.g.r. in U_n, the number of operations which leave fixed all the points of some linear space of dimension $n - k$ is the coefficient of t^k in the product*

$$\prod_{i=1}^{n} (1 + m_i t)$$

where $m_1, m_2, \ldots, m_n$ are the exponents of the group.

5.4 *If $\mathfrak{G}$ is an irreducible u.g.g.r. in U_n which is generated by n reflections, then these reflections can be chosen and ordered in such a way that their product has period $h = m_n + 1$, and so that the characteristic roots of this product are*

$$\exp(2\pi i m_r/h) \qquad (r = 1, 2, \ldots, n),$$

where the m_i are the exponents of the group.

Before proceeding to the proofs of these propositions it is convenient to discuss them in more detail.

Of these theorems, 5.1 is perhaps of the greatest theoretical interest since it gives a characterisation of the u.g.g.r. in terms of their invariants. In the real case, when the transformations are orthogonal, Coxeter has shown **(10)** that (a) implies (b), and that any invariant form of $\mathfrak{G}$ can be expressed rationally in terms of n forms satisfying condition (b). But it does not appear from Coxeter's argument that this expression is necessarily integral. Coxeter also obtains Σm_i for the number of reflections in the real case. Our proof of 5.1, which occupies §§6–8, runs along the following lines. It is clear that condition (a) implies (b) for a reducible u.g.g.r. if it does so for each irreducible component. Since, from Coxeter's results, it holds for the Euclidean groups, it is sufficient to verify it for each of the irreducible unitary g. g. r. which are not equivalent to orthogonal groups. This verification occupies §6, and is based either on explicit construction of a set of invariant forms, or upon known results for the associated collineation groups. Next, in §7, we prove that (b) implies (c) by showing that any set of n invariant forms which satisfy (b) is necessarily basic. Since the degrees of the invariant forms of a basic set are evidently unique, this establishes the invariance of the exponents m_i. Finally, in §8, we prove that (c) implies (a), and show in the course of the proof that the number of reflections of $\mathfrak{G}$ is Σm_i.

It is of interest to note, in this connection, that Frame has shown **(10a)** how these basic invariant forms may be used to compute the character table of the group.

Coxeter gives **(10)**, for the real case, a deduction of 5.2 from the first part of the proof of 5.1 by a very simple argument due to Racah. This argument extends at once to the unitary groups; the additional complication due to multiplicities greater than unity if the group contains p-fold reflections with $p > 2$ is not serious. We omit details of this proof.

Theorem 5.3 implies that the number of reflections in $\mathfrak{G}$ is Σm_i and that the order of $\mathfrak{G}$ is $\Pi(m_i + 1)$, both of which results are implicit in 5.1. We have been unable to extend the argument to deduce the remaining properties expressed by 5.3 from 5.1. Since 5.3 holds for reducible groups if it holds for each irreducible component, it is enough to verify the result for each of the irreducible groups listed in Table VII. This is done in §9 and Table VIII.

The result expressed by 5.4 is in a somewhat different category. In the first place it only applies to a restricted class of u.g.g.r., since both the condition of irreducibility and the condition of being generated by n reflections (and not $n + 1$) are relevant. In the real case, a more precise result has been given by Coxeter (**6**; **10**), who shows that the various products formed from n generating reflections arranged in different orders are all conjugate in $\mathfrak{G}$, and therefore have the same characteristic roots. His argument (**6**, p. 602) depends upon the fact that the graph (**9**, p. 84) of any real reflection group is a *tree*, i.e., contains no *circuits*. When the group is unitary the graph must contain (**24**, p. 368) either a circuit or a numbered node (i.e., a p-fold reflection with $p > 2$) and so Coxeter's argument is no longer valid. In general, therefore, the various products of generators of a unitary g. g. r. in different orders are not necessarily conjugate, and it is necessary to select the appropriate generators and their order. Consequently, to establish 5.4 we write down a set of generating reflections which have the required property. This is done in §10.

For a real group the exponents occur in pairs such that

$$m_r + m_{n-r} = h \qquad (r = 1, 2, \ldots, [\tfrac{1}{2}n]).$$

This is not true for unitary groups in general, nor is the corollary $\Sigma m_i = \frac{1}{2}nh$ (**10**, p. 772).

6. The invariants of the irreducible u.g.g.r. In this section we consider in turn the irreducible u.g.g.r. which are not expressible as groups of orthogonal transformations in Euclidean space, and show that in each case we can find an algebraically independent set of invariants, the product of whose degrees is equal to the order of the group $\mathfrak{G}$. The groups are numbered as in Table VII.

(2) Reference to the explicit form of the operations of $G(m, p, n)$ in §2 shows that a suitable set of invariants consists of the elementary symmetric functions of $x_1^m, x_2^m, \ldots, x_n^m$ of degrees $1, 2, \ldots, n - 1$, together with $(x_1x_2 \ldots x_n)^q$.

(3) The cyclic group of order m in U_1 is generated by a single m-fold reflection, and so there is clearly a single invariant, which may be taken to be x_1^m.

Consider next the groups in U_2. The invariant theory of the binary linear groups is given in detail by Klein (**15**). For each of the three collineation groups (tetrahedral, octahedral, icosahedral) there are three invariant forms which may be denoted by f, h, t; h is the Hessian of f, and t the Jacobian of f and h. Any invariant of the collineation group is expressible integrally and rationally in terms of f, h and t, and there is a syzygy expressing t^2 as a polynomial in f and

h. These forms are relative invariants of any linear group which has this collineation group associated with it. The determination of the absolute invariants is easily accomplished by examining the effect on the various forms of a pair of generating operations of $\mathfrak{G}$.

For the tetrahedral group f, h, t are of respective degrees 4, 4, 6, and t^2 is a linear combination of f^3 and h^3. Under the generating transformations S_1 and T_1 of the binary tetrahedral group given in §4, the forms f, h, t are absolutely unaltered by S_1, and receive factors ω^2, ω, 1 respectively under the operation T_1. (Precisely this means that if T_1 is represented as a transformation $x' = Tx$, then $f(x'_1, x'_2) \equiv \omega^2 f(x_1, x_2)$ and so on). We now examine each of the groups of Table I which have the tetrahedral group as the corresponding collineation group and note the effect on f, h, t of the generators S, T of the u.g.g.r. Hence we may pick out the invariant forms of the group. The results are listed in Table IV.

TABLE IV

Group	Order	Sf	Sh	St	Tf	Th	Tt	Invariants	Degrees
4	24	1	1	1	1	ω^2	1	f, t	4, 6
5	72	ω	ω	1	1	ω^2	1	f^3, t	12, 6
6	48	1	1	-1	1	ω^2	1	f, t^2	4, 12
7	144	ω	ω	-1	1	ω^2	1	f^3, t^2	12, 12

The columns headed Sf, Sh, St, Tf, Th, Tt give the factors by which the forms f, h, t are multiplied under operation by S or T.

For the octahedral group, f, h, t have respective orders 6, 8, 12, and under S_1 and T_1 are multiplied by $(-1, 1, -1)$ and $(1, 1, 1)$ respectively. Hence the results for the groups associated with the octahedral group are as listed in Table V.

TABLE V

Group	Order	Sf	Sh	St	Tf	Th	Tt	Invariants	Degrees
8	96	$-i$	1	1	1	1	1	h, t	8, 12
9	192	1	1	-1	$-i$	1	-1	h, t^2	8, 24
10	288	$-i$	ω	1	1	ω^2	1	h^3, t	24, 12
11	576	1	1	-1	$-i$	ω^2	-1	h^3, t^2	24, 24
12	48	1	1	-1	1	1	1	f, h	6, 8
13	96	1	1	-1	-1	1	1	f^2, h	12, 8
14	144	1	1	-1	1	ω^2	1	f, t^2	6, 24
15	288	1	1	-1	-1	ω^2	1	f^2, t^2	12, 24

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For the icosahedral group, f, h, t have respective orders 12, 20, 30 and are absolute invariants under S_1 and T_1. Hence the results for the remaining groups in U_2 are as listed in Table VI, in which $\eta = \exp(2\pi i/5)$.

TABLE VI

Group	Order	Sf	Sh	St	Tf	Th	Tt	Invariants	Degrees
16	600	η	1	1	1	1	1	h, t	20, 30
17	1200	1	1	-1	η	1	-1	h, t^2	20, 60
18	1800	η	ω^2	1	1	ω	1	h^3, t	60, 30
19	3600	1	ω^2	-1	η	1	-1	h^3, t^2	60, 60
20	360	1	1	1	1	ω	1	f, t	12, 30
21	720	1	1	-1	1	ω	1	f, t^2	12, 60
22	240	1	1	-1	1	1	1	f, h	12, 20

The results for the u.g.g.r. in U_n with $n > 2$ are obtained (with one exception, the group of order 108.9! in U_6) as follows. Each such group $\mathfrak{G}$ corresponds to a collineation group $\mathfrak{G}'$ whose invariants have been determined explicitly (we shall give precise references below). To determine the invariants of $\mathfrak{G}$ it is therefore sufficient to examine the behaviour of these invariants of $\mathfrak{G}'$ under the generating reflections of $\mathfrak{G}$. These operations correspond to homologies in $\mathfrak{G}'$ which form a single conjugate set of cyclic subgroups (except in the case of the Hessian group, where there are two sets of homologies). Since conjugate operations of $\mathfrak{G}$ have the same effect on any relative invariant, and since the conjugate cyclic subgroups corresponding to the homologies in $\mathfrak{G}'$ determine conjugate cyclic subgroups corresponding to the reflections in $\mathfrak{G}$, it is sufficient to consider the behaviour of the forms for a single reflection in $\mathfrak{G}$ [or for a single reflection of each of the two types in the case of the symmetry group of *3(24)3(18)2*]. The classical expressions for these groups in each case possess a homology whose equations are given by a matrix of very simple (diagonal or monomial) form, and so the necessary verification is almost immediate. It seems unnecessary to give further details.

(24) The invariants of the collineation group of order 168 in S_2 consist **(14)** of a quartic, its Hessian (a sextic), a covariant of degree 14, and the Jacobian of these, which breaks up into the 21 axes of the homologies in the group. For the corresponding u.g.g.r. of order 336 in U_3, the forms of degrees 4, 6, 14 are absolute invariants, while their Jacobian changes sign for half the operations of the group **(5; 18)**.

(25) and (26) The Hessian group of order 216 in S_2 is the collineation group that leaves invariant the nine inflections of a pencil of cubic curves in the plane. The invariant forms of this collineation group comprise a sextic $I_{(6)}$, a form $I_{(9)}$ of degree 9 representing the harmonic polars of the inflections, and two forms

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$I_{(12)}$, $I'_{(12)}$ representing respectively the four degenerate cubics and the four equianharmonic cubics in the pencil. A syzygy connects $I_{(12)}^3$ with $I_{(6)}$, $I_{(9)}$ and $I'_{(12)}$ (**16**; **18**, p. 253). For the two corresponding u.g.g.r. in U_3 of order 648 and 1296 respectively, the invariant forms may be taken to be $I_{(6)}$, $I_{(9)}$, $I'_{(12)}$ and $I_{(6)}$, $I'_{(12)}$, $I_{(18)}$ respectively, where

$$I_{(18)} \equiv 432\ I_{(9)}^2 - I_{(6)}^3 + 3I_{(6)}I'_{(12)}.$$

The Jacobian of $I_{(6)}$, $I_{(9)}$, $I'_{(12)}$ is a multiple of the square of $I_{(12)}$; that of $I_{(6)}$, $I'_{(12)}$, $I_{(18)}$ is a multiple of the product $I_{(9)}I_{(12)}^2$.

(27) The collineation group of order 360 in S_2 has invariant forms of degrees 6, 12, 30 with a Jacobian of degree 45 (**29**; **18**, p. 254), related in the same way as the forms corresponding to the group of order 168. These forms, of degrees 6, 12, 30 are invariant for the corresponding u.g.g.r. of order 2160 in U_3.

(29) and (31) The invariant forms for the collineation group of order 11520 in S_3 are of degrees 8, 12, 20, 24 and 60, the latter form being the Jacobian of the first four and having reducible square. The form of degree 24 can be taken to be the product of six quartic forms, and the collineation group of order 1920 in S_3 is the group keeping one of these forms fixed; its fundamental system consists of invariants of degrees 4, 8, 12, 20 together with their Jacobian (of degree 40). The forms of degrees 8, 12, 20, 24 are invariants for the group (31), and those of degrees 4, 8, 12, 20 for (29) (**16**).

(32) For the collineation group of order 25920 in S_3, Maschke (**17**) obtains invariants of degrees 12, 18, 24 and 30, the Jacobian of these being the square of a form of degree 40 representing the primes of the 40 homologies of period three contained in the group. These forms of degrees 12, 18, 24, 30 are invariant for the corresponding u.g.g.r. (32).

(33) For the collineation group of order 25920 in S_4 Burkhardt (**4**) obtains invariants of degrees 4, 6, 10, 12, 18. These forms are invariants for the corresponding u.g.g.r. of order 51840. Their Jacobian, of degree 45, consists of the invariant primes of the 45 homologies in $\mathfrak{G}'$.

(34) Finally, the group $[2\ 1; 3]^3$ of order 108.9! in U_6 possesses a system of invariants of degrees 6, 12, 18, 24, 30, 42 whose Jacobian is of degree 126. The existence of these forms was indicated by Todd (**26**) and the slight reservation expressed there about their possible interdependence can be settled by a calculation like that made by Coxeter (**10**, p. 777) showing that for a certain special set of values of the variables the Jacobian of these forms does not vanish.

The remaining groups, which are all Euclidean, have been discussed by Coxeter (**10**). In Table VII we summarise the results, by listing, in the seventh column of the table, the degrees of the invariant forms for all the irreducible u.g.g.r. It is easily verified that their product is equal to the order of the group and so we conclude:

Any irreducible u.g.g.r. in U_n possesses a set of n algebraically independent invariant forms, the product of whose degrees is equal to the order of the group.

The extension of this result to reducible groups is almost immediate.

7. Basic sets of invariants. In this section we prove that the condition 5.1 (b) implies 5.1 (c). More precisely, we prove the following:

Let $\mathfrak{G}$ be a finite group of linear substitutions on n variables, and suppose that $\mathfrak{G}$ possesses a set of n algebraically invariant forms $I_1, I_2, \ldots, I_n$ of degrees $m_1 + 1, m_2 + 1, \ldots, m_n + 1$ such that $\Pi(m_i + 1) = g$, the order of $\mathfrak{G}$. Then any invariant form of $\mathfrak{G}$ is expressible rationally and integrally in terms of $I_1, I_2, \ldots, I_n$.

In the first place we show that any invariant form of $\mathfrak{G}$ may be expressed *rationally* in terms of $I_1, I_2, \ldots, I_n$. This may be done (exactly as in **(10)**) by observing that the solutions of the equations

$$I_i(x_1, x_2, \ldots, x_n) = c_i \qquad (i = 1, 2, \ldots, n)$$

where the c_i are constants, form a *single* set of conjugate points under $\mathfrak{G}$ (if the constants are sufficiently general). Consequently any invariant of $\mathfrak{G}$, necessarily an algebraic function of $I_1, I_2, \ldots, I_n$ is a single-valued, and hence a rational function of these forms.

Let $(\xi_1, \xi_2, \ldots, \xi_n)$ be a point of the space of the n variables $(x_1, x_2, \ldots, x_n)$ whose transforms under $\mathfrak{G}$ are all distinct and which therefore does not make the Jacobian of $I_1, I_2, \ldots, I_n$ vanish. Let $\beta_i = I_i(\xi_1, \xi_2, \ldots, \xi_n)$. Then the system of equations

7.1 $$I_i(x_1, x_2, \ldots, x_n) = \beta_i$$

has a system of $g = \Pi(m_i + 1)$ distinct solutions, and these solutions are isolated, since the Jacobian of $I_1, I_2, \ldots, I_n$ does not vanish for any of them. It follows from the general theory of elimination that these exhaust all the solutions of 7.1, and in particular there are no 'infinite' solutions, (i.e., solutions of the homogeneous equations

$$I_i(x_1, x_2, \ldots, x_n) = \beta_i x_0^{m_i+1}$$

with $x_0 = 0$). Consequently *for any finite set of constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ the equations*

7.2 $$I_i(x_1, x_2, \ldots, x_n) = \alpha_i \qquad (i = 1, 2, \ldots, n)$$

have at least one finite solution (no 'infinite solutions' being possible since these would also be solutions of 7.1).

Now suppose that $J(x_1, x_2, \ldots, x_n)$ is an invariant of $\mathfrak{G}$ that is expressible *rationally*, but not *integrally*, in terms of $I_1, I_2, \ldots, I_n$. We shall show that this assumption leads to a contradiction.

Let $\phi(y_1, y_2, \ldots, y_n)$ and $\psi(y_1, y_2, \ldots, y_n)$ be polynomials with no common factor, such that

7.3 $$\phi(I_1, I_2, \ldots, I_n) \equiv \psi(I_1, I_2, \ldots, I_n)\ J(x_1, x_2, \ldots, x_n)$$

identically in $x_1, x_2, \ldots, x_n$. By our assumption $\psi(y_1, y_2, \ldots, y_n)$ is of degree greater than zero, that is, is not a constant. Let $\psi_1(y_1, y_2, \ldots, y_n)$ be an irreducible factor of $\psi(y_1, y_2, \ldots, y_n)$ and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be any set of constants such that

$$\psi_1(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0.$$

Let $x'_1, x'_2, \ldots, x'_n$ be a solution (which we have shown exists) of 7.2. Then, for $x_i = x'_i$, the right hand side of 7.3 is zero, since $\psi_1(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$. Hence

$$\phi(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$$

since $x'_1, x'_2, \ldots, x'_n$ satisfy 7.2. Thus $\phi(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$ whenever $\psi_1(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$, and so $\psi_1(y_1, y_2, \ldots, y_n)$ is a factor of $\phi(y_1, y_2, \ldots, y_n)$. This is a contradiction. Thus 7.3 can hold only if $\psi(y_1, y_2, \ldots, y_n)$ is a constant, i.e., if $J(x_1, x_2, \ldots, x_n)$ is a polynomial in $I_1, I_2, \ldots, I_n$.

8. Characterisation of the u.g.g.r. In this section we complete the proof of Theorem 5.1 by proving that (c) implies (a). That is, we prove:

Any finite group of unitary transformations on n variables which possesses a basic set of invariant forms $I_1, I_2, \ldots, I_n$ is a u.g.g.r.

Let $\mathfrak{G}$ be such a group, g its order, and $m_i + 1$ the degree of $I_i (i = 1, 2, \ldots, n)$. The proof depends on a theorem of Molien (**22**; **5**, p. 300) which states that, in any group of linear transformations, if a_r denote the number of linearly independent invariant forms of $\mathfrak{G}$ of order r, then

$$\sum_0^\infty a_r \lambda^r \equiv \frac{1}{g} \sum_s \frac{1}{(1 - \omega_1{}^s\lambda) \ldots (1 - \omega_n{}^s\lambda)} \equiv \frac{1}{g} \sum_S \frac{1}{|\mathbf{I} - \mathbf{S}\lambda|}$$

where $\omega_1{}^s, \omega_2{}^s, \ldots, \omega_n{}^s$ denote the characteristic roots of the operation S of $\mathfrak{G}$, and the sums on the right extend over all the operations of the group.

Now if the invariants $I_1, I_2, \ldots, I_n$ form a basic set (so that they are algebraically independent, and every invariant can be expressed as a polynomial in them) then the linearly independent forms of degree r may be taken to be just the power products of $I_1, I_2, \ldots, I_n$ of degree r in the variables. Thus the function

$$\prod_{i=1}^n (1 - \lambda^{m_i+1})^{-1}$$

is also a generating function for the number of independent invariant forms of a given degree and so,

8.1 $$g \prod_{i=1}^n \frac{1}{(1 - \lambda^{m_i+1})} \equiv \sum_s \frac{1}{(1 - \omega_1{}^s\lambda) \ldots (1 - \omega_n{}^s\lambda)}.$$

On the right of 8.1 there is only one term with denominator $(1 - \lambda)^n$, corresponding to the identical element of $\mathfrak{G}$. Hence, on multiplying both sides of 8.1 by $(1 - \lambda)^n$ and then putting $\lambda = 1$ we obtain

8.2 $$g = \prod_{i=1}^n (m_i + 1).$$

This proves, incidentally, that the product of the degrees of a basic set of invariant forms is necessarily equal to the order of the group.

We now subtract $(1 - \lambda)^{-n}$ from each side of 8.1, multiply by $(1 - \lambda)^{n-1}$ and again put $\lambda = 1$. After a brief calculation we find that the left hand side reduces to $\frac{1}{2}\Sigma m_i$. The only terms on the right which can contribute non-zero terms to the sum are those whose denominators involve the factor $(1 - \lambda)^{n-1}$. These terms correspond to operations S with $n - 1$ characteristic roots equal to unity, i.e., to *unitary reflections*. Suppose that π is the fixed prime corresponding to such a reflection, then the aggregate of all reflections for which π is the fixed prime are the elements (other than the identity) of a cyclic group of some order p say, generated by a p-fold reflection S with $n - 1$ characteristic roots unity and the other one a primitive pth root of unity. The sum of the corresponding terms in the expression under consideration is thus

$$(1 - \theta)^{-1} + (1 - \theta^2)^{-1} + \ldots + (1 - \theta^{p-1})^{-1} = \tfrac{1}{2}(p - 1)$$

where $\theta = \exp(2\pi i/p)$. By summing over all the reflecting primes we obtain $\frac{1}{2}\Sigma m_i = \frac{1}{2}\Sigma(p - 1)$. Hence *the number of reflections in* $\mathfrak{G}$ *is equal to* Σm_i.

Let $\mathfrak{H}$ be the subgroup of $\mathfrak{G}$ generated by these reflections. Then $\mathfrak{H}$ is a u.g.g.r., and consequently, by the results of §§6, 7, possesses a basic set of n invariant forms $J_1, J_2, \ldots, J_n$ of degrees $\mu_1 + 1, \mu_2 + 1, \ldots, \mu_n + 1$, such that $\Pi(\mu_i + 1) = h$, the order of $\mathfrak{H}$. Since $\mathfrak{H}$ possesses this basic set of invariant forms, the number of reflections in $\mathfrak{H}$ is $\Sigma\mu_i$ and so

8.3 $$\sum \mu_i = \sum m_i.$$

Now each of the forms $I_1, I_2, \ldots, I_n$, being invariant for $\mathfrak{G}$, is invariant for the subgroup $\mathfrak{H}$, and is therefore expressible as a polynomial in $J_1, J_2, \ldots, J_n$ (since these last forms constitute a basic set for $\mathfrak{H}$). Hence we may write

$$I_i = \phi_i(J_1, J_2, \ldots, J_n) \qquad (i = 1, 2, \ldots, n)$$

where the ϕ_i are polynomials. Since each of the sets $I_1, I_2, \ldots, I_n$ and $J_1, J_2, \ldots, J_n$ are algebraically independent forms in $x_1, x_2, \ldots, x_n$, the Jacobian of $I_1, I_2, \ldots, I_n$ with respect to $J_1, J_2, \ldots, J_n$ is not identically zero. Consequently there is a permutation $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ of $(1, 2, \ldots, n)$ such that the term

$$\frac{\partial I_{\sigma_1}}{\partial J_1} \frac{\partial I_{\sigma_2}}{\partial J_2} \cdots \frac{\partial I_{\sigma_n}}{\partial J_n}$$

is not zero. This implies

$$m_{\sigma_i} + 1 \geqslant \mu_i + 1 \qquad (i = 1, 2, \ldots, n)$$

and so, by 8.3, $m_{\sigma_i} = \mu_i$. Therefore the order $\Pi(\mu_i + 1)$ of $\mathfrak{H}$ is equal to the order $\Pi(m_i + 1)$ of $\mathfrak{G}$, so that $\mathfrak{H}$ must coincide with $\mathfrak{G}$. The group $\mathfrak{G}$ is therefore generated by the reflections it contains.

This completes the proof of Theorem 5.1.

9. Types of operation in u.g.g.r. In this section we prove Theorem 5.3. This theorem may be restated in the form

$$g_r = s_r(m_1, m_2, \ldots, m_n)$$

where g_r is the number of operations of $\mathfrak{G}$ that leave a space of $n - r$ dimensions invariant, and s_r is the value of the elementary symmetric function of degree r in the exponents of $\mathfrak{G}$.

We are unable to give an explanation of this property in general terms, except for $r = 1$, which has already been discussed in §8, and for the trivial case $r = 0$ corresponding to the identity of $\mathfrak{G}$. Consequently it is necessary to verify the theorem for each group in Table VII.

This verification is divided, for convenience, into three parts:

(a) Groups $[3^{n-1}]$ and $G(m, p, n)$ in U_n.

(b) The u.g.g.r. in U_2.

(c) The u.g.g.r. in U_n ($n > 2$) except those in (a).

We proceed to examine each of these in turn.

(a) *The groups* $[3^{n-1}]$ *and* $G(m, p, n)$ *in* U_n.

The group $G(m, p, n)$ consists (cf. §2) of the transformations

9.1 $$x'_i = \theta^{\nu_i} x_{\sigma_i}$$

where $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is an arbitrary permutation σ of $(1, 2, \ldots, n)$, θ is a primitive mth root of unity, and

9.2 $$\nu_1 + \nu_2 + \ldots + \nu_n \equiv 0 \pmod{p}.$$

As we have seen, the exponents m_i for this group take the values

9.3 $$rm - 1 \quad (r = 1, 2, \ldots, n-1), \qquad nq - 1,$$

where $q = m/p$. When $m = p = 1$, the group reduces to the symmetry group of the regular simplex α_{n-1} lying in the prime $\Sigma x_i = 0$. Considered as a group in U_n, Σx_i is a linear invariant and the corresponding exponent m_i is zero. (More pedantically, $G(1, 1, n)$ is the direct product of $[3^{n-1}]$ with a cyclic group of order one in U_1, consisting of the identical operation!) Our treatment will therefore apply, not only to $G(m, p, n)$ but also to the group $[3^{n-1}]$.

Let $g_r(m, p, n)$ denote the number of operations of $G(m, p, n)$ which leave fixed every point of some space of dimension $n - r$ ($r = 0, 1, \ldots, n$), so that $g_0(m, p, n) = 1$, and let

$$G(m, p, n;\ t) \equiv \sum_{r=0}^{n} g_r(m, p, n)\, t^r.$$

Any operation leaving fixed every point of a space of dimension $n - r$ has all but r of its characteristic roots equal to unity, and conversely. By forming the characteristic equation of the matrix of transformation 9.1 we see that this will happen if and only if there are exactly $n - r$ cycles in the permutation σ with the property that the sum of the corresponding exponents ν_i is congruent to zero (mod m).

Let us consider a particular permutation σ with $n - r + s$ cycles, $n - r$ of which are designated in advance as the ones for which the exponent sum is congruent to zero (mod m). Let $\rho_1, \rho_2, \ldots, \rho_s$ be the residues of the exponent sums in the remaining s cycles. Then evidently, all but one of the exponents in each cycle can be chosen at will; the last exponent is then completely determined. Thus, for given $\rho_1, \rho_2, \ldots, \rho_s$ there are m^{r-s} ways of choosing the exponents ν_i.

The residues $\rho_1, \rho_2, \ldots, \rho_s$ are subject to the conditions

$$\rho_1 \not\equiv 0, \rho_2 \not\equiv 0, \ldots, \rho_s \not\equiv 0 \pmod{m},$$
$$\rho_1 + \rho_2 + \ldots + \rho_s \equiv 0 \pmod{p},$$

the latter arising from 9.2. Let $\phi(s)$ be the number of solutions of this set of congruences. If $\rho_1, \rho_2, \ldots, \rho_{s-1}$ are given arbitrary values not congruent to zero (mod m), then ρ_s is determined modulo p. If $\rho_1 + \rho_2 + \ldots + \rho_{s-1} \not\equiv 0 \pmod{p}$ then $\rho_s \not\equiv 0 \pmod{p}$ and so there are $m/p = q$ possible values for ρ_s, while if $\rho_1 + \rho_2 + \ldots + \rho_{s-1} \equiv 0 \pmod{p}$ then there are only $q - 1$ possible values (since $\rho_s \equiv 0 \pmod{m}$ is excluded). Consequently

$$\phi(s) = q(m-1)^{s-1} - \phi(s-1)$$

and, since $\phi(1) = q - 1$, we find that

$$\begin{aligned}\phi(s) &= q[(m-1)^{s-1} - (m-1)^{s-2} + \ldots + (-1)^s (m-1)] - (-1)^s [q-1],\\ &= q[(m-1)^{s-1} - (m-1)^{s-2} + \ldots + (-1)^{s-1}] + (-1)^s,\\ &= q[(m-1)^s - (-1)^s]/m + (-1)^s,\\ &= \frac{(m-1)^s + (-1)^s (p-1)}{p}.\end{aligned}$$

It follows that

$$g_r(m, p, n) = \sum_{s=0}^{r} \binom{n-r+s}{s} m^{r-s} \left[\frac{(m-1)^s + (-1)^s (p-1)}{p}\right] a_{r-s}(n),$$

where $a_{r-s}(n)$ is the number of operations in $\mathfrak{S}_n$ containing exactly $n - r + s$ cycles. Hence

$$G(m, p, n; t) = \sum_{r=0}^{n} \sum_{s=0}^{r} \binom{n-r+s}{s} m^{r-s} \left[\frac{(m-1)^s + (-1)^s (p-1)}{p}\right] a_{r-s}(n)\, t^r,$$

and so, upon putting $r = s + u$,

$$\begin{aligned}G(m, p, n; t) &= \sum_{u=0}^{n} \sum_{s=0}^{n-u} \binom{n-u}{s} m^u \left[\frac{(m-1)^s + (-1)^s (p-1)}{p}\right] a_u(n)\ t^{s+u}\\ &= \sum_{u=0}^{n} \left[\frac{\{1 + (m-1)t\}^{n-u} + (p-1)(1-t)^{n-u}}{p}\right] a_u(n)\, m^u t^u\end{aligned}$$

9.4
$$= \frac{1}{p}\Bigg[\{1 + (m-1)t\} \sum_{u=0}^{n} a_u(n) \left(\frac{mt}{1 + (m-1)t}\right)^u + (p-1)(1-t)^n \sum_{u=0}^{n} a_u(n) \left(\frac{mt}{1-t}\right)^u\Bigg].$$

Now, from the recurrence relation

$$a_u(n+1) = a_u(n) + n\, a_{u-1}(n),$$

which may be verified by considering the partitions of $\mathfrak{S}_n$ which arise by sup-

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pressing a fixed symbol in the permutations of $\mathfrak{S}_{n+1}$, it may be shown by induction that

$$\sum_{u=0}^{n} a_u(n)\, \xi^u = \prod_{r=1}^{n-1} (1 + r\xi).$$

Hence, from 9.4,

$$G(m, p, n;\ t)$$

$$= \frac{1}{p}\left[\{1+(m-1)t\} \prod_{r=1}^{n-1} \{1+(m-1)t+rmt\} + (p-1)(1-t) \prod_{r=1}^{n-1} (1-t+rmt) \right]$$

$$= \frac{1}{p}\left(\prod_{r=1}^{n-1} \{1 + (rm - 1)\, t\} \right)\left(\{1 + (nm - 1)\, t\} + (p - 1)(1 - t) \right)$$

$$= \prod_{r=1}^{n-1} [1 + (rm - 1)\, t] \,.\, [1 + (nq - 1)\, t],$$

and so

9·5 $$G(m, p, n;\ t) = \prod_{i=1}^{n} (1 + m_i t),$$

where the m_i are the exponents of the group in 9.3. Hence Theorem 5.3 is verified for these two groups.

(b) *The u.g.g.r. in* U_2 (*nos.* (4) *to* (22) *in Table VII*).

The verification of Theorem 5.3 for these groups is trivial, since in enumerating the groups in §4 we stated the number of reflections, and in each case this will be found to be equal to $m_1 + m_2$ where m_1 and m_2 have the values indicated in Table VII. Also, as the order of the group is $(m_1 + 1)(m_2 + 1)$, it follows that $g_2 = m_1 m_2$.

(c) *The u.g.g.r. in* U_n (*nos.* (23) *to* (37) *in Table VII*).

In order to determine the values of the g_r for these groups, we employ two different methods.

The first method depends upon examination of a polytope associated with the group and upon determining the number of operations of each type from the known geometry of the figure. This method can be applied easily only in a small number of dimensions ($n = 3, 4$) after which it is simpler to use the second method. In order to explain this it is necessary to make some definitions.

Let S be any operation of the group $\mathfrak{G}$ and suppose that it leaves invariant a linear space of dimension $n - r$ (and no space of higher dimension), then we shall call this linear space the *axis* of S. Thus the axis of a reflection is of $n - 1$ dimensions, and the axis of the identity consists of the whole space. All the operations of $\mathfrak{G}$ which leave a particular linear subspace L_{n-r} invariant (at least) form a subgroup $\mathfrak{H}$ of $\mathfrak{G}$ which may be called the subgroup associated with that subspace. $\mathfrak{H}$ is a u.g.g.r. in r dimensions, and g_r (computed for $\mathfrak{H}$), which is the number of operations of $\mathfrak{H}$ which leave no more than a single point invariant, is evidently the number of operations of $\mathfrak{G}$ which leave *only* the L_{n-r}

invariant, i.e., have the L_{n-r} as their axis. We call this number the *multiplier* of $\mathfrak{H}$.

Turning our attention now to the collineation group $\mathfrak{G}'$ in S_{n-1} corresponding to $\mathfrak{G}$, we see that, for most of the groups with $n > 2$, the reflections of $\mathfrak{G}$ are in 1-1 correspondence with the homologies of $\mathfrak{G}'$ (§3). The reflections of $\mathfrak{G}$ that leave a space L_{n-r} invariant correspond to a set of homologies in $\mathfrak{G}'$ whose centres span a subspace S_{r-1} in S_{n-1}. Thus to each axis L_{n-r} corresponds a subspace of the configuration of centres of homologies of the collineation group, and since the latter have been enumerated for many of the larger groups (**11**; **12**; **13**; **25**), the determination of the former is straightforward.

By way of example, consider the group $[2\ 1;\ 3]^3$. The associated collineation group is of order 6,531,840 and we refer to Miss Hamill's description (**12**) of the configuration formed by the centres of the 126 homologies. The lines of the configuration are of two types: 2835 e-lines, each of which corresponds to an axis L_4 associated with the subgroup which consists of the direct product of two groups of order two, and 1680 κ-lines each of which corresponds to an axis L_4 associated with the dihedral group of order six. Considering the planes, spaces, etc. of the configuration we may identify and enumerate all the axes of $\mathfrak{G}$. Determination of the subgroup associated with each axis from the configuration of vertices in S_{r-1} is straightforward, but it is worth noting that the configurations β_r and γ_r of (**13**) correspond to axes associated with the groups $[3^{1,1,r-2}]$ and $[3^r]$ respectively, illustrating that β_2 and γ_2 are identical (**13**, p. 58).

In Table VIII at the end of this paper we list all the axes of each type for the groups (23) to (37) by the method outlined above. The different subgroups $\mathfrak{H}$ associated with the axes are given in the second column of the table, and the multipliers in the third. The groups are denoted by the symbols of Table VII, and, as usual, the sign $\times$ implies that the direct product is to be taken. In addition, the symbol $[\Pi_n]$ is used for the symmetry group of the polytope Π_n.

10. The product of the generators. We now come to the verification of Theorem 5.4 which asserts that, in the case where $\mathfrak{G}$ is an irreducible u.g.g.r. in U_n generated by n reflections, these can be chosen in such a way that the period of their continued product is $h = m_n + 1$, and the characteristic roots of the corresponding matrix are $\exp(2\pi i m_r/h)$ $(r = 1, 2, \ldots, n)$, where m_i are the exponents of the group.

This has been established for the real groups by Coxeter (**10**). It is trivial for the group generated by a single p-fold reflection in U_1, and in the case of the complex regular polygons in U_2 it may be verified that the pairs of generating reflections given in §4 have the required property.

The remaining groups to be considered are $G(m, m, n)$ and $G(m, 1, n)$, the groups of orders 336 and 2160 in U_3, the symmetry groups of the regular complex polytopes

$$3(24)3(24)3, \quad 3(24)3(18)2, \quad 3(24)3(24)3(24)3$$

and the groups $[2\ 1;\ 1]^4$, $[2\ 1;\ 2]^3$, $[2\ 1;\ 3]^3 \equiv [3\ 1;\ 2]^3$.

$G(m, 1, n)$ is the symmetry group of the regular polytope $\gamma_n{}^m$ and may be generated **(24)** by the m-fold reflection

$$Q: \qquad x_1' = \theta x_1,\ x_i' = x_i \qquad (i = 2, 3, \ldots, n;\ \theta^{-1} = \exp(2\pi i/m))$$

and by the 2-fold reflections

$$R_i: \qquad x_i' = x_{i+1},\ x_{i+1}' = x_i \qquad (i = 1, 2, \ldots, n-1)$$

$$x_j' = x_j \qquad (j \neq i,\ i+1).$$

The product $QR_1R_2 \ldots R_{n-1}$ is the transformation

$$x_1' = \theta x_n,\quad x_2' = x_1,\quad x_3' = x_2,\quad \ldots,\quad x_n' = x_{n-1}.$$

The characteristic equation of the corresponding matrix is

$$\lambda^n - \theta = 0$$

and so the characteristic roots are the n values of $\theta^{1/n}$, or

$$\epsilon^{rm-1} \qquad (r = 1, 2, \ldots, n),$$

where $\epsilon = \exp(2\pi i/mn)$.

$G(m, m, n)$ is the symmetry of the polytope $\frac{1}{m}\gamma_n{}^m$ and may be generated **(24)** by the reflections R_i together with the 2-fold reflection

$$S: \qquad x_1' = \theta^{-1}x_2,\quad x_2' = \theta x_1,\quad x_i' = x_i \qquad (i = 3, 4, \ldots, n).$$

The product $SR_1R_2 \ldots R_{n-1}$ is the transformation

$$x'_1 = \theta^{-1}x_1,\quad x_2' = \theta x_n,\quad x_3' = x_2,\quad x_4' = x_3,\quad \ldots,\quad x'_n = x_{n-1}.$$

The characteristic equation of the corresponding matrix is

$$(\lambda - \theta^{-1})(\lambda^{n-1} - \theta) = 0$$

and so the characteristic roots are θ^{-1} and the values of $\theta^{1/(n-1)}$, or

$$\epsilon^{n-1},\ \epsilon^{rm-1} \qquad (r = 1, 2, \ldots, n-1),$$

where $\epsilon = \exp(2\pi i/m(n-1))$.

The group of order 336 in U_3 **(19)** contains 2-fold reflections in 21 planes:

$$x_i = 0,\quad x_i \pm x_j = 0,\quad \alpha x_i \pm x_j \pm x_k = 0,$$

where (i, j, k) is any permutation of (1, 2, 3) and α is either root of the equation $t^2 - t + 2 = 0$. We select

$$\alpha = \tfrac{1}{2}(1 - i\sqrt{7}) = -\ (\beta^4 + \beta^2 + \beta), \qquad \beta = \exp(2\pi i/7),$$

so that

$$\bar{\alpha} = \tfrac{1}{2}(1 + i\sqrt{7}) = -\ (\beta^3 + \beta^5 + \beta^6).$$

The group is generated by the reflections

$$10.1 \qquad \mathbf{R}_1: \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix},\quad \mathbf{R}_2: \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix},\quad \mathbf{R}_3: \tfrac{1}{2}\begin{pmatrix} 1 & -1 & -\alpha \\ -1 & 1 & -\alpha \\ -\bar{\alpha} & -\bar{\alpha} & 0 \end{pmatrix}$$

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in the primes

$$x_2 - x_3 = 0, \quad x_3 = 0, \quad x_1 + x_2 + \alpha x_3 = 0,$$

respectively. The matrix of the product $R_1R_2R_3$ is

$$\frac{1}{2}\begin{pmatrix} 1 & -1 & -\alpha \\ \bar{\alpha} & \bar{\alpha} & 0 \\ -1 & 1 & -\alpha \end{pmatrix}.$$

Its characteristic equation is

$$\lambda^3 - \bar{\alpha}\lambda^2 - \alpha\lambda + 1 = 0,$$

and its characteristic roots are ζ^3, ζ^5, ζ^{13} where $\zeta = \exp(2\pi i/14)$.

The group of order 2160 in U_3 **(19)** contains 2-fold reflections in 45 planes:

$$x_i = 0, \quad x_i \pm \omega x_i = 0, \quad x_i \pm \gamma x_i \pm \gamma^2 x_k = 0,$$
$$x_i \pm \omega\gamma^2 x_j \pm \omega^2\gamma x_k = 0, \quad x_i \pm (1 - \omega^2\gamma)\, x_i \pm \omega x_k = 0,$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$ and γ is either root of the equation $t^2 + t - 1 = 0$. We substitute

$$\gamma = \tfrac{1}{2}(-1 + \sqrt{5}) = 2\cos(2\pi/5) = \tau^{-1},$$

where $\tau = \tfrac{1}{2}(1 + \sqrt{5}) = -2\cos(4\pi/5)$. The group is generated by the reflections

10.2 $$\mathbf{R}_1\colon \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad \mathbf{R}_2\colon \frac{1}{2}\begin{pmatrix} 1 & -\omega\tau & -\omega^2\tau^{-1} \\ -\omega^2\tau & -\tau^{-1} & -\omega \\ -\omega\tau^{-1} & -\omega^2 & \tau \end{pmatrix}, \quad \mathbf{R}_3\colon \begin{pmatrix} & -\omega^2 & \\ -\omega & & \\ & & 1 \end{pmatrix}$$

in the primes $x_1 = 0$, $\omega^2\tau x_1 + \tau^2 x_2 + \omega x_3 = 0$, $x_1 + \omega^2 x_2 = 0$ respectively. The matrix of the product $R_1R_2R_3$ is

$$\frac{1}{2}\begin{pmatrix} -\omega^2\tau & \omega^2 & \omega^2\tau^{-1} \\ \omega\tau^{-1} & \omega\tau & -\omega \\ 1 & \tau^{-1} & \tau \end{pmatrix}.$$

Its characteristic equation is

$$\lambda^3 + \omega^2\tau\,\lambda^2 + \omega\tau\,\lambda + 1 = 0$$

and its characteristic roots are ζ^5, ζ^{11}, ζ^{29}, where $\zeta = \exp(2\pi i/30)$.

The symmetry group of the polyhedron *3(24)3(24)3* **(23)** contains 3-fold reflections in the 12 planes:

$$x_i = 0, \quad x_1 + \omega^j x_2 + \omega^k x_3 = 0 \qquad (i, j, k = 1, 2, 3).$$

It is generated by the reflections

10.3 $$\mathbf{R}_1\colon \begin{pmatrix} 1 & & \\ & 1 & \\ & & \omega^2 \end{pmatrix}, \quad \mathbf{R}_2\colon \frac{-i}{\sqrt{3}}\begin{pmatrix} \omega & \omega^2 & \omega^2 \\ \omega^2 & \omega & \omega^2 \\ \omega^2 & \omega^2 & \omega \end{pmatrix}, \quad \mathbf{R}_3\colon \begin{pmatrix} 1 & & \\ & \omega^2 & \\ & & 1 \end{pmatrix}$$

in the planes $x_3 = 0$, $x_1 + x_2 + x_3 = 0$, $x_2 = 0$ respectively. The characteristic roots of $\mathbf{R}_1$ and $\mathbf{R}_3$ are $(1, 1, \omega^2)$ and of $\mathbf{R}_2$ are $(1, 1, \omega)$. The matrix of the product $R_1 R_2^{-1} R_3$ is

$$\frac{i}{\sqrt{3}}\begin{pmatrix} \omega^2 & 1 & \omega \\ \omega & \omega & \omega \\ 1 & \omega^2 & \omega \end{pmatrix}.$$

Its characteristic equation is

$$\lambda^3 - \omega^2\lambda^2 + \omega\lambda - 1 = 0,$$

and its characteristic roots are ζ^5, ζ^8, ζ^{11} where $\zeta = \exp(2\pi i/12)$.

The symmetry group **(23)** of the regular polyhedron $3(24)3(18)2$ (or its reciprocal) contains in addition to the above 3-fold reflections, 2-fold reflections in the nine planes

$$x_i - \omega^k x_j = 0, \qquad (i, j, k = 1, 2, 3).$$

It is generated by the 3-fold reflections R_1, R_2 above together with the 2-fold reflection

10.4 $$\mathbf{S}: \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}$$

in the plane $x_2 - x_3 = 0$. The matrix of the product $SR_1R_2^{-1}$ is

$$\frac{i}{\sqrt{3}}\begin{pmatrix} \omega^2 & \omega & \omega \\ 1 & 1 & \omega \\ \omega & \omega^2 & \omega \end{pmatrix}.$$

Its characteristic equation is

$$\lambda^3 + \omega = 0,$$

and its characteristic roots are ζ^5, ζ^{11}, ζ^{17} where $\zeta = \exp(2\pi i/18)$.

The symmetry group of the regular polytope $3(24)3(24)3(24)3$ **(8; 23)** contains 80 3-fold reflections in 40 primes:

$$x_i = 0, \quad x_1 + \omega^j x_2 + \omega^k x_3 = 0, \quad x_1 - \omega^j x_2 - \omega^k x_4 = 0$$
$$x_1 - \omega^j x_3 + \omega^k x_4 = 0, \quad x_2 - \omega^j x_3 - \omega^k x_4 = 0$$
$$(i = 1, 2, 3, 4; \quad j, k = 1, 2, 3).$$

The group is generated by the 3-fold reflections:

10.5 $$\mathbf{R}_1: \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \omega^2 & \\ & & & 1 \end{pmatrix}, \quad \mathbf{R}_2: \frac{-i}{\sqrt{3}}\begin{pmatrix} \omega & \omega^2 & \omega^2 & 0 \\ \omega^2 & \omega & \omega^2 & 0 \\ \omega^2 & \omega^2 & \omega & 0 \\ 0 & 0 & 0 & i\sqrt{3} \end{pmatrix}$$
$$\mathbf{R}_3: \begin{pmatrix} 1 & & & \\ & \omega^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \mathbf{R}_4: \frac{-i}{\sqrt{3}}\begin{pmatrix} \omega & -\omega^2 & 0 & -\omega^2 \\ -\omega^2 & \omega & 0 & \omega^2 \\ 0 & 0 & i\sqrt{3} & 0 \\ -\omega^2 & \omega^2 & 0 & \omega \end{pmatrix}$$

in the primes $x_3 = 0$, $x_1 + x_2 + x_3 = 0$, $x_2 = 0$, $-x_1 + x_2 + x_4 = 0$ respectively. The characteristic roots of $\mathbf{R}_1$, $\mathbf{R}_3$ are $(1, 1, 1, \omega^2)$ and of $\mathbf{R}_2$, $\mathbf{R}_4$ are $(1, 1, 1, \omega)$. The matrix of the product $R_1R_2^{-1}R_3R_4^{-1}$ is

$$\frac{-i}{\sqrt{3}}\begin{pmatrix} 0 & \omega & -\omega & -\omega^2 \\ -\omega & -\omega & -\omega & 0 \\ \omega & 0 & -\omega & \omega^2 \\ \omega & -\omega & 0 & -\omega^2 \end{pmatrix}.$$

Its characteristic equation is

$$\lambda^4 - \omega^2\lambda^3 + \omega\lambda^2 - \lambda + \omega^2 = 0,$$

and its characteristic roots are ζ^{11}, ζ^{17}, ζ^{23}, ζ^{29}, where $\zeta = \exp(2\pi i/30)$.

The group $[2\ 1;\ 1]^4$ can be generated **(24**, 373**)** by 2-fold reflections P_2, P_1, Q_1, R_1 in the primes $x_1 + x_2 + x_3 + x_4 = 0$, $x_1 - ix_2 = 0$, $x_1 - x_2 = 0$, $x_2 - x_3 = 0$. The matrix of the product $P_2P_1Q_1R_1$ is

$$\frac{1}{2}\begin{pmatrix} i & -1 & i & -1 \\ -i & -1 & -i & -1 \\ -i & 1 & i & -1 \\ -i & -1 & i & 1 \end{pmatrix}.$$

Its characteristic equation is

$$\lambda^4 - i\lambda^3 - \lambda^2 + i\lambda + 1 = 0,$$

and its characteristic roots are ζ^3, ζ^7, ζ^{11}, ζ^{19} where $\zeta = \exp(2\pi i/20)$.

The group $[2\ 1;\ 2]^3$ can be generated **(24**, 373**)** by 2-fold reflections P_1, P_2, Q_1, R_1, R_2 in the primes

$$x_2 - x_3 = 0, \quad x_3 - x_4 = 0, \quad x_1 - x_2 = 0, \quad x_1 - \omega x_2 = 0,$$
$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0$$

(using coordinates in six dimensions). The matrix of the product $R_2R_1Q_1P_1P_2$ is

$$\frac{1}{3}\begin{pmatrix} 2\omega & -1 & -1 & -\omega^2 & -1 & -1 \\ -\omega & -1 & -1 & 2\omega^2 & -1 & -1 \\ -\omega & 2 & -1 & -\omega^2 & -1 & -1 \\ -\omega & -1 & 2 & -\omega^2 & -1 & -1 \\ -\omega & -1 & -1 & -\omega^2 & 2 & -1 \\ -\omega & -1 & -1 & -\omega^2 & -1 & 2 \end{pmatrix}.$$

Its characteristic equation, after division by the extraneous factor $1 - \lambda$ arising from the extra dimension, is

$$\lambda^5 - \omega\lambda^4 + \omega^2\lambda^3 + \omega\lambda^2 - \omega^2\lambda + 1 = 0,$$

and its characteristic roots are ζ^3, ζ^5, ζ^9, ζ^{11}, ζ^{17} where $\zeta = \exp(2\pi i/18)$.

The group $[3\ 1;\ 2]^3$ can be generated **(24**, 373**)** by the above five reflections together with the 2-fold reflection P_3 in the prime $x_4 - x_5 = 0$. The matrix of the product $R_2R_1Q_1P_1P_2P_3$ is

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$$\frac{1}{3}\begin{pmatrix} 2\omega & -1 & -1 & -1 & -\omega^2 & -1 \\ -\omega & -1 & -1 & -1 & 2\omega^2 & -1 \\ -\omega & 2 & -1 & -1 & -\omega^2 & -1 \\ -\omega & -1 & 2 & -1 & -\omega^2 & -1 \\ -\omega & -1 & -1 & 2 & -\omega^2 & -1 \\ -\omega & -1 & -1 & -1 & -\omega^2 & 2 \end{pmatrix}.$$

Its characteristic equation is

$$\lambda^6 - \omega\lambda^5 + \omega^2\lambda^4 - \lambda^3 + \omega\lambda^2 - \omega^2\lambda + 1,$$

and its characteristic roots are ζ^5, ζ^{11}, ζ^{17}, ζ^{23}, ζ^{29}, ζ^{41} where $\zeta = \exp(2\pi i/42)$.

This completes the verification of Theorem 5.4 for all the irreducible u.g.g.r. generated by n reflections.

11. Abstract definitions of the finite unitary groups generated by n reflections. We now give, either explicitly or by reference, abstract definitions of all the u.g.g.r. which are generated by n reflections. In each case the abstract definition takes the form of n generators (corresponding to the n generating reflections of the unitary group) and a set of relations that they satisfy. (In certain cases it will be convenient to introduce further elements of the group into the definition in order to simplify it. Such elements will always be denoted by Z_i.)

The definitions of the real groups ($G(2, 2, n)$, $G(2, 1, n)$ and nos. (1), (23), (28), (30), (35), (36), (37) of Table VII) have been given by Coxeter **(8; 9)** who shows how they may be read off from the graphical symbol for the spherical simplex that forms a fundamental region for the group.

For the symmetry groups of the regular polygons (nos. (4), (5), (6), (8), (9), (10), (14), (16), (17), (18), (20), (21)) the definitions are implicit in the tables of §4.

Shephard **(24)** has given definitions for $G(m, m, n)$, $G(m, 1, n)$ and nos. (29), (33), (34) by extending Coxeter's graphical notation to these unitary groups and reading off the definitions in an analogous manner.

The abstract definitions of the remaining groups are as follows:

(24) The operations R_1, R_2, R_3 of 10.1 satisfy the relations

$$R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^4 = (R_2R_3)^4 = (R_3R_1)^3 = (R_1R_2R_1R_3)^3 = 1,$$

and this is an abstract definition for the u.g.g.r. of order 336 in U_3. The subgroup $\{R_3R_1, R_1R_2\}$ is the simple G_{168}, since it is $(3, 4 \mid 4, 3) \sim (3, 3 \mid 4, 4)$ in the notation of **(7**, pp. 78, 83**)**.

(27) The operations R_1, R_2, R_3 of 10.2 satisfy the relations

$$R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^3 = (R_2R_3)^3 = (R_3R_1)^4 = (R_1R_2R_1R_3)^5 = 1,$$

and this is an abstract definition for the u.g.g.r. of order 2160 in U_3. The subgroup $\{R_1R_2, R_2R_3\}$ is $(3, 3 \mid 4, 5)$ in the notation of **(7**, p. 85**)**.

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(25) The operations R_1, R_2, R_3 of 10.3 satisfy the relations

$$R_1^3 = R_2^3 = R_3^3 = 1, \quad R_1R_3 = R_3R_1,$$
$$(R_1R_2)^2 = Z_1, \quad Z_1R_1 = R_1Z_1, \quad Z_1^2 = 1,$$
$$(R_2R_3)^2 = Z_2, \quad Z_2R_2 = R_2Z_2, \quad Z_2^2 = 1,$$

and this is an abstract definition for the symmetry group (of order 648) of the regular polyhedron *3(24)3(24)3*.

(26) The operations S, R_1, R_2 of 10.3 and 10.4 satisfy the relations

$$S^2 = R_1^3 = R_2^3 = 1, \quad SR_2 = R_2S,$$
$$(SR_1)^2 = Z_1, \quad Z_1R_1 = R_1Z_1, \quad Z_1^3 = 1,$$
$$(R_1R_2)^2 = Z_2, \quad Z_2R_2 = R_2Z_2, \quad Z_2^2 = 1,$$

and this is an abstract definition for the symmetry group (of order 1296) of the regular polyhedron *3(24)3(18)2* or its reciprocal.

(32) The operations R_1, R_2, R_3, R_4 of 10.5 satisfy the relations

$$R_1^3 = R_2^3 = R_3^3 = R_4^3 = 1, \quad R_1R_3 = R_3R_1, \quad R_1R_4 = R_4R_1, \quad R_2R_4 = R_4R_2,$$
$$(R_1R_2)^2 = Z_1, \quad Z_1R_1 = R_1Z_1, \quad Z_1^2 = 1,$$
$$(R_2R_3)^2 = Z_2, \quad Z_2R_2 = R_2Z_2, \quad Z_2^2 = 1,$$
$$(R_3R_4)^2 = Z_3, \quad Z_3R_3 = R_3Z_3, \quad Z_3^2 = 1,$$

and this is an abstract definition for the symmetry group (of order 155520) of the regular polytope *3(24)3(24)3(24)3*.

Each of the above definitions has been checked by the Todd-Coxeter method of enumeration by cosets (**27**). It is worth noting that the definitions of the groups (25), (26) and (32) are what might have been expected by applying the rules of (**24**, p. 374) and the definitions of the groups $G(3, 3, 2)$ and (4) to the extended Coxeter graph (**24**, p. 368).

The authors wish to thank Professor Coxeter for removing some redundant relations. He observes that the definition of (26) still contains one such relation: $Z_1^3 = 1$. In fact, the remaining relations imply

$$Z_1^3 = Z_1^3 R_1^{-3} = (Z_1R_1^{-1})^3 = (SR_1S)^3 = SR_1^3S = 1.$$

Thus a sufficient set of generating relations for this G_{1296} is

$$S^2 = R_1^3 = R_2^3 = 1, \quad SR_2 = R_2S,$$
$$(SR_1)^2 = (R_1S)^2,$$
$$(R_1R_2)^2 = (R_2R_1)^2 = Z, \quad Z^2 = 1.$$

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TABLE VII

LIST OF IRREDUCIBLE U.G.G.R. IN U_n

No.	n	Symbol	Polytope	g	g'	$m_i + 1$	Reference
1	n	$[3^{n-1}]$	α_n	$(n+1)!$	$(n+1)!$	$2, 3, \ldots, n+1$	**(10)**
2	n	$G(m, p, n)$		$qm^{n-1}n!$	$m^{n-1}n!/d$*	$m, 2m, \ldots, (n-1)m, qn$	§§2, 6
3	1	$[\ ]^m$	m-line	m	1	m	§§3, 6
4	2		*3(24)3*	24	12	4, 6	§4
5	2		*3(72)3*	72	12	6, 12	"
6	2		*3(48)2*	48	12	4, 12	"
7	2			144	12	12, 12	"
8	2		*4(96)4*	96	24	8, 12	"
9	2		*4(192)2*	192	24	8, 24	"
10	2		*4(288)3*	288	24	12, 24	"
11	2			576	24	24, 24	"
12	2			48	24	6, 8	"
13	2			96	24	8, 12	"
14	2		*3(144)2*	144	24	6, 24	"
15	2			288	24	12, 24	"
16	2		*5(600)5*	600	60	20, 30	"
17	2		*5(1200)2*	1200	60	20, 60	"
18	2		*5(1800)3*	1800	60	30, 60	"
19	2			3600	60	60, 60	"
20	2		*3(360)3*	360	60	12, 30	"
21	2		*3(720)2*	720	60	12, 60	"
22	2			240	60	12, 20	"
23	3	[3, 5]	{3, 5}	120	60	2, 6, 10	**(10)**
24	3			336	168	4, 6, 14	§§3, 6
25	3		*3(24)3(24)3*	648	216	6, 9, 12	"
26	3		*3(24)3(18)2*	1296	216	6, 12, 18	"
27	3			2160	360	6, 12, 30	"
28	4	[3, 4, 3]	{3, 4, 3}	1152	576	2, 6, 8, 12	**(10)**
29	4	$[2\ 1;1]^4$	$(\frac{1}{4}\gamma_3^4)^{+1}$	7680	1920	4, 8, 12, 20	§§3, 6
30	4	[3, 3, 5]	{3, 3, 5}	14400	7200	2, 12, 20, 30	**(10)**
31	4		$(\frac{1}{2}\gamma_3^4)^{+1}$	64.6!	11520	8, 12, 20, 24	§§3, 6
32	4		*3(24)3(24)3(24)3*	216.6!	36.6!	12, 18, 24, 30	"
33	5	$[2\ 1;2]^3$	$(\frac{1}{3}\gamma_4^3)^{+1}$	72.6!	36.6!	4, 6, 10, 12, 18	"
34	6	$[2\ 1;3]^3$	$(\frac{1}{3}\gamma_5^3)^{+1}$	108.9!	18.9!	6, 12, 18, 24, 30, 42	"
35	6	$[3^{2,2,1}]$	2_{21}	72.6!	72.6!	2, 5, 6, 8, 9, 12	**(10)**
36	7	$[3^{3,2,1}]$	3_{21}	8.9!	4.9!	2, 6, 8, 10, 12, 14, 18	"
37	8	$[3^{4,2,1}]$	4_{21}	192.10!	96.10!	2, 8, 12, 14, 18, 20, 24, 30	"

* $m = pq,\ m > 1,\ m > 1,\ d = (p, n)$.

The u.g.g.r. in U_n are all generated by n reflections except for the following: $G(m, p, n)$ for $p \neq 1, m$; the groups in U_2 (nos. 7, 11, 12, 13, 15, 19, 22) which are not the symmetry groups of regular polygons; and the group of the polytope $(\frac{1}{2}\gamma_3^4)^{+1}$ in U_4.

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TABLE VIII

Types of Operation in U.G.G.R.

Dimension of axis	Subgroup $\mathfrak{H}$	Multiplier of $\mathfrak{H}$	Group (23)	(24)	(25)	(26)	(27)	(28)	(29)	(30)	(31)	(32)	(33)	(34)	(35)	(36)	(37)
		$g_n =$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$n-1$	[ ]	1	15	21		9	45	24	40	60	60		45	126	36	63	120
	$[\ ]^3$	2			12	12						40					
		$g_{n-1} =$	15	21	24	33	45	24	40	60	60	80	45	126	36	63	120
$n-2$	[ ] × [ ]	1	15					72	120	450	360		270	2,835	270	945	3,780
	[3]	2	10	28			120	32	160	200	320		240	1,680	120	336	1,120
	[4]	3		21			45	18	30								
	[5]	4	6				36			72							
	$[\ ]^2 \times [\ ]$	2				36											
	[3(24)3]	15			9	9						90					
	$[\ ]^3 \times [\ ]^3$	4			12							240					
	$G(3, 1, 2)$	10				12											
	$G(4, 2, 2)$	9									30						
		$g_{n-2} =$	59	119	183	327	519	190	530	1,138	1,270	2,310	750	6,195	510	1,617	6,020
$n-3$	[ ] × [ ] × [ ]	1											270	11,340	540	4,095	37,800
	[3] × [ ]	2						96	160	600	960		720	30,240	720	5,040	40,320
	[5] × [ ]	4								360							
	$[3^2]$	6							160	300	480		540	11,340	270	1,260	7,560
	[4, 3]	15						24	40								
	[5, 3]	45								60							
	$[3(24)3] \times [\ ]^3$	30										360					
	[3(24)3(24)3]	440										40					
	$G(3, 3, 3)$	20											40	560			
	$G(4, 4, 3)$	42							20								
	$G(4, 2, 3)$	105									60						
		$g_{n-3} =$						552	2,720	7,140	11,100	28,400	5,750	151,060	3,600	21,735	163,800
$n-4$	[ ] × [ ] × [ ] × [ ]	1														3,780	113,400
	[3] × [ ] × [ ]	2												45,360	1,080	15,120	302,400
	[3] × [3]	4												30,240	120	3,360	67,200
	$[3^2] \times [\ ]$	6											540	68,040	540	8,820	151,200
	$[3^3]$	24											216	27,216	216	2,016	24,192
	$G(2, 2, 4)$	45											45	2,835	45	315	3,150
	$G(3, 3, 3) \times [\ ]$	20												5,040			
	$G(3, 3, 4)$	240											40	1,680			
		$g_{n-4} =$											20,019	1,904,679	13,089	162,939	2,616,558

Dimension of axis	Subgroup $\mathfrak{H}$	Multiplier of $\mathfrak{H}$	Group (23)	(24)	(25)	(26)	(27)	(28)	(29)	(30)	(31)	(32)	(33)	(34)	(35)	(36)	(37)
$n-5$	$[3] \times [\ ] \times [\ ] \times [\ ]$	2														5,040	604,800
	$[3] \times [3] \times [\ ]$	4													360	10,080	403,200
	$[3^2] \times [\ ] \times [\ ]$	6														7,560	453,600
	$[3^2] \times [3]$	12												45,360		5,040	302,400
	$[3^3] \times [\ ]$	24												27,216	216	6,048	241,920
	$[3^4]$	120												18,144	36	1,344	40,320
	$G(2, 2, 4) \times [\ ]$	45														945	37,800
	$G(2, 2, 5)$	420												3,402	27	378	7,560
	$G(3, 3, 4) \times [\ ]$	240												5,040			
	$G(3, 3, 5)$	3,520												672			
	$[2\ 1;\ 2]^3$	25,245												126			
		$g_{n-5} =$												11,559,534	22,284	663,957	24,693,480
$n-6$	$[3] \times [3] \times [\ ] \times [\ ]$	4															604,800
	$[3^2] \times [3] \times [\ ]$	12														5,040	604,800
	$[3^3] \times [\ ] \times [\ ]$	24															362,880
	$[3^2] \times [3^2]$	36															151,200
	$[3^3] \times [3]$	48														2,016	241,920
	$[3^4] \times [\ ]$	120														1,008	120,960
	$[3^5]$	720														288	34,560
	$G(2, 2, 3) \times [3]$	90															50,400
	$G(2, 2, 5) \times [\ ]$	420														378	45,360
	$G(2, 2, 6)$	4,725														63	3,780
	$[3^{1,2,2}]$	12,320														28	1,120
		$g_{n-6} =$														1,289,963	130,085,780
$n-7$	$[3^3] \times [3] \times [\ ]$	48															241,920
	$[3^3] \times [3^2]$	144															120,960
	$[3^5] \times [\ ]$	720															34,560
	$[3^6]$	5,040															8,640
	$G(2, 2, 5) \times [3]$	840															30,240
	$G(2, 2, 7)$	62,370															1,080
	$[3^{1,2,2}] \times [\ ]$	12,320															3,360
	$[3^{1,2,3}]$	765,765															120
		$g_{n-7} =$															323,507,400
		$g_0 =$	45	195	440	935	1,595	385	4,389	6,061	33,649	124,729	25,245	25,569,445	12,320	765,765	215,656,441
Order of group $= \Sigma g_i =$			120	336	648	1,296	2,160	1,152	7,680	14,400	46,080	155,520	51,840	39,191,040	51,840	2,903,040	696,729,600

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