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A GEOMETRICALLY ABERRANT BANACH SPACE WITH UNIFORMLY NORMAL STRUCTURE

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A geometrically aberrant Banach space with uniformly normal structure, a modification of Brown's example, is given.

A Banach space X is said to have normal structure if for each non-trivial bounded convex subset K there exists a point $p \in K$ such that

$$\sup(\|p - x\|\,;x \in K) < \operatorname{diam} K.$$

A Banach space X is said to have uniformly normal structure if $N(X) = \sup(r_A(A)\,;A \subset X,$ convex, bounded, $\operatorname{diam} A = 1)$, where

$$r_A(A) = \inf(\sup(\|x - y\|\,;y \in A)\; x \in A).$$

Normal structure and uniformly normal structure have been significant in the development of fixed point theory. Considerable research has been directed into finding geometrical conditions which imply normal structure and uniformly normal structure.

A Banach space X is said to be uniformly rotund in every direction if for any given $z \neq 0$ and $\varepsilon > 0$ there exists $\delta(\varepsilon, z) > 0$ such that $|\lambda| < \varepsilon$ for $x, y, \|x\| = \|y\| = 1$ and $x - y = \lambda z$ when $\|x + y\| > 2 - \delta$.

A Banach space X is said to be locally uniformly rotund if for any given x, $\|x\| = 1$ and $\varepsilon > 0$ there exists a $\delta(\varepsilon, x) > 0$ such that $\|x - y\| < \varepsilon$ for $\|y\| = 1$, when $\|x + y\| > 2 - \delta$.

A Banach space is said to be k-uniformly rotund ($k - UR$ see Sullivan [**9**]) if for any $\varepsilon > 0$ there exists $\delta^k(\varepsilon) > 0$ such that for any $x_i \in X$, $\|x_i\| \leqslant 1$, $i = 1, \ldots, k+1$, with $\left\|1/(k+1)\sum_{i=1}^{k+1} x_i\right\| > 1 - \delta^k$, then $V(x_1, \ldots, x_{k+1}) < \varepsilon$, where

$$V(x_1, \ldots, x_{k+1}) = \sup\left\{\begin{vmatrix} 1 & \ldots & 1 \\ f_1(x_1) & \ldots & f_1(x_{k+1}) \\ \vdots & \ddots & \vdots \\ f_k(x_1) & \ldots & f_k(x_{k+1}) \end{vmatrix}\,;\; \begin{matrix} f_i \in X^*, \|f_i\| \leqslant 1 \\ i = 1, \ldots, k \end{matrix}\right\}.$$

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A Banach space X is said to be uniformly Kadec-Klee if for every $\varepsilon > 0$ there exists a δ, $0 < \delta < 1$, such that for every sequence (x_n), $\|x_n\| \leqslant 1$ which converges weakly to x and $\inf(\|x_n - x_m\| \; n \neq m) \geqslant \varepsilon$ we have $\|x\| < \delta$. A reflexive and uniformly Kadee-Klee Banach space is said to be nearly uniformly convex (NUC;see Huff [5]).

A Banach space X is said to be weakly uniformly Kadec-Klee if there exists an $0 < \varepsilon < 1$ and $\delta > 0$ such that for every sequence $(x_n)\,\|x_n\| \leqslant 1$ which converges weakly to x and $\inf(\|x_n - x_m\| \; n \neq m) \geqslant \varepsilon$ we have $\|x\| \leqslant 1 - \delta$.

Brown [2] devised an equivalent norm $\|\cdot\|_B$ on $(l_2, \|\cdot\|_2)$ such that its restriction to $M = \{(a_i) \in l_2 \colon a_1 = 0\}$ remains the original l_2-norm $\|\cdot\|_2$ and its restriction to $M_k =$ span $\{e_1, e_k\}$, $k \geqslant 3$, is an $l_{p(k)}$ norm, where $\{e_k\}$ is the natural basis on l_2 and $p(k) \to +\infty$ as $k \to +\infty$, $p(3) = 16$. Giles, Sims and Swaminatham [4] proved that $(l_2, \|\cdot\|_B)$ has normal structure.

In this paper, modifing Brown's example, we give another equivalent norm $\|\cdot\|_a$ on $(l_2, \|\cdot\|_2)$ such that $(l_2, \|\cdot\|_a)$ is non-URED, non-LUR, non-wUKK and non-KUR, but $(l_2, \|\cdot\|_a)$ has uniformly normal structure.

Let $R_2(x) = \|(0, a_2, \dots)\|_2$,

$$R_k(x) = \left(|a_1|^{p(k)} + |a_k|^{p(k)}\right)^{\frac{1}{p(k)}}$$

where $x = (a_1, \dots, a_n, \dots) \in l_2$, $k \geqslant 3$, $p(k) \uparrow +\infty$, $p(3) = 5$.

Then, for $x = (a_1, \dots, a_n, \dots) \in l_2$, we put

$$\|x\|_a = \sup(R_k(x) \; 2 \leqslant k < +\infty), \text{ and}$$
$$\|x\|_t = \max(|a_1|, R_2(x)).$$

We have $(l_2, \|\cdot\|_t) \cong (R^1 \oplus l_2)_\infty$ ($\cong$ denotes linear isometric).
Since for $k \geqslant 3$, $x = (a_1, \dots, a_n, \dots) \in l_2$,

$$R_k(x) = \left(|a_1|^{p(k)} + |a_k|^{p(k)}\right)^{\frac{1}{p(k)}} \leqslant \left(|a_1|^5 + |a_k|^5\right)^{\frac{1}{5}} \leqslant \left(|a_1|^5 + R_k(x)^5\right)^{\frac{1}{5}} \leqslant 2^{\frac{1}{5}} \|x\|_t,$$

and $R_2(x) \leqslant \|x\|_t$, so

$$\|x\|_a \leqslant 2^{\frac{1}{5}} \|x\|_t.$$

Obviously, $\|x\|_t \leqslant \|x\|_a$ for $x \in l_2$, therefore

$$\|x\|_t \leqslant \|x\|_a \leqslant 2^{\frac{1}{5}} \|x\|_t, \text{ for } x \in l_2.$$

From this, we get

$$d((l_2, \|\cdot\|_t), (l_2, \|\cdot\|_a)) \leqslant 2^{\frac{1}{5}},$$

where $d(.,.)$ denotes Mazur distance.

It is easy to see that, as the proof of Giles etcetra, $(l_2, \|\cdot\|_a)$ is non-URED and non-LUR.

Also, $(l_2, \|\cdot\|_a)$ is non-wUKK. In fact, let $s_k = \frac{e_1+e_k}{\|e_1+e_k\|}$, then, since $e_1 + e_k \xrightarrow{w} e_1$, and $\|e_1 + e_k\| = 2^{\frac{1}{p(k)}} \longrightarrow 1$, we have

$$x_k \xrightarrow{w} e_1.$$

But

$$\varliminf_{k,h,k\neq h} \|x_k - x_h\|_a = \varliminf_{k,h,k\neq h} \|e_k - e_h\|_a = \sqrt{2},$$

and $\|e_1\|_a = 1$. So $(l_2, \|\cdot\|_a)$ is non-wUKK.

Since KUR spaces are NUC spaces [**10**], and NUC spaces are wUKK spaces, then $(l_2, \|\cdot\|_a)$ is non-KUR space.

Since

$$\left\| \frac{e_1 + e_K}{\|e_1 + e_k\|} \pm \frac{e_1 - e_k}{\|e_1 - e_k\|} \right\| \to 2,$$

$(l_2, \|\cdot\|_a)$ is not uniformly non square (for definition see [**3**]).

Nevertheless, $(l_2, \|\cdot\|_a)$ has uniformly normal structure from the following results.

THEOREM. (Lin, Yu and Zan [**7**]) *If X_i, $i = 1, \dots, n$, has uniformly normal structure (UNS), then $\left(\sum_{i=1}^{n} \oplus X_i\right)_\infty$ has uniformly normal structure.*

PROOF: It is enough to prove that $X = (X_1 \oplus X_2)_\infty$ has UNS, if X_1 and X_2 have UNS.

Let K be a bounded convex subset of X, define the projection $P_i : X \to X_i$, $i = 1, 2$, then there exists $x_i \in K_i = P_i(K)$, $i = 1, 2$, such that

$$\sup(\|x_i - y\|; y \in K_i) \leqslant N(X_i)\operatorname{diam}(K).$$

Choose $u_1 \in X_2$ such that $(x_1, u_1) \in K$ and $u_2 \in X_1$ such that $(u_2, x_2) \in K$. Let $w = \left(\frac{x_1+u_2}{2}, \frac{x_2+u_1}{2}\right)$, then for any $(x, y) \in K$,

$$\begin{aligned}
&\left\| \frac{1}{2}(x_1 + u_2) - x, \frac{1}{2}(u_1 + x_2) - y \right\| \\
&\qquad = \max\left(\left\| \frac{1}{2}(x_1 + u_2) - x \right\|, \left\| \frac{1}{2}(u_1 + x_2) - y \right\| \right) \\
&\qquad \leqslant \max\left(\frac{1}{2}(\|x - x_1\| + \|u_2 - x\|), \frac{1}{2}(\|x_2 - y\| + \|u_1 - y\|) \right) \\
&\qquad \leqslant \max\left(\frac{1}{2}(1 + N(X_1)) \operatorname{diam}(K), \frac{1}{2}(1 + N(X_2))\operatorname{diam}(K) \right) \\
&\qquad \leqslant \frac{1}{2} \max((1 + N(X_1)), (1 + N(X_2))) \operatorname{diam}(K).
\end{aligned}$$

Then X has UNS and $N(X) \leqslant \frac{1}{2}\max((1+N(X_1)),(1+N(X_2)))$.

Corollary 1. *$(R^1 \oplus l_2)_\infty \cong (l_2, \|\cdot\|_t)$ has UNS and $N((l_2,\|\cdot\|_t)) \leqslant \max\left(\frac{3}{4}, \frac{1}{4}(2+\sqrt{2})\right) = \frac{1}{4}(2+\sqrt{2})$.*

Proof: Since $N(R^1) = \frac{1}{2}$ and $N((l_2,\|\cdot\|_2)) = \frac{\sqrt{2}}{2}$.

Remark: It is worth noting that $\|x\|_t \leqslant \|x\|_2 \leqslant \sqrt{2}\,\|x\|_t$, but $(l_2,\|\cdot\|_t)$ has UNS.

Corollary 2. *If $p > 5\left(> \left(\log_2 \frac{4}{2+\sqrt{2}}\right)^{-1}\right)$ and $d((l_2,\|\cdot\|_t), X) < 2^{\frac{1}{4}}$, then X has UNS.*

Proof: Since $p > \left(\log_2 \frac{4}{2+\sqrt{2}}\right)^{-1}$, then

$$N(X) \leqslant d((l_2,\|\cdot\|_t), X)N((l_2,\|\cdot\|_t)).$$

Corollary 3. *$(L_2,\|\cdot\|_a)$ has UNS.*

Proof: By Corollary 2 and $d((l_2,\|\cdot\|_a),(l_2,\|\cdot\|_t)) < 2^{\frac{1}{5}}$.

In [**1**], Bernal and Sullivan proved that if for all $x \in l_2$,

$$\frac{1}{\beta}\|x\|_2 \leqslant \|x\| \leqslant \|x\|_2,$$

where $1 \leqslant \beta < \sqrt{2}$, then $(l_2,\|\cdot\|)$ has normal structure.

In [**7**], using the estimation of $\delta^k(\varepsilon)$, it had been proved that if $1 \leqslant \beta < \sqrt{2}$ then $(l_2,\|\cdot\|)$ has UNS. This was also proved in [**8**].

Since $\|e_1 + e_k\|_a \to 1$, $\|e_1 + e_k\|_2 = \sqrt{2}$, so $(l_2,\|\cdot\|_a)$ has $\beta \geqslant \sqrt{2}$, hence $1 \leqslant \beta < \sqrt{2}$ is not necessary for UNS.

From the above results it seems that UNS does not necessarily imply that the space has good geometric properties.

References

[1] J. Bernal and F. Sullivan, 'Banach spaces that have normal structure and are isomorphis to a Hilbert space', *Proc.A.N.S.* **90** **No.4** (1984), 550–555.

[2] A.L. Brown, 'Arotund reflexive space having a subspace of two codimension which has a discontinuous metric projection.', *Mich. Math. J.* **21** (1974), 145–151.

[3] M.M. Day, *Normed Linear Spaces* (Spring-Verlag, 1973).

[4] J.R. Giles, B. Sims and S. Saminatham, 'A geometrically aberrant Banach space with normal structure.', *Bull. Aust. Math. Soc.* **31** (1985), 75–81.

[5] R. Huff, 'Banach spaces which are nearly uniformly convex', *Rocky Mountains J. of Math.* **10** (1980), 743–749.

[6] W.A. Kirk, *Proc. of Research Workshop of Banach spaces theory*, in, Edited by B.L. Lin, pp. 113–128 (The Univ. of Iowa, 1981).

[7] B.L. Lin, Yu Xin Tai and Zang Jun, *On the uniformly normal structure* **no. 1** (J.E.C.N.U., 1985).

[8] E. Maluta, 'Uniformly normal structure and related coefficients', *Pacific J. Math* **110 No 1** (1984), 125–143.

[9] F. Sullivan, 'A generalization of uniformly rotund Banach spaces', *Canad. J. Math.* **31** (1979), 628–636.

[10] Yu Xin tai, *KUR spaces are NUC spaces* **24**: *Kexue Tong bao*, pp. 1473–1475 (Academia Science China, 1983).

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