28

Technologies for evaluating the Wilson coefficients

There are nice expositions of these methods in the literature [431–434,3,362,45,399]. We shall not try to replace these discussions, but for a pedagogical reason, we shall repeat within our proper style some of these results. Let us remind ourselves that the evaluation of the Green's function of some local colourless currents is reduced to its evaluation in external gluons or/and quark fields, assuming that the field is weak, namely its momentum is much smaller than the characteristic scale of the problem. In this way, the expansion in power series à la Wilson makes sense.

28.1 Fock–Schwinger fixed-point technology

28.1.1 Fock-Schwinger gauge

Let us now come to the methods for evaluating the Wilson coefficients appearing in the SVZ-expansion. Among others, the Fock–Schwinger method is the most convenient one in practice [431]. It corresponds to the choice of the Fock–Schwinger gauge [435,319]:

$$(x - x_0)A^a_\mu(x) = 0 \,, \tag{28.1}$$

used often in QED. $A^a_\mu(x)$ is the four-potential and x_0 is an arbitrary choice of coordinate which plays the role of a gauge. As Eq. (28.1) breaks explicitly the translation invariance, its restoration (cancellation of the x_0 terms) for gauge-invariant quantities provides a double check of the validity of the calculation. Unfortunately, due to algebraic complications, most calculations have been done in the special choice $x_0 = 0$ of the gauge.

28.1.2 Gluon fields and condensates

Using the identity:

$$A^\mu(x) = \frac{\partial}{\partial x^\mu}[A^\rho(x)x_\rho] - x_\rho \frac{A^\rho(x)}{\partial x^\mu} \,, \tag{28.2}$$

and from Eq. (28.1) at $x_0 = 0$:

$$x_\rho \frac{A^\rho}{\partial x^\mu} = x_\rho G^{\rho\mu} + x_\rho \frac{A^\mu(x)}{\partial x_\rho} \,, \tag{28.3}$$

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one can deduce:

$$A^\mu(x) + x_\rho \frac{A^\mu(x)}{\partial x_\rho} = x_\rho G^{\rho\mu} \ . \tag{28.4}$$

By substituting $x \equiv \alpha z$ in the previous equation, it is easy to realize that this equation is a full derivative:

$$\frac{d}{d\alpha}[\alpha A_\mu(\alpha z)] \ , \tag{28.5}$$

which gives after integration:

$$A^\mu(x) = \int_0^1 d\alpha \ \alpha G^{\rho\mu}(\alpha x) x_\rho \ , \tag{28.6}$$

which expresses the gauge field $A^a_\mu(x)$ in terms of the gluon-strength tensor $G^a_{\mu\nu}$. One can use Eq. (28.6) by Taylor expanding $G_{\rho\mu}$ around $x^\mu = 0$:

$$A^a_\mu = \sum_{x=0}^{\infty} \frac{1}{n!(n+2)} x^\rho x^{\nu_1} \cdots x^{\nu_n} \partial_{\nu_1} \cdots \partial_{\nu_n} \ G^a_{\rho\mu}\big|_{x=0} \ . \tag{28.7}$$

By Taylor expanding $A^\mu(x)$, the gauge condition $x_\mu A^\mu(x) = 0$ becomes:

$$x_\mu \left[A^\mu(x) + x_{\nu_1} \partial_{\nu_1} A^\mu(x)(0) + \frac{1}{2} x_{\nu_1} x_{\nu_2} \partial_{\nu_1} \partial_{\nu_2} A^\mu(x)(0) + \cdots \right] = 0 \ , \tag{28.8}$$

for all x and leads to:

$$\begin{aligned} x_\mu A^\mu(x)(0) &= 0 \ , \\ x_\mu x_{\nu_1} \partial_{\nu_1} A^\mu(x)(0) &= 0 \ , \\ x_\mu x_{\nu_1} x_{\nu_2} \partial_{\nu_1} \partial_{\nu_2} A^\mu(x)(0) &= 0 \ . \end{aligned} \tag{28.9}$$

Therefore:

$$\begin{aligned} x_{\nu_1} \partial_{\nu_1} G(0) &= x_{\nu_1} [D_{\nu_1}, \ G(0)] \ , \\ x_{\nu_1} x_{\nu_2} \partial_{\nu_1} \partial_{\nu_2} G(0) &= x_{\nu_1} x_{\nu_2} \partial_{\nu_1} [D_{\nu_2}, \ G(0)] = x_{\nu_1} x_{\nu_2} [D_{\nu_1}, \ [D_{\nu_2}, \ G(0)]], \ldots \end{aligned} \tag{28.10}$$

and then the useful formula:

$$A^\mu(x) = \sum_{x=0}^{\infty} \frac{1}{n!(n+2)} x^\rho x^{\nu_1} \cdots x^{\nu_n} \left[D_{\nu_1}, \ \left[D_{\nu_2}, \ \left[\ldots \left[D_{\nu_n}, \ G^a_{\rho\mu}\big|_{x=0}\right] \cdots \right]\right]\right] . \tag{28.11}$$

One can immediately form the gluon normal ordered condensate:

$$\begin{aligned} A_\mu(x) A_\nu(y) &= \frac{1}{4} x^\lambda y^\rho G_{\lambda\mu} G_{\rho\nu} + \cdots \\ &= \frac{1}{4d(d-1)} x^\lambda y^\rho [g_{\lambda\rho} g_{\mu\nu} - g_{\lambda\nu} g_{\mu\rho}] G^{\alpha\beta} G_{\alpha\beta} + \cdots \ , \end{aligned} \tag{28.12}$$

where $d = 4 - \epsilon$ is the space-time dimension.

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28.1.3 Light quark fields and condensates

Analogous arguments can be used for the quark fields. One obtains the Taylor expansion:

$$\psi(x)=\sum_n \frac{1}{n!}x^{\nu_1}\cdots x^{\nu_n}D_{\nu_1}\ D_{\nu_2}\cdots D_{\nu_n}\psi|_{x=0}$$
$$\bar\psi(x)=\sum_n \frac{1}{n!}x^{\nu_1}\cdots x^{\nu_n}\bar\psi D^\dagger_{\nu_1}\ D^\dagger_{\nu_2}\cdots D^\dagger_{\nu_n}|_{x=0}\ , \tag{28.13}$$

with:

$$\bar\psi(0)\partial^\dagger_{\nu_1}=\partial_{\nu_1}\bar\psi\ . \tag{28.14}$$

From the previous expressions, one can also form the normal-ordered quark condensate:

$$\begin{aligned}\langle\ :\bar\psi_{i,\alpha}(x)\psi_{i,\alpha}(0):\ \rangle=&\frac{1}{4N}\delta_{\alpha\beta}\Bigg[\left(\delta_{ij}+\frac{i}{4}m\ x^\mu(\gamma_\mu)_{ij}\right)\langle:\bar\psi\psi:\rangle\\&-\frac{i}{16}x^2\left(\delta_{ij}+\frac{i}{6}m\ x^\mu(\gamma_\mu)_{ij}\right)\left\langle:\bar\psi\sigma^{\mu\nu}\frac{\lambda_a}{2}G^a_{\mu\nu}\psi:\right\rangle\\&+\frac{i}{288}x^2\ x^\mu(\gamma_\mu)_{ij}g^2\left\langle:\bar\psi\gamma_\rho\frac{\lambda_a}{2}\psi\sum_f\bar\psi_f\gamma^\rho\frac{\lambda_a}{2}\psi_f:\right\rangle\Bigg].\end{aligned} \tag{28.15}$$

This expression tells us that one should be careful in evaluating the Wilson coefficients of high-dimension condensates as the *propagation* of the $\langle\bar\psi\psi\rangle$ condensate induces extra-contributions due to the mixed and four-quark condensates. This effect has been one of the main source of errors in the existing QSSR literature.

28.1.4 Mixed quark-gluon condensate

By combining the Taylor expressions of the quark and gluon fields, one can form the normal-ordered mixed quark-gluon condensate:

$$\begin{aligned}\langle\ :\bar\psi_i(x)A_\rho(z)\psi_j(0):\ \rangle=&\frac{1}{2}z^\mu\langle\ :\bar\psi G_{\mu\rho}\psi:\ \rangle+\frac{1}{2}x^\nu z^\mu\langle\ :\bar\psi D^\dagger_\nu G_{\mu\rho}\psi:\ \rangle+\cdots\\=&\frac{z^\mu}{96}\Bigg[\left[\sigma_{\mu\rho}-\frac{m}{2}(x_\mu x_\rho-x_\rho x_\mu)+\frac{i}{2}mx^\nu\sigma_{\mu\rho}x_\nu\right]_{ij}\\&\times\langle\ :\bar\psi\sigma_{\tau k}G^{\tau k}\psi:\ \rangle+\left[i\left(-\frac{2}{3}z_\mu\gamma_\rho+\frac{2}{3}z_\rho\gamma_\mu\right)+\frac{1}{2}x^\nu\gamma_\nu\sigma_{\mu\rho}\right]_{ij}\\&\times g^2\left\langle\ :\bar\psi\gamma_\rho\frac{\lambda_a}{2}\psi\sum_f\bar\psi_f\gamma^\rho\frac{\lambda_a}{2}\psi_f:\right\rangle\Bigg].\end{aligned} \tag{28.16}$$

This expression also indicates that the *propagation* of the mixed quark-gluon condensate induces a quartic condensate. Here, one should remark that the non-local condensates used

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in some literature can be identified with the LHS of Eqs. (28.15) and (28.16). In this framework, the Wilson coefficients of these non-local condensates differ from the standard SVZ expansion.

28.1.5 Gluon propagator

For a complete calculational purpose, one also likes to have the expression of the propagators. We only quote their expressions in this gauge. The gluon propagator reads:

$$\begin{aligned} D_{\mu\nu}(q) = & \int d^4x e^{iqx} D_{\mu\nu}(x,0) \\ & = \frac{g_{\mu\nu}}{q^2} + g\frac{2}{q^4}G_{\mu\nu} + g\frac{4i}{q^6}(qD)G_{\mu\nu} - g\frac{2i}{3q^6}g_{\mu}\nu D_\alpha G_{\alpha\beta}\, q^\beta \\ & + g\frac{2}{q^8}(qD)D_\alpha G_{\alpha\beta}g^\beta g_{\mu\nu} + g\frac{2}{q^8}(q^2D^2G_{\mu\nu} - 4(qD)^2G_{\mu\nu}) \\ & + g^2\frac{1}{2q^8}g_{\mu\nu}(q^2G_{\alpha\beta}G^{\alpha\beta} - 4(q_\alpha G_{\alpha\beta})^2) + g^2\frac{4}{q^6}G_{\mu\alpha}G_{\alpha\nu}\ , \end{aligned} \tag{28.17}$$

where:

$$G_{\mu\nu} \equiv G^{ab}_{\mu\nu} = G^a_{\mu\nu}\lambda^b = f^{abc}G^c_{\mu\nu}(0)\ . \tag{28.18}$$

28.1.6 Quark propagator

The quark propagator satisfies the Dirac equation:

$$\left(i\frac{\partial}{\partial x_\mu}\gamma_\mu + gA^\mu(x)\gamma_\mu - M\right)S(x,y) = \delta^{(4)}(x-y)\ , \tag{28.19}$$

where: $A^\mu \equiv (\lambda_a/2)A^\mu_a$ and M is the quark mass. Under the condition that the position of the field is much smaller than the characteristic distance $x-y$, one can have the Taylor expansion:

$$\begin{aligned} iS(x,y) \ = \ & iS^{(0)}(x,y) + g\int d^4z\, iS^{(0)}(x,z)\, i\hat{A}(z)\, iS^{(0)}(z,y) \\ & + g^2\int d^4z'\, d^4z\, iS^{(0)}(x,z')\, i\hat{A}(z')\, iS^{(0)}(z',z)\, i\hat{A}(z)\, iS^{(0)}(z,y) \\ & + \cdots\ , \end{aligned} \tag{28.20}$$

where $S^{(0)}(x,y)$ is the free quark propagator, and $\hat{A} \equiv A^\mu\gamma_\mu$. This expression shows explicitly how many times the quark from the point y scatters 0,1,... external fields before annihilating at $x=0$.

We shall consider the case of the heavy quark propagators in the next section due to the subtlety that the quark and gluon condensates are related to each other through Eq. (27.52).

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Let us now consider the massless quark propagator in external fields. It reads in the x-space:

$$2\pi^2 S(x,y) = \frac{\hat{r}}{(r^2)^2} - \frac{1}{4}\frac{r^\alpha}{r^2}\tilde{G}_{\alpha\mu}(0)\gamma_\mu\gamma_5 + \left\{\frac{i}{2}\frac{\hat{r}}{(r^2)^2}y_\rho x_\mu G_{\rho\mu}(0) - \frac{1}{96}\frac{\hat{r}}{(r^2)^2}(x^2y^2-(xy)^2)G_{\mu\nu}(0)G^{\mu\nu}(0)\right\} + \text{operators of higher dimensions}\,, \tag{28.21}$$

where:

$$r = x - y \qquad G_{\mu\nu} \equiv g\frac{\lambda^a}{2}G^a_{\mu\nu} \qquad \tilde{G}_{\alpha\mu} = \frac{1}{2}\epsilon_{\alpha\mu\nu\beta}G^{\nu\beta}\,. \tag{28.22}$$

For two-point correlators without derivative currents, only the first two terms are operative in the evaluation of the gluon condensate effects, while the other terms contribute in the case of three-point functions or current with derivatives. The extension of this expression including higher-dimension gluon operators can be done. For completeness, this expression is:

$$\begin{aligned} S(p) = \int d^4x\, e^{ipx}S(x,o) &= \frac{1}{\hat{p}} - \frac{p^\alpha}{p^4}g\tilde{G}_{\alpha\beta}\gamma^\beta\gamma^5 \\ &+ \frac{2}{3}g\frac{1}{p^6}[p^2D^\alpha G_{\alpha\beta}\gamma^\beta - \hat{p}D^\alpha G_{\alpha\beta}\,p^\beta - p_\nu D^\nu D^\alpha G_{\alpha\beta}\gamma^\beta - 3ip_\nu D^\nu D^\alpha\tilde{G}_{\alpha\beta}\gamma^\beta\gamma^5] \\ &+ \frac{2}{p^8}\Big[ip^2 ip_\nu D^\nu D^\alpha G_{\alpha\beta}\gamma^\beta - i\,\hat{p}p_\nu D^\nu D^\alpha G_{\alpha\beta}\gamma^\beta - i\,(p_\nu D^\nu)^2\,p^\alpha G_{\alpha\beta}\gamma^\beta \\ &+ 2\,(p_\nu D^\nu)^2\,p^\alpha\tilde{G}_{\alpha\beta}\gamma^\beta\gamma^5 - \frac{1}{2}D_\nu D^\nu p^2 p^\alpha\tilde{G}_{\alpha\beta}\gamma^\beta\gamma^5\Big] \\ &+ \frac{1}{p^8}g^2[-2\hat{p}p^\alpha G_{\alpha\beta}G^{\beta\nu}p_\nu + p^2p^\alpha(G_{\alpha\beta}G^{\beta\nu} + G^{\beta\nu}G_{\alpha\beta})\gamma_\nu \\ &- p^2p^\alpha(G_{\alpha\beta}G^{\beta\nu} - G^{\beta\nu}G_{\alpha\beta})\gamma_\nu] + \cdots \end{aligned} \tag{28.23}$$

where here $G_{\alpha\beta} = (\lambda_a/2)G^a_{\alpha\beta}$. The Wilson coefficient of the gluon condensate having dimension D is proportional to p^{-D+1}.

28.2 Application of the Fock–Schwinger technology to the light quarks pseudoscalar two-point correlator

In order to illustrate the discussions in the previous sections and chapters, let us consider the two-point correlator:

$$\Psi_5(q^2) \equiv i\int d^4x\, e^{iqx}\,\langle 0|\mathcal{T}J_P(x)\,(J_P(0))^\dagger\,|0\rangle\,, \tag{28.24}$$

where:

$$J_P = (m_i + m_j)\bar{\psi}_i(i\gamma_5)\psi_j \tag{28.25}$$

is the light quark pseudoscalar current. The lowest order perturbative result for massive quarks and the two loop expression for massless quarks has been discussed for illustration in previous chapters. Here, we shall discuss explicitly the evaluation of the non-perturbative contributions.

28.2.1 Quark condensate $\langle\ :\bar{\psi}\psi:\ \rangle$

In order to compute the Wilson coefficient, one can start from the Wick's theorem and leave one pair of $\langle\ :\bar{\psi}\psi:\ \rangle$ without contraction. Therefore:

$$\begin{aligned}\Psi_5(q^2) = &(m_u+m_d)^2(\gamma_5)_{ij}(\gamma_5)_{kl}(-i)\int d^4x\, e^{iqx}\\ &\times[d(x)_{\alpha j}\bar{d}(o)_{\beta k}\langle\ :\bar{u}(x)_{\alpha i}u(0)_{\beta l}:\ \rangle + u(0)_{\beta l}\bar{u}(x)_{\alpha i}\langle\ :\bar{d}(0)_{\beta k}d(x)_{\alpha j}:\ \rangle]\,.\end{aligned} \tag{28.26}$$

Using the definition of the propagator:

$$\begin{aligned}\bar{\psi}^F_{\alpha i}(x)\psi^{F'}_{\beta j}(y) &= i\delta_{\alpha\beta}\delta^{FF'}S_{ij}(x-y)\\ &= i\delta_{\alpha\beta}\delta^{FF'}\int\frac{d^4p}{(2\pi)^4}\delta_{ij}(p)e^{-ip(x-y)}\,,\end{aligned} \tag{28.27}$$

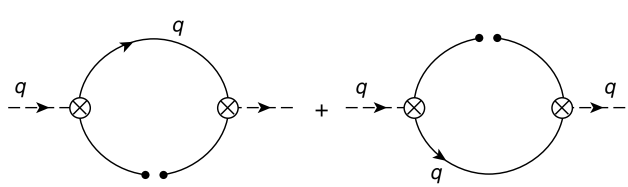
with:

$$S^F_{ij}(p) = \frac{1}{\hat{p}-m_F+i\epsilon'}\,, \tag{28.28}$$

one obtains:

$$\begin{aligned}\Psi_5(q^2) = &(m_u+m_d)^2\int d^4x\int\frac{d^4p}{(2\pi)^4}e^{-i(p-q)x}\\ &\times[\langle\ :\bar{u}(x)_{\alpha i}u(0)_{\beta l}:\ \rangle[\gamma_5 S^d(p)\gamma_5]_{il} + \langle\ :\bar{d}(0)_{\beta k}d(x)_{\alpha j}:\ \rangle[\gamma_5 S^d(p)\gamma_5]_{kj}]\,.\end{aligned} \tag{28.29}$$

In terms of Feynman diagrams, Eq. (28.29) reads:



(28.30)

where • • means that the two-quark fields condense at the same point, so that a Taylor expansion in x_μ of $\langle\ :\bar{\psi}(x)\psi(0):\ \rangle$ makes sense. Using Eq. (28.15), wherein we shall limit

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ourselves to the first two terms of the expansion, one obtains:

$$\Psi_5(q^2) = (m_u + m_d)^2 \frac{3}{12} \times \left[\langle\ : \bar{u}u :\ \rangle \left\{ \mathbf{Tr}\ \gamma_5 S^d(q)\gamma_5 - \frac{1}{4} m_u \left[-\frac{\partial}{\partial p_\lambda} \mathbf{Tr}(\gamma_5 S^d(q)\gamma_5\gamma^\lambda) \right]_{p=q} \right\} \times (u \longleftrightarrow d) \right]. \tag{28.31}$$

Using the property:

$$-\frac{\partial}{\partial p_\lambda} S(p) = S(p)\gamma_\lambda S(p)\,, \tag{28.32}$$

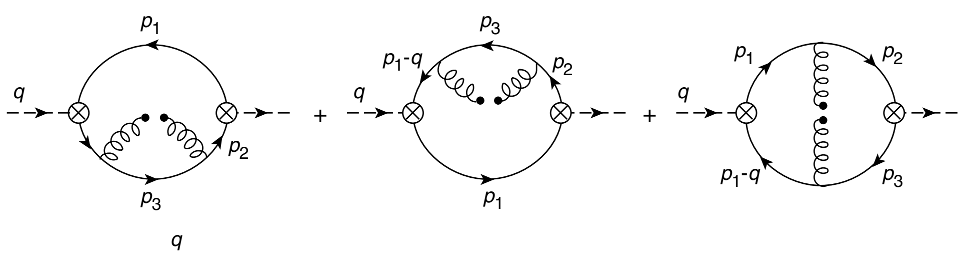
one can deduce the final result:

$$\Psi_5(q^2)|_{\bar\psi\psi} = \frac{(m_u+m_d)^2}{q^2} \left[\left(m_d - \frac{m_u}{2} \right) \langle\ : \bar{u}u :\ \rangle\ +\ (u \longleftrightarrow d) \right]. \tag{28.33}$$

The minus sign is due to the γ_5 chirality flip which acts on the term $\partial/\partial p^\lambda$. This implies that for the scalar current, one has to change this minus sign.

28.2.2 Gluon condensate $\langle\ : \alpha_s G^2 :\ \rangle$

The evaluation of the effect of the gluon condensate can be done by using the previous expression of the quark propagators in external fields. Diagramatically, one has to compute (Fig. 28.2):



(28.34)

As usual, we apply Wick's theorem where all quark fields should be contracted but not the gluon ones. The notation • • means again that the gluon fields are put at the same point, such that the previous Taylor expansion in Eq. (28.20) is valid. Using Feynman rules, one can deduce:

$$\Psi_5(q^2)|_G = (m_u+m_d)^2(-i)\frac{g^2}{2} \int d^4y\, d^4z \int \prod i = 1^3 \frac{d^4 p_1}{(2\pi)^4} \langle\ : A^a_\lambda(y) A^a_\rho(z) :\ \rangle$$
$$\times \left[\mathbf{Tr}[\gamma_5 S(p_1+q)\gamma^\lambda S(p_3)\gamma^\rho S(p_2)\gamma_5 S(p_1)] e^{i(q+p_1-p_3)y+i(p_3-p_2)z} \right.$$
$$+ \mathbf{Tr}[\gamma_5 S(p_1)\gamma_5 S(p_2)\gamma^\rho S(p_3)\gamma^\lambda S(p_1-q)] e^{i(-p_1+q+p_3)y+i(p_2-p_3)z}$$
$$\left. + \mathbf{Tr}[\gamma_5 S(p_1)\gamma^\rho S(p_2)\gamma_5 S(p_3)\gamma^\lambda S(p_1-q)] e^{i(q-p_1+p_3)y+i(p_1-p_2)z} \right], \tag{28.35}$$

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where we have omitted the flavour indices u, d as we shall work in the massless quark limit. Now, one takes advantage of Eq. (28.12), which is valid in the Schwinger gauge. Substituting it in Eq. (28.35), one gets:

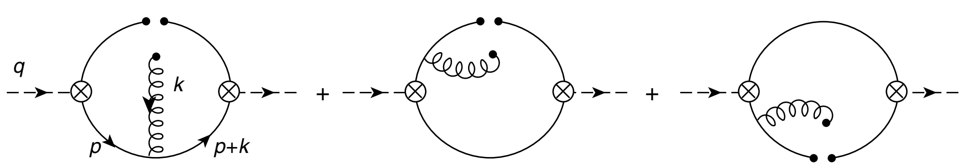
$$\Psi_5(q^2)|_G = (m_u + m_d)^2(-i)\frac{1}{16d(d-1)}\langle g^2 G_a^{\mu\nu} G_{\mu\nu}^a\rangle[g_{\nu\tau}g_{\lambda\rho} - g_{\nu\rho}g_{\lambda\tau}]\int\frac{d^4p_1}{(2\pi)^4}$$
$$\times\left[2\frac{\partial}{\partial p_{1\nu}}\frac{\partial}{\partial p_{2\tau}}\mathbf{Tr}[\gamma_5 S(p_1+q)\gamma^\lambda S(p_3)\gamma^\rho S(p_2)\gamma_5 S(p_1)]_{p_2=p_3=p_1+q}\right.$$
$$\left.+\frac{\partial}{\partial p_{3\nu}}\frac{\partial}{\partial p_{2\tau}}\mathbf{Tr}[\gamma_5 S(p_1)\gamma^\rho S(p_2)\gamma_5 S(p_3)\gamma^\lambda S(p_1-q)]_{p_2=p_1=p_3=p_1-q}\right], \tag{28.36}$$

where we have used the fact that the two self-energy-like diagrams give the same contribution. Using Eq. (28.32) and the properties of Dirac matrices and Feynman integrals given in Appendices D and F and in that of QSSR1, one can deduce:

$$\Psi_5(q^2)|_G = -\frac{1}{8\pi}\frac{(m_u+m_d)^2}{q^2}\left\langle\alpha_s G_a^{\mu\nu}G_{\mu\nu}^a\right\rangle . \tag{28.37}$$

28.2.3 Mixed quark-gluon condensate

This contribution corresponds to the diagram:



(28.38)

As before, one again writes the Wick product where two quark fields should be contracted. The first diagram gives:

$$\Psi_5(q^2)\Big|_M^{(1)} = (m_u+m_d)^2\int d^4x\, d^4y\, e^{iqx}\int\frac{d^4p}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}$$
$$\times\, g\langle\ :\bar u(x)_{\alpha i}A_\mu^a u(0)_{\beta l}:\ \rangle(\gamma^\mu)_{mn}(\gamma_5)_{ij}(\gamma_5)_{kl}$$
$$\times\, e^{-i[p(x-y)+(p+k)y]}S_{jm}^d(p)S_{nk}^d(p+k)\ +\ (u\longleftrightarrow d)\,. \tag{28.39}$$

We use now Eq. (28.16), the property in Eq. (28.32) and we do the Dirac algebra.

The self-energy-like diagram can be obtained by considering the *propagation* of the $\langle\bar\psi\psi\rangle$ condensate in a weak external field as given in Eq. (28.15). Using iteratively the property in Eq. (28.32) and doing as usual the Dirac algebra, one obtains the desired result.

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The sum of the mixed quark-gluon condensate contributions is:

$$\Psi_5(q^2)|_M = -\frac{(m_u + m_d)^2}{2(q^2)^2} g[m_d \langle \bar{u} G u\rangle + m_u \langle \bar{d} G d\rangle] \,, \tag{28.40}$$

with the shorthand notation:

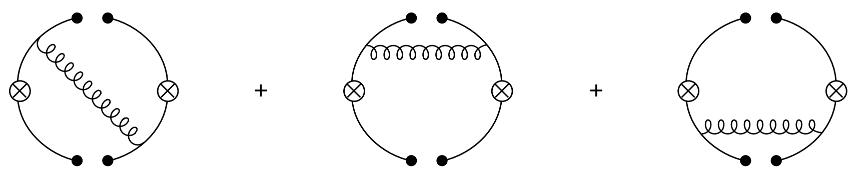
$$g\langle \bar{\psi} G \psi\rangle \equiv g \left\langle \; : \bar{\psi} \sigma^{\mu\nu} \frac{\lambda^a}{2} G^a_{\mu\nu} \psi : \; \right\rangle . \tag{28.41}$$

The result for the scalar current can be deduced from Eq. (28.40) by changing the overall -$(m_d + m_u)^2$ factor with $(m_d - m_u)^2$.

28.2.4 Four-quark condensates

Two classes of diagrams contribute to the four-quark condensates.

- **Class 1**: is that where the gluon fields once contracted give a hard momentum gluon propagator:

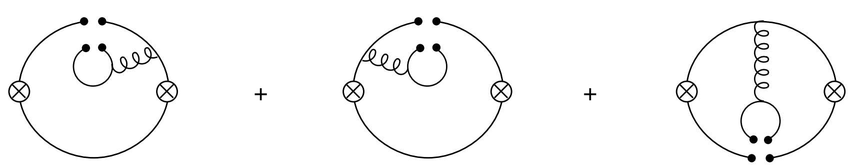


(28.42)

The computation of these diagrams can be done using standard perturbation theory, namely by writing the Wick product, contracting the gluon fields and two pairs of quark fields and by taking the vacuum expectation values (v.e.v) of the four-quark operators. Then, one obtains:

$$\Psi_5(q^2)\Big|^{(1)}_{4\psi} = \frac{(m_u + m_d)^2}{2(q^2)^2} \pi \alpha_s \left\langle \; : \bar{u} \sigma^{\mu\nu} \gamma_5 \frac{\lambda^a}{2} u - \bar{d} \sigma^{\mu\nu} \gamma_5 \frac{\lambda^a}{2} d : \; \right\rangle^2 . \tag{28.43}$$

- **Class 2** is that where the momentum of the gluon propagator is zero. This contribution is represented by the diagrams:



(28.44)

The first two diagrams are generated by the propagation of the $\langle \; : \bar{\psi}\psi : \; \rangle$ condensate in a weak external field as given in Eq. (28.15). The third diagram is generated by the mixed quark-gluon

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condensate as in Eq. (28.16). Evaluation of these diagrams leads to:

$$\Psi_5(q^2)|^{(2)}_{4\psi} = \frac{(m_u+m_d)^2}{2(q^2)^2}\frac{\pi\alpha_s}{6}\left\langle\ :\left(\bar{u}\gamma^\mu\frac{\lambda^a}{2}u+\bar{d}\gamma^\mu\frac{\lambda^a}{2}d\right)\sum_{u,d,s}\bar{\psi}\gamma_\mu\frac{\lambda^a}{2}\psi :\ \right\rangle . \tag{28.45}$$

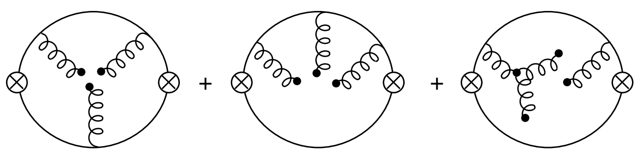
If one uses the vacuum saturation and the $SU(2)_F$ flavour symmetry of the quark condensates, the sum of the four-quark contributions reads:

$$\Psi_5(q^2)|_{4\psi} = \frac{(m_u+m_d)^2}{2(q^2)^2}\frac{112}{27}\rho\pi\alpha_s\langle\ :\bar{\psi}\psi:\ \rangle^2 , \tag{28.46}$$

where ρ measures the deviation from the vacuum saturation estimate.

28.2.5 Triple gluon condensate

The contribution of the triple gluon condensate $g\langle\ :f_{abc}G^a_{\mu\nu}G^b_{\nu\rho}G^c_{\rho\mu}:\ \rangle$ has been evaluated in [436,432] and comes from the diagrams:



(28.47)

One can use here the quark propagator in the external field (Eq. (28.23)) and write the gluon fields in terms of the field strengths as in Eqs. (28.11) and (28.12) in order to form the triple condensate. The calculation can be done using standard perurbation theory. In the chiral limit $(m_{i,j}=0)$, the effect of the triple gluon condensate vanishes for any quark-bilinear currents.

28.3 Fock–Schwinger technology for heavy quarks

28.3.1 General procedure

The technology differs slightly from the light quark one as we can no longer neglect the quark mass M which is the most important scale in the OPE. Moreover, due to the Wigner–Weyl realization of chiral symmetry for the heavy quark systems, the heavy quark condensate vanishes as $1/M$ and is correlated to the gluon condensate as in Eq. (27.52), which is the most important non-perturbative scale in the heavy quark sector.

The Fock–Schwinger gauge [435,319] remains the most convenient working gauge and the momentum space is also the most convenient working space [431]. Let the generic heavy quark two-point correlator:

$$\Psi_\Gamma(q^2) = i\int d^4x\ e^{iqx}\ \langle 0|\mathcal{T} J_\Gamma(x)\,(J_\Gamma(0))^\dagger\,|0\rangle , \tag{28.48}$$

where:

$$J_\Gamma = \bar{\psi}_i(\Gamma)\psi_j \tag{28.49}$$

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and Γ is any Dirac matrices. The non-perturbative contributions to the correlator are typically of the form:

$$\Psi_\Gamma(q^2,\ M^2) = \langle gG\cdots G\rangle \int \frac{d^4k}{(2\pi)^4} \frac{\mathbf{Tr}(\Gamma\ldots,\ \hat{k},\ \hat{q},\ m,\ldots\Gamma\ldots)}{(k^2-M^2)[(k+q)^2-M^2]^n}\ . \tag{28.50}$$

The trace can be done using some algebraic programs. It is convenient to express the result as inverse powers of $k^2 - M^2$ and $(k+q)^2 - M^2$. After a Feynman parametrization, one encounters integrals of the form:

$$I_n^{\alpha\beta}(q^2, M^2) = \int_0^1 dx \frac{x^\alpha(1-x)^\beta}{[-q^2x(1-x)+M^2]^n}\ . \tag{28.51}$$

By noting the symmetry $x \to (1-x)$, one can re-expand the previous integral in $x(1-x)$ and deduce the recursive relation:

$$I_n^{\alpha\alpha} \equiv I_n^\alpha = \frac{1}{Q^2}\left(I_{n-1}^{\alpha-1} - M^2 I_n^{\alpha-1}\right)\ . \tag{28.52}$$

This leads to the basic integral:

$$J_n = \int_0^1 \frac{dx}{[1-x(1-x)q^2/M^2]^n}\ , \tag{28.53}$$

which reads:

$$J_n = \frac{(2n-3)!!}{(n-1)!}\left[\left(\frac{v^2-1}{2v^2}\right)^n v^{1/2}\log\frac{v+1}{v-1} + \sum_{k=1}^{n-1}\frac{(k-1)!}{(2k-1)!!}\left(\frac{v^2-1}{2v^2}\right)^{n-k}\right], \tag{28.54}$$

where:

$$v \equiv \left(1-\frac{4M^2}{q^2}\right)^{1/2}\ . \tag{28.55}$$

28.3.2 $D = 4$ gluon condensate of the electromagnetic correlator

We use the Fock–Schwinger gauge in order to express the gluon fields in terms of the field strengths as in Eq. (28.12). The algorithm is very similar to the one used for the light quarks. The first two self-energy-like diagrams normalized to $\langle\alpha_s G^2\rangle$ give [431]:

$$C_G^a = -\frac{1}{96\pi}\frac{1}{q^4}\left[2\frac{(5v^4+3)}{v^4} + \frac{(v^2-1)^2(5v^2+3)}{v^5}\log\frac{v-1}{v+1}\right]. \tag{28.56}$$

The vertex-like diagram contributes as:

$$C_G^b = \frac{1}{48\pi}\frac{1}{q^4}\left[2\frac{2(v^2+1)}{v^2} + \frac{(v^2-1)^2}{v^3}\log\frac{v-1}{v+1}\right], \tag{28.57}$$

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where one can notice that each set of diagrams develops a non-transverse part:

$$q^\mu q^\nu \Pi^b_{\mu\nu} = -\frac{1}{16\pi}\frac{1}{q^4}\frac{(1-v^2)}{v^2}\left[1 + \frac{(v^2+1)}{2v}\log\frac{v-1}{v+1}\right], \tag{28.58}$$

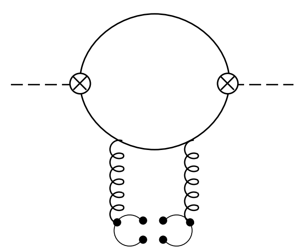
which vanishes in the sum. One can also express the sum of the transverse contribution in terms of the basic integral in Eq. (28.53):

$$C_4 \equiv C^a_G + C^b_G = \frac{1}{24\pi}\frac{1}{q^4}(-1 + 3J_2 - 2J_3) , \tag{28.59}$$

which is a useful compact expression for further analysis.

28.3.3 $D = 6$ *condensates of the electromagnetic correlator*

The *light* four-quark condensates contribute through the diagram:

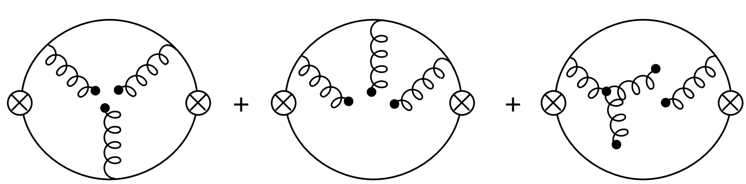


(28.60)

via the equation of motion of the gluon fields:

$$gJ^\mu_a \equiv D_\mu G^{\mu\nu}_a = -\frac{g^2}{2}\sum_{u,d,s}\bar\psi\gamma_\nu\frac{\lambda^a}{2}\psi , \tag{28.61}$$

while the triple gluon condensate contributes via the diagrams in Eq. (28.3.3):



(28.62)

In order to reach the desired result, it is useful to express the v.e.v:

$$\langle\ :D_\alpha D_\beta G_{\mu\nu}G_{\rho\sigma}:\ \rangle,\quad \langle\ :D_\alpha G_{\mu\nu}D_\beta G_{\rho\sigma}:\ \rangle,\quad \left\langle\ :g^3 f_{abc}G^a_{\mu\nu}G^b_{\alpha\beta}G^c_{\rho\sigma}:\ \right\rangle. \tag{28.63}$$

In so doing, one uses the colour trace due to two and three λ matrices, the previous gluon field equation of motion and the Bianchi identity. After a lengthy but straightforward

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algebraic manipulation, one can express the result in terms of the two condensates:

$$C_{3G}\langle g^3 f_{abc} G^a_{\mu\nu} G^b_{\nu\rho} G^c_{\rho\mu}\rangle \,, \qquad C_{JJ}\langle g^4 J^a_\mu J^a_\mu\rangle \,, \tag{28.64}$$

where the Wilson coefficients are [1,433]:

$$\begin{aligned} C_{3G} &= \frac{1}{72\pi^2 q^6}\left[\frac{2}{15} + 4J_2 - \frac{31}{3}J_3 + \frac{43}{5}J_4 - \frac{12}{5}J_5 + \frac{q^2}{10M^2}\right], \\ C_{JJ} &= \frac{1}{36\pi^2 q^6}\left[\frac{41}{45} + \left(\frac{2}{3} - \frac{q^2}{3M^2}\right)J_1 - J_2 - \frac{4}{9}J_3 - \frac{26}{15}J_4 + \frac{8}{5}J_5 + \frac{3q^2}{5M^2}\right]. \end{aligned} \tag{28.65}$$

28.3.4 Matching the heavy and light quark expansions

It is instructive to compare the coefficient functions obtained directly from a light quark expansion and from the heavy quark one by taking the limit $v = 1$. In order to be explicit, let us consider the coefficient of the gluon condensate $\langle\ :\alpha_s G^2:\ \rangle$. In the light quark-expansion, one obtains [411]:

$$\begin{aligned} G^a_G(m=0) &= 0 \,, \\ G^b_G(m=0) &= \frac{1}{12\pi}\frac{1}{q^4}\langle\ :\alpha_s G^2:\ \rangle \,. \end{aligned} \tag{28.66}$$

If one takes naively the heavy quark result, one obtains from Eqs. (28.56) and (28.57):

$$\begin{aligned} G^a_G(v\to 1) &= -\frac{1}{6\pi}\frac{1}{q^4}\langle\ :\alpha_s G^2:\ \rangle \,, \\ G^b_G(v\to 1) &= \frac{1}{12\pi}\frac{1}{q^4}\langle\ :\alpha_s G^2:\ \rangle \,, \end{aligned} \tag{28.67}$$

which shows that the two limits do not coincide (!). This discrepancy can be restored by including the effect of the quark condensate which is known to be correlated to that of the gluon through Eq. (27.52).

One obtains in the two cases [411]:

$$\begin{aligned} C_\psi(m=0) &= \frac{2}{q^4} m\langle\ :\bar\psi\psi:\ \rangle \,, \\ C_\psi(v) &= \frac{8}{3}\frac{v+2}{(v+1)^2}\frac{1}{q^4} m\langle\ :\bar\psi\psi:\ \rangle \,, \end{aligned} \tag{28.68}$$

where the two results coincide for $v \to 1$. Using the relation in Eq. (27.56), one can introduce the non-normal-ordered quark and gluon condensates, where an extra gluon condensate term has been induced by the quark condensate. This term cancels the extra part in $C^a_G(v\to 1)$.

This lesson just tells us that one cannot directly take the $v = 1$ limit of the heavy quark correlator in order to get the light-quark result without paying attention to the *masked*

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contribution of the quark condensate, which induces a gluon condensate effect. Some other similar relations and properties hold for higher dimension condensates.

28.3.5 Cancellation of mass singularities

Let us now discuss another example related to the previous subtlety of the quark and gluon condensates.

Let the example of the correlator of the vector current built from one light and one heavy quark fields:

$$J^\mu(x)^i_j = \bar\psi_i \gamma^\mu \psi_j \ . \tag{28.69}$$

By keeping the quark mass terms and taking the limit $-q^2 \to \infty$ after integration, one obtains for the transverse part [437]:

$$C^T_G = \frac{1}{12\pi}\left(1 - \frac{m_i}{m_j} - \frac{m_j}{m_i}\right)\frac{1}{q^4}\langle \alpha_s G^2\rangle \tag{28.70}$$

which exhibits a dangerous mass singularity. The *normal ordered* quark condensate contribution is:

$$C_\psi = \frac{1}{q^4}\langle\ : m_i \bar\psi_j \psi_j + m_j \bar\psi_i \psi_i :\ \rangle \ , \tag{28.71}$$

Expressing it in terms of the *non-normal ordered* quark condensate as defined Eq. (27.56) and adding it to the previous gluon condensate contribution, one obtains the IR stable result:

$$C^T_G = \frac{1}{12\pi}\frac{1}{q^4}\langle\ : \alpha_s G^2 :\ \rangle \ . \tag{28.72}$$

and:

$$C_\psi = \frac{1}{q^4}\langle\, m_i \bar\psi_j \psi_j + m_j \bar\psi_i \psi_i\, \rangle(\nu) \ , \tag{28.73}$$

However, the natural question to ask is the commutativity of the operation by taking the limit $m_{i,j} = 0$ before the loop integration. A positive answer to this question can only be provided if one treats the IR integral in dimensional regularization and if one removes the $1/\epsilon$-pole at the *very end* of the calculation.

Indeed, in this calculation, one encounters integrals of the type:

$$\begin{aligned} I &\equiv \int \frac{d^n l}{(2\pi)^n}\left(\frac{q^2}{l^2 + i\epsilon}\right)^a \left(\frac{q^2}{(l+q)^2 + i\epsilon}\right)^b , \\ &= \left(\frac{-q^2}{4\pi}\right)^{n/2} \frac{\Gamma(a+b-n/2)\Gamma(n/2-a)\Gamma(n/2-b)}{\Gamma(a)\Gamma(b)\Gamma(n-a-b)} \end{aligned} \tag{28.74}$$

where the IR singularity is transformed into $1/\epsilon$-pole, which can be removed. In general, the extension of this method for the calculation of the Wilson coefficients of higher dimension

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condensates can be easily done provided one takes care of the mixing of the operators under renormalizations as discussed earlier.

28.4 The plane wave method

This method exploits the fact that the Wilson's expansion is an operator identity, namely that one can single out a given operator by sandwiching it between appropriate states. Let us consider the two-point correlator associated with the quark current:

$$J^\Gamma(x) = \bar\psi\Gamma\psi \ , \tag{28.75}$$

characterized by the Dirac matrix Γ, and which possesses the generic OPE (omitting Lorentz indices):

$$\Pi(q^2) \simeq C_1 1 + C_m\bar\psi\psi + G_g G^2 + D_G \left\{ G_{\alpha\delta}G^\alpha_\beta q^\delta q^\beta = \frac{1}{4}q^2G^2 \right\} \ . \tag{28.76}$$

The first unit term corresponds to the usual perturbative calculation, which one obtains by sandwiching the correlator between the vacua. The next term is obtained by sandwiching the correlator between one-quark states and corresponds to the quark-current scattering amplitude shown in Fig. 28.1.

The Wilson coefficient C_G can be obtained by sandwiching the correlator between one-gluon states. Therefore, the problem reduces to the evaluation of the forward gluon scattering amplitude on a colour-singlet current. From Lorentz invariance, this amplitude can be decomposed as:

$$\begin{aligned} T^{\mu\nu}(q,k) &\equiv i\int d^4x e^{iqx}\langle k,\mu|\mathcal{T}J^\Gamma(x)\left(J^\Gamma(0)\right)^\dagger|k,\nu\rangle \\ &= F_1^{\mu\nu}C(q,k) + F_2^{\mu\nu}D(q,k) \ , \end{aligned} \tag{28.77}$$

where:

$$F_1^{\mu\nu} = 4(k^2g^{\mu\nu} - k^\mu k^\nu) \equiv \langle k,\mu|G^2|k,\nu\rangle \ , \tag{28.78}$$

and:

$$\begin{aligned} F_2^{\mu\nu} &= 2[k^2q^\mu q^\nu - (k.q)(q^\mu k^\nu + q^\nu k^\mu) + g^{\mu\nu}(k\cdot q)^2] - q^2(k^2g^{\mu\nu} - k^\mu k^\nu) \\ &\equiv \langle k,\mu|G_{\alpha\delta}G^{\alpha\beta}q^\delta q^\beta - \frac{1}{4}q^2G^2|k,\nu\rangle \ . \end{aligned} \tag{28.79}$$

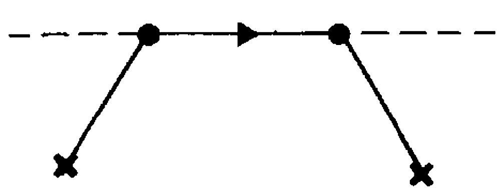

Fig. 28.1. 'Weak' quark (full line)-current (dashed line) scattering amplitude.

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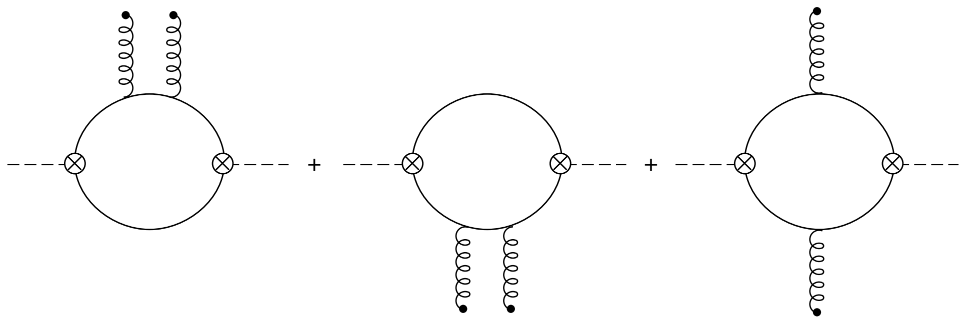
They correspond to the diagrams:



(28.80)

A comparison of Eqs (28.76) and (28.77)–(28.79) gives:

$$D_G(q^2) = C_G(q, k)|_{k_\alpha=0}. \tag{28.81}$$

In practice, the plane wave method is convenient when one has external weak quark fields as in Fig. 28.1. In the case of many 'weak' external gluon fields, the extraction of a particular operator from various possible candidates having the same dimensions becomes very difficult. In one sense, this is the main inconvenience of this method.

28.5 On the calculation in a covariant gauge

The evaluation of the Wilson coefficients can be also obtained in a covariant gauge. Unlike the usual perturbative term, and the quark condensate term, which are easily obtained in this gauge, the evaluation of the Wilson coefficients of the gluon condensates is much more cumbersome in this gauge than in that of Fock–Schwinger. A published evaluation of the gluon condensate contribution in this gauge can be found in [626]. As applications of this method, we give at the end of this part a compilation of QCD two-point functions useful for further analysis.

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