FULLY DEVELOPED TURBULENCE, INTERMITTENCY AND MAGNETIC FIELDS

Uriel Frisch
C.N.R.S.
Observatoire de Nice, France

1. INTRODUCTION

Turbulence is usually associated with the idea of chaos, i.e. erratic behaviour of some observable quantity. Let me stress that there are at least two different kinds of chaos.

Temporal chaos is known to appear in certain systems having only a few degrees of freedom. Take for example the Lorenz model (Lorenz, 1963) which has three degrees of freedom. It is a crude truncation of the Rayleigh-Bénard problem with only one Fourier component in the X- and the Z- directions so that the motion can in no way be considered as spatially chaotic. Nevertheless, there is a strong numerical evidence that the temporal spectrum becomes continuous when the Rayleigh number crosses a certain threshold, an indication that temporal chaos has developed. This kind of chaos can appear on the largest scales of the system which makes it easy to observe. Possible candidates for such theories are irregular variable stars, the geodynamo, etc.,...

Very different is the problem of fully developed turbulence which is essentially a spatio-temporal chaos : when the Reynolds number goes to infinity all space and time scales, down to infinitesimal are excited. Such chaos may develop in a finite time and has universal scaling properties (e.g. a power-law energy spectrum). In the astrophysical context fully developed turbulence may not always be directly observable because of lack of resolution of the small scales. But it will always manifest itself indirectly through a drastic modification of the transport properties.

In the next two sections we try to give simple phenomenological insight into universal properties of fully developed turbulence, particularly the question of intermittency, or in other words spottiness of the small scales. Intermittency is very much at the center of present theoretical studies (Kraichnan, 1974, Frisch, Lesieur and Sulem, 1976, Frisch, Sulem and Nelkin, 1977). Experimentally, it is rather difficult to observe because the small scales carry very little energy. However, magnetic fields which are very sensitive to small-scale velocity gradients can be used as tracers of the small scales (in the MHD case). It is therefore of great interest to note that recent high resolution observations of the small-scale solar magnetic field indicate a very intermittent structure (Stenflo, 1975). Non magnetic intermittent turbulence being rather poorly understood it seems premature to consider the MHD case in detail. However, many overall features are probably common to both cases in particular the steepening of the spectrum. The reader interested in the non intermittent aspects of MHD turbulence such as the non linear dynamo effect is referred to Pouquet et al. (1976).

2. KOLMOGOROV 1941 REVISITED

Big whorls have little whorls
Which feed on their velocity
And little whorls have lesser whorls
And so on to viscosity.

L.F. Richardson, 1922

By the Kolmogorov 1941 (in short K41) theory, we mean the general class of arguments developed by Kolmogorov, Obukhov, Onsager and others which has led in particular to the 5/3 law (see Batchelor, 1953, and Monin and Yaglom, 1975, for review). The 5/3 law may be derived from dimension analysis, but more insight is gained from a simple dynamical argument borrowed from Kraichnan (1972, page 213). We define the energy spectrum $E(k)$ as the kinetic energy per unit mass and unit wavenumber k. It is a convenient simplification, with no significant loss of generality, to consider a discrete sequency of scales or "eddies"

$$\ell_n = \ell_o\, 2^{-n} \qquad n = 0,\ 1,\ 2,\ \ldots \tag{2.1}$$

and of wavenumbers $k_n = \ell_n^{-1}$. The kinetic energy per unit mass in scales $\sim \ell_n$ is defined as

$$E_n = \int_{k_n}^{k_{n+1}} E(k)\, dk\,. \tag{2.2}$$

Let us assume that we have a state of statistically stationary turbulence where energy is introduced into the fluid at scales $\sim \ell_o$, and is then transferred successively to scales $\sim \ell_1$, $\sim \ell_2$, ..., until some scale ℓ_d is reached where dissipation is able to compete with non linear transfer (Fig. 1). If we now make the essential assumption that <u>eddies of any generation are space filling</u>, as indicated in Fig. 1, we may write

$$E_n \sim v_n^2, \tag{2.3}$$

where v_n is a velocity characteristic of n-th generation eddies (in short, n-eddies).

In Eq. (2.3) and subsequently, factors of the order of unity will be systematically dropped except when such factors would cumulate multiplicatively in successive cascade steps. Note that v_n is <u>not</u> the velocity with which n-eddies move with respect to the reference frame of the mean flow, this being mostly due to advection by the largest eddies. It is rather a typical velocity difference across a distance $\sim \ell_n$, the latter being the only dynamically significant quantity. (In this respect the "velocity" in the second line of Richardson's poem is misleading). We now define the eddy turnover time

$$t_n \sim \ell_n/v_n . \tag{2.4}$$

The quantity t_n^{-1} may be considered as the typical shear in scales $\sim \ell_n$, and therefore defines the characteristic rate at which excitation at scales $\sim \ell_n$ is fed into scales $\sim \ell_{n+1}$.

There are, however, at least two important exceptions to this statement. First we may define a viscous dissipation time

$$t_n^{\text{diss}} \sim \ell_n^2/\nu , \tag{2.5}$$

where ν is the kinematic viscosity of the fluid. If

$$t_n^{\text{diss}} << t_n , \tag{2.6}$$

then transfer is no longer able to compete with dissipation, and most of the excitation in scales $\sim \ell_n$ is lost to viscosity. Second, if

$$t_n >> t_o = \ell_o/v_o , \tag{2.7}$$

then the shear acting on scales $\sim \ell_n$ comes mostly from scales $\sim \ell_o$, and t_o should be used instead of t_n as a dynamical time.

Assuming that neither of these two exceptions applies, (this may be checked a posteriori) we make the fundamental assumption that in a time of the order of t_n a sizeable fraction of the energy in scales $\sim \ell_n$ is transferred to scales $\sim \ell_{n+1}$. The rate of transfer of energy per unit mass from n-eddies to (n+1)-eddies is then given by

$$\varepsilon_n \sim E_n/t_n \sim v_n^3/\ell_n . \tag{2.8}$$

Since we assume a stationary process in which energy is introduced at scales $\sim \ell_o$ and removed at scales $\sim \ell_d$, conservation of energy requires that

$$\varepsilon_n \equiv \bar{\varepsilon} , \qquad \ell_d \le \ell_n \le \ell_o . \tag{2.9}$$

https://doi.org/10.1017/S0252921100112539 Published online by Cambridge University Press

Notice that $\bar{\varepsilon}$ can be thought of as a rate of energy injection, a rate of energy transfer or a rate of energy dissipation. From the point of view of inertial range dynamics, the second of these three definitions is the most relevant. Using (2.8) and (2.9) we solve for v_n and E_n:

$$v_n \sim (\bar{\varepsilon}\, \ell_n)^{1/3}, \quad E_n \sim (\bar{\varepsilon}\, \ell_n)^{2/3} , \tag{2.10}$$

which by Fourier transformation yields the K41 spectrum

$$E(k) \sim \bar{\varepsilon}^{\,2/3}\, k^{-5/3} . \tag{2.11}$$

The eddy turnover time of Eq. (2.4) is given by

$$t_n \sim \bar{\varepsilon}^{\,-1/3}\, \ell_n^{\,2/3} . \tag{2.12}$$

Equating (2.12) to the viscous diffusion time (2.5) determines the Kolmogorov microscale

$$\ell_d = (\nu^3/\bar{\varepsilon}\,)^{1/4} . \tag{2.13}$$

Eq. (2.13) gives the length scale at which the cascade is terminated by viscous dissipation.

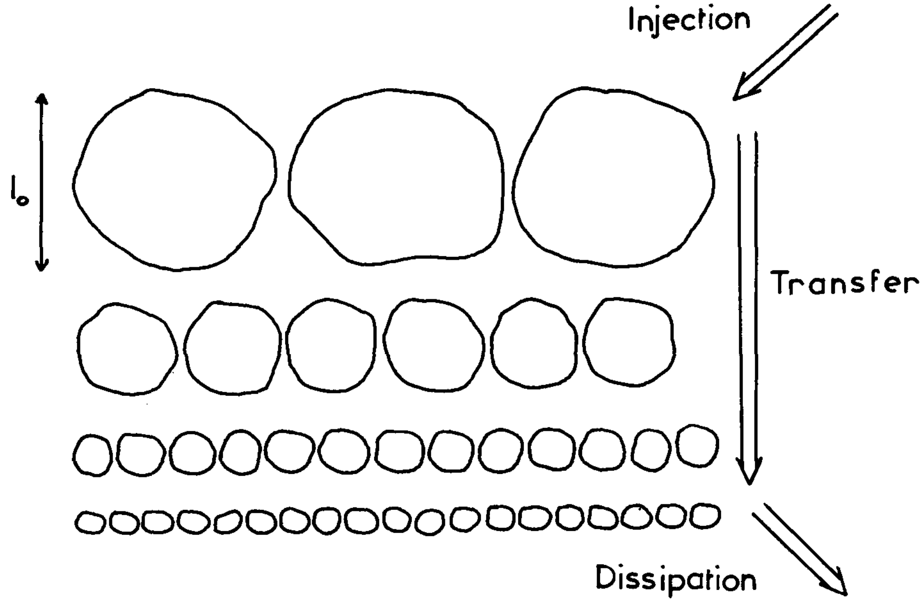

Fig. 1. The energy cascade according to the Kolmogorov 1941 theory. Notice that at each step the eddies are space-filling.

https://doi.org/10.1017/S0252921100112539 Published online by Cambridge University Press

3. INTERMITTENCY : THE β- MODEL

Big whorls have little whorls
Which feed on their vorticity
And little whorls have lesser whorls
Which hardly ever can you see.

Since the first experiments of Batchelor and Townsend (1949) there has been strong evidence that the small scale structures of turbulence become less and less space filling as scale size decreases. (Kuo and Corrsin, 1972; see also Monin and Yaglom, 1975, for review). Dynamically this spottiness of the small scales can be made plausible by a simple vortex stretching argument. Consider the point M within a large scale structure which at the initial time T_o has the largest vorticity amplitude $|\omega|$. This point is also likely to have a large velocity gradient $|\nabla v| \sim |\omega|$. The straining action of the velocity gradient on the vorticity may then be described by a crude form of the vorticity equation

$$\frac{D|\omega|}{Dt} \sim |\omega|^2 \ ; \tag{3.1}$$

hence it is expected that the vorticity downstream of M will rise to very large values (possibly infinite at zero viscosity) in a time of the order of the large eddy turnover time $t_o \sim |\omega|^{-1}$.

Even if the vorticity at time T_o has a very flat spatial distribution, the nonlinearity of Eq. (3.1) will lead to a very narrowly peaked spatial distribution at time $T_o + t_o$. So we see that small scale structures may be generated in a very localized fashion. This argument can be made fully rigorous for the Burgers equation, but not for the Navier-Stokes equation (Léorat, 1975). For the Navier-Stokes equation there is the important complication that the velocity gradient at a point $\underset{\sim}{x}$ is not related in any simple way to the vorticity at $\underset{\sim}{x}$; instead it is given by a Poisson integral with a fairly substantial local contribution, but also with some coupling to nearby points. This could smooth out the vorticity peak, but the smallest scale structures will still have some tendency not to occur uniformly.

Assuming that the small eddies do indeed become less and less space filling, let us now define the β-model. At each step of the cascade process any n-eddy of size $\ell_n = \ell_o 2^{-n}$ produces on the average N (n+1)-eddies. If the largest eddies are space filling, after n generations only a fraction

https://doi.org/10.1017/S0252921100112539 Published online by Cambridge University Press

$$\beta_n = \beta^n \qquad (\beta = N/2^3 \leq 1) \qquad (3.2)$$

of the space will be occupied by active fluid (see Fig. 2). Furthermore we assume that (n+1)-eddies are positionally correlated with n-eddies by embedding or attachment. (For the sake of pictorial clarity this feature is not included in Fig. 2).

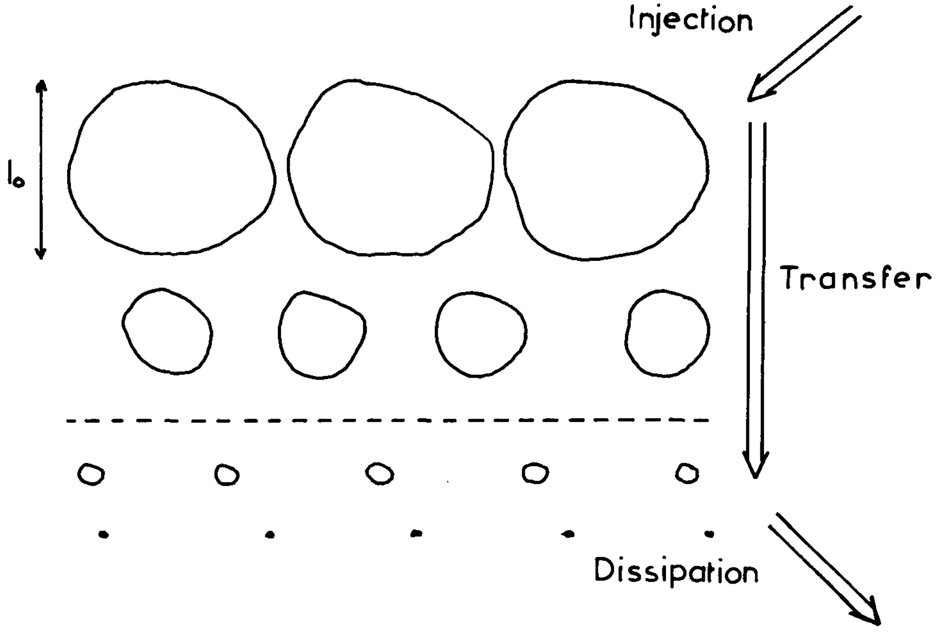

Fig. 2. The energy cascade for intemittent turbulence. Notice that the eddies become less and less space filling.

It is straightforward to work out the modification to K41 in the β-model. Let v_n now denote a typical velocity difference over a distance $\sim \ell_n$ in an active region. The kinetic energy per unit mass on scales $\sim \ell_n$ is then given by

$$E_n \sim \beta_n v_n^2 \qquad . \qquad (3.3)$$

https://doi.org/10.1017/S0252921100112539 Published online by Cambridge University Press

The characteristic dynamical time for transfer of energy from active n-eddies to smaller scales is still given by the turnover time $t_n = \ell_n/v_n$ as in K41 : the generation of (n+1)-eddies arises from the internal dynamics of the n-eddy in which it is embedded. We can express the rate of energy transfer from n-eddies to (n+1)-eddies exactly as in K41, and as in K41 this quantity must be independent of n in the inertial range:

$$\varepsilon_n \sim E_n/t_n \sim \beta_n \ v_n^3/\ell_n \sim \bar{\varepsilon} \ . \tag{3.4}$$

Defining

$$\mu = - \log_2 \beta \ ,$$

we combine Eqs. (3.2 - 3.4) to obtain

$$v_n \sim \bar{\varepsilon}^{\,1/3} \ell_n^{\,1/3} (\ell_n/\ell_o)^{-\mu/3} , \tag{3.5}$$

$$t_n \sim \bar{\varepsilon}^{\,-1/3} \ell_n^{\,2/3} (\ell_n/\ell_o)^{\mu/3} , \tag{3.6}$$

$$E_n \sim \bar{\varepsilon}^{\,2/3} \ell_n^{\,2/3} (\ell_n/\ell_o)^{+\mu/3} , \tag{3.7}$$

and

$$E(k) \sim \bar{\varepsilon}^{\,2/3} k^{-5/3} (k \ \ell_o)^{-\mu/3} . \tag{3.8}$$

All the intermittency corrections may be expressed in terms of the self-similarity dimension $D = 3 - \mu$, a special case of Mandelbrot's (1975) fractal dimension, which is related to the number of offspring by

$$N = 2^D \ . \tag{3.9}$$

That D can suitably be called a dimension is made clear by Fig. 3 which shows three very familiar objects : a unit interval, a square and a cube which have dimensions D equal to 1, 2 and 3 respectively. If we reduce the linear dimensions of these objects by a factor of 2, as in the cascade process, the number of offspring needed to reconstruct the original object is 2^D. For more complicated self-similar objects a natural interpolation is $N = 2^D$, where D need no longer take only integer values. (Some rather exotic examples can be found in Mandelbrot (1975).) It has been shown by Mandelbrot (1974) that D is also the Hausdorff dimension of the dissipative structures in the limit of zero viscosity. D = 2 would correspond to sheet-like structures, but in view of the experimental value of the exponent for the dissipation correlation function a more likely value is $D \approx 2.5$ (See Frisch et al 1977).

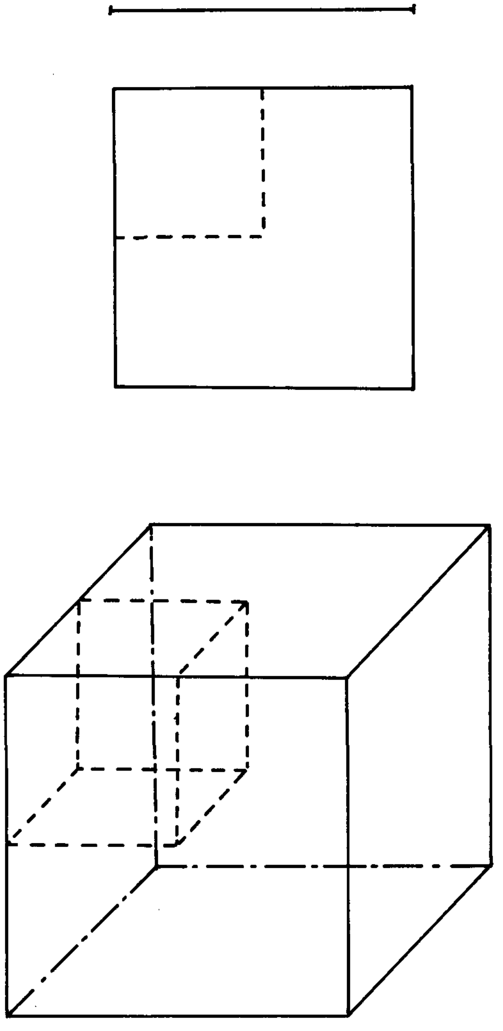

Fig. 3 When the linear dimensions of a D-dimensional object are reduced by a factor λ (here 2), λ^D pieces are needed to reconstruct the original. More exotic examples with non integer D, such as probably occur in turbulence, may be found in Mandelbrot (1975).

https://doi.org/10.1017/S0252921100112539 Published online by Cambridge University Press

Remark 3.1 Equation (3.8) relating the correction to the 5/3 exponent of the K41 theory and the fractal dimension was first derived by Mandelbrot (1976) using the Novikov-Stewart (1964) model.

Remark 3.2 Since $0 \leq D \leq 3$ the corrections to K41 can not make the spectral exponent larger than 8/3. This same upper bound can be derived rigorously from the Navier-Stokes equation for finite energy turbulence (Sulem and Frisch, 1975). This reference also gives a heuristic argument to show that $D \geq 2$.

Remark 3.3 When (3.4) is used for the largest scales we obtain

$$\bar{\varepsilon} \sim v_o^3/\ell_o , \tag{3.10}$$

the same result as in the non-intermittent case. This is important since (3.10) is frequently used in practical calculations. Intermittency may, however, be of practical importance in other ways, particularly when chemical reactions are involved (Herring, 1973).

Dissipation scale

Equating the turnover time (3.6) to the viscous diffusion time ℓ_n^2/ν we obtain the dissipation scale

$$\ell_d \sim \ell_o R^{-3/(4-\mu)} , \tag{3.11}$$

where we have introduced the Reynolds number

$$R = \ell_o v_o/\nu \sim \bar{\varepsilon}^{1/3} \ell_o^{4/3} \nu^{-1} . \tag{3.12}$$

Singularities

Both the K41 and the β-model imply that the three dimensional Euler equation (Navier-Stokes with zero viscosity) leads to a singularity in a finite time. Indeed, if we start with very smooth initial conditions, say only large eddies, then the complete hierarchy of eddies, down to infinitesimal scales should be established in a time

$$t_* \sim \sum_{n=0}^{\infty} t_n \sim \ell_o/v_o . \tag{3.13}$$

https://doi.org/10.1017/S0252921100112539 Published online by Cambridge University Press

Since there is now no viscous cutoff the enstrophy given by

$$\Omega = \int_0^\infty k^2 E(k)\, dk \tag{3.14}$$

will become infinite at time t_*. There is in fact some numerical evidence that such singularities exist (Orszag, 1976a, b). There are also a few known exact solutions which display singularities at a finite time, but these solutions are badly behaved at large distances (Childress and Spiegel, 1976). Finally various stochastic models or low order closures of the statistical Euler equation can be shown to produce singularities in a finite time (Lesieur and Sulem, 1976; André and Lesieur, 1977).

NOTE ON THE M H D CASE

The K41 theory can be easily modified to account for the effect of Alfvén waves. It then yields a $k^{-3/2}$ spectrum (Kraichnan, 1965; Pouquet et al., 1976). How intermittency can be handled in the MHD case is not yet clear, but it is again likely to steepen the spectrum. That the spectral exponent can become as large as 2.5 as suggested by certain solar observations (Harvey, 1976) is a possibility which cannot be ruled out.

Finally we note that singularities should appear in the MHD case as well as in the non magnetic case. There are even some indications that they are present in two-dimensional MHD flows (Pouquet, 1976) although they are known to be absent in the non magnetic two-dimensional case (Wolibner, 1933). The presence of singularities at a finite time in the MHD equation implies that magnetic field line reconnection at high kinetic and magnetic Reynolds number occurs in a time which does not depend on the magnetic diffusivity : it is essentially the large eddy turnover time.

https://doi.org/10.1017/S0252921100112539 Published online by Cambridge University Press

REFERENCES

André, J.C. & Lesieur, M. 1977, Evolution of high Reynolds number turbulence, J. Fluid Mech., to appear

Batchelor, G.K. & Townsend, A.A. 1949, Proc. Roy. Soc. A 199, 238

Batchelor, G.K. 1953, Theory of homogeneous turbulence, Cambridge U. Press

Childress, S. & Spiegel, E. 1976, Private communication

Frisch, U., Lesieur, M. & Sulem, P.L. 1976, Phys. Rev. Lett. 37, 895

Frisch, U., Sulem, P.L. & Nelkin, M. 1977, A simple dynamical model of intermittent fully developed turbulence, submitted J. Fluid Mech.

Harvey, J.W. 1976, Private communication

Herring, J. 1973, Private communication

Kolmogorov, A.N. 1941, C. R. Acad. Sci. USSR 30, 301, 538

Kraichnan, R.H. 1965, Phys. Fluids 8, 1385

Kraichnan, R.H. 1972 in "Statistical Mechanics : New concepts, New problems, New applications", Rice, S.A., Fried, K.F. & Light, J.C. Eds., University of Chicago Press, Chicago

Kraichnan, R.H. 1974, J. Fluid Mech. 62, 305

Kraichnan, R.H. 1975, J. Fluid Mech. 67, 155

Kuo, A.Y. & Corrsin, S. 1971, J. Fluid Mech. 50, 285

Léorat, J. 1975, Thesis, Observatoire de Meudon, Meudon, France

Lesieur, M. & Sulem, P.L. 1976, "Les équations spectrales en turbulence homogène et isotrope : quelques résultats théoriques et numériques" in "Proc. Journées Mathématiques sur la Turbulence", Temam, R. ed., Springer Lecture Notes in Math., to appear

Lorenz, E.N. 1963, J. Atmos. Sci. 20, 130

Mandelbrot, B. 1974, J. Fluid Mech. 62, 331

Mandelbrot, B. 1975, "Les Objets Fractals : Forme, Hasard et Dimension", Flammarion, Paris

Mandelbrot, B. 1976, "Intermittent turbulence and fractal dimension : kurtosis and the spectral exponent 5/3 + B" in "Proc. Journées Mathématiques sur la Turbulence", Orsay, June 1974, Temam, R. ed., Springer Lecture Notes in Mathematics, to appear

Monin, A.S. & Yaglom, A.M. 1975, Statistical Fluid Mechanics, vol. 2 , MIT Press

Novikov, E.A. & Stewart, R.W. 1964, Isvestia Akad. Nauk USSR, Ser. Geophys. n° 3, p. 408

Orszag, S. 1976a, "Statistical Theory of Turbulence" in "Proc. Les Houches 1973", Balian, R. ed., North Holland, to appear

Orszag, S. 1976b, Private communication

Pouquet, A. 1976, "Remarks on two dimensional MHD turbulence", preprint

Pouquet, A., Frisch, U. and Léorat, J. 1976, J. Fluid Mech. 77, 321

Stenflo, J.O. 1976, "Influence of magnetic fields on solar hydrodynamics : experimental results", to appear in "Proc. of IAU Coll. n° 36" held at Nice, September 1976

Sulem, P.L. & Frisch, U. 1975, J. Fluid Mech. 72, 417

Wolibner, W. 1933, Math. Zeitschr. 37, 698

https://doi.org/10.1017/S0252921100112539 Published online by Cambridge University Press