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A THEOREM ON THE SPECTRAL RADIUS OF THE SUM OF TWO OPERATORS AND ITS APPLICATION

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In the present paper a theorem on the spectral radius of the sum of linear operators is established. The application of this theorem to a functional differential equation of neutral type is also given.

1. INTRODUCTION

In the monograph [5] the following theorem on the spectral radius of the sum of two operators is given:

THEOREM. (Equation 21 p.426 of [5].) *Let A, B be linear and bounded operators mapping a Banach space $(X, \|\cdot\|)$ into itself. If these operators are commutative, that is, $ABx = BAx$ for each $x \in X$, then*

$$r(A+B) \leqslant r(A) + r(B),$$

where $r(A)$, $r(B)$ and $r(A+B)$ denote the spectral radii of A, B and $A+B$ respectively.

As the following example from the monograph [4] shows, the assumption of the commutativity is important:

Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It is easy to check, that $r(A) = 0$, $r(B) = 0$ while $r(A+B) = 1$. In this case the operators A and B are not commutative. The aim of our paper is to give a sufficient condition, different from the global commutativity, under which the inequality $r(A+B) \leqslant r(A) + r(B)$ holds.

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2. A theorem on the spectral radius of the sum of two linear operators

Let $(X, \|\cdot\|, \prec)$ denote a Banach space of elements $x \in X$, with a binary relation $\prec$. We shall assume that:

1° the relation $\prec$ is reflexive and transitive,

2° the norm $\|\cdot\|$ is monotonic, that is, if $\Theta \prec x \prec y$, then $\|x\| \leqslant \|y\|$,

3° if $x \prec y$, then $x + z \prec y + z$ for $x, y, z \in X$.

Theorem 1. *In the space considered above, let the linear and bounded operators $A\colon X \to X$, $B\colon X \to X$ be given. We shall assume that the following conditions are satisfied:*

4° *if $\Theta \prec x$, then $\Theta \prec Ax$ and $\Theta \prec Bx$,*

5° *there exists an element $x_0 \in X$, $\Theta \prec x_0$ such that:*

(a) $r(B) = \lim_{n\to\infty} \|B^n x_0\|^{1/n}$ and $r(A+B) = \lim_{n\to\infty} \|(A+B)^n x_0\|^{1/n}$,

(b) $BA^jB^kx_0 \prec A^jB^{k+1}x_0$ for $j = 1, 2, \ldots,\ k = 0, 1, \ldots$.

Under the assumptions 4° − 5° the inequality

$$r(A+B) \leqslant r(A) + r(B) \tag{1}$$

holds.

Proof: It is easy to see that, in view of our assumptions, we get for an arbitrary $n \in N$:

$$\Theta \prec (A+B)^n x_0 \prec \sum_{i=0}^{n} \binom{n}{i} A^i B^{n-i} x_0.$$

Hence, from 2° and the property of a norm of a linear operator, we obtain:

$$\|(A+B)^n x_0\| \leqslant \left\| \sum_{i=0}^{n} \binom{n}{i} A^i B^{n-i} x_0 \right\| \leqslant \sum_{i=0}^{n} \binom{n}{i} \|A^i\|\, \|B^{n-i} x_0\|. \tag{2}$$

The next step of our proof is, in principle, a repetition of a fragment from the monograph [**5**, p.450]. Thus, in view of Gelfand formula (see for example [**3**, **5**]), we have:

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}, \tag{3}$$

and from 5°:

$$r(B) = \lim_{n\to\infty} \|B^n x_0\|^{1/n} \quad \text{and} \quad r(A+B) = \lim_{n\to\infty} \|(A+B)^n x_0\|^{1/n}. \tag{4}$$

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Let us take arbitrary numbers p and q such that

$$p > r(A) \quad \text{and} \quad q > r(B). \tag{5}$$

There exists an integer number $K > 0$ such that for $n \geqslant K$

$$\|A^n\|^{1/n} < p \quad \text{and} \quad \|B^n x_0\|^{1/n} < q,$$

hence

$$\|A^n\| < p^n \quad \text{and} \quad \|B^n x_0\| < q^n \quad \text{for} \quad n \geqslant K.$$

Moreover, there exist constants s and t such that for all $n \in N$:

$$\|A^n\|^{1/n} \leqslant s \quad \text{and} \quad \|B^n x_0\|^{1/n} \leqslant t.$$

Thus, for $n > 2K$, we have:

$$\begin{aligned}
\|(A+B)^n x_0\| &\leqslant \sum_{i=0}^{n-K} \binom{n}{i} \|A^i\| \, \|B^{n-i} x_0\| + \sum_{i=n-K+1}^{n} \binom{n}{i} \|A^i\| \, \|B^{n-i} x_0\| \\
&< \sum_{i=0}^{K-1} \binom{n}{i} s^i q^{n-i} + \sum_{i=K}^{n-K} \binom{n}{i} p^i q^{n-i} + \sum_{i=n-K+1}^{n} \binom{n}{i} p^i t^{n-i} \\
&= \sum_{i=0}^{K-1} \binom{n}{i} p^i q^{n-i} \left(\frac{s}{p}\right)^i + \sum_{i=K}^{n-K} \binom{n}{i} p^i q^{n-i} \\
&\quad + \sum_{i=n-K+1}^{n} \binom{n}{i} p^i q^{n-i} \left(\frac{t}{q}\right)^{n-i} \\
&\leqslant \sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} M,
\end{aligned}$$

where $M = \max_{0 \leqslant i \leqslant K-1} (s/p)^i + 1 + \max_{0 \leqslant i \leqslant K-1} (t/q)^i$.

Then $\|(A+B)^n x_0\|^{1/n} < (p+q)^n M$ and M does not depend on n.

Hence

$$\lim_{n\to\infty} \|(A+B)^n x_0\|^{1/n} \leqslant \lim_{n\to\infty} [(p+q) M^{1/n}] = p + q.$$

Since the numbers p and q were chosen arbitrary, in view of (3), (4) and (5), we get:

$$r(A+B) \leqslant r(A) + r(B).$$

This completes the proof of Theorem 1. $\square$

REMARK 1. For the matrices A and B in Section 1 the conditions of Theorem 1 of course do not apply. Indeed it is readily seen that 5°(b) only holds for $x_0 = (0, 0)$, but then 5°(a) is contradicted.

REMARK 2. Returning to our theorem, as x_0 we may choose, for example, an interior element of a normal cone in Banach spaces where such an element exists. In this case the spectral radius of a linear bounded operator A can be calculated by the formula [4]:

$$r(A) = \lim_{n\to\infty} \|A^n x_0\|^{1/n}.$$

3. AN EXAMPLE ILLUSTRATING THEOREM 1 IN THE CASE OF THE SPACE OF CONTINUOUS FUNCTIONS

In the space of continuous functions on the interval $[0, T]$ with the norm $\|u\| = \max_{[0,T]}|u(t)|$, we shall say that $x \prec y$ if and only if $x(t) \leqslant y(t)$ for each $t \in [0, T]$. Obviously, this relation satisfies the conditions 1° - 3°.

Let us consider two linear operators

$$(A_1u)(t) = L_1 \int_0^{h(t)} u(s)ds$$

and

$$(A_2u)(t) = L_2 u(H(t)).$$

Suppose that the functions h, $H\colon [0, T] \to [0, T]$ are continuous, $h(H(t)) \leqslant h(t)$ for each $t \in [0, T]$ and L_1, L_2 are positive constants. In the space mentioned above we choose a cone K of non-negative functions. Such a cone is normal and $\operatorname{int} K \neq \emptyset$. As an element $x_0 \in \operatorname{int} K$ we take $x_0(t) \equiv 1$ on $[0, T]$. We shall show that the operators A_1 and A_2 satisfy the assumption 5°(b) of Theorem 1, that is

$$A_2A_1^jA_2^kx_0 \prec A_1^jA_2^{k+1}x_0, \qquad j = 1, 2, \dots, \quad k = 0, 1, \dots .$$

Let us notice that:

$$\left(A_2A_1^jA_2^kx_0\right)(t) = L_1^jL_2^{k+1} \int_0^{h(H(t))} \left(\int_0^{h(t_1)} \dots \int_0^{h(t_{j-1})} 1\, dt_j \dots\right) dt_1,$$

and

$$(A_1A_2^{k+1}x_0)(t) = L_1^jL_2^{k+1} \int_0^{h(t)} \left(\int_0^{h(t_1)} \dots \int_0^{h(t_{j-1})} 1\, dt_j \dots\right) dt_1.$$

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In view of the inequality $h(H(t)) \leqslant h(t)$ for $t \in [0, T]$ we have:

$$L_1^j L_2^{k+1} \int_0^{h(H(t))} \left(\int_0^{h(t_1)} \cdots \int_0^{h(t_{j-1})} 1\, dt_j \ldots \right) dt_1$$
$$\leqslant L_1^j L_2^{k+1} \int_0^{h(t)} \left(\int_0^{h(t_1)} \cdots \int_0^{h(t_{j-1})} 1\, dt_j \ldots \right) dt_1,$$

which means that $A_2 A_1^j A_2^k x_0 \prec A_1^j A_2^{k+1} x_0$.

It is easy to verify that the assumptions 4° and 5° (a) are also satisfied. Hence we obtain $r(A_1 + A_2) \leqslant r(A_1) + r(A_2)$.

4. An application of Theorem 1

In this part of our paper we shall show an application of Theorem 1 to an initial value problem for a differential equation of neutral type.

Let us consider the following Cauchy problem:

$$\text{(6)} \qquad \begin{aligned} &x'(t) = f(t, x(h(t)), x'(H(t))), \quad t \in [0, T] \\ &x(0) = 0. \end{aligned}$$

Suppose that:

6° $f \colon [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and satisfies the Lipschitz condition, that is, for all (t, x_1, x_2), $(t, y_1, y_2) \in [0, T] \times \mathbb{R}^2$

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \leqslant L_1(t)\,|x_1 - y_1| + L_2(t)\,|x_2 - y_2|,$$

where the functions L_i, $i = 1, 2$ are continuous and positive on the interval $[0, T]$,

7° $\max\limits_{[0,T]} L_2(t) < 1$,

8° $h, H : [0, T] \to [0, T]$ are continuous,

9° for each $t \in [0, T]$ the inequality $h(H(t)) \leqslant h(t) \leqslant t$ holds.

Theorem 2. *If the assumptions 6° - 9° are satisfied, then the problem (6) has exactly one solution in the space of continuous functions on the interval* $[0, T]$.

Proof: In the sequel we need the following fixed point theorem from [7]:

Let $(X, \|\cdot\|, \prec, m)$ denote a Banach space of elements $x \in X$, with a binary relation $\prec$ and a mapping $m \colon X \to X$. We shall assume that the conditions 1° and 2° are satisfied and moreover

$$3' \qquad \Theta \prec m(x) \quad \text{and} \quad \|m(x)\| = \|x\| \quad \text{for each} \quad x \in X.$$

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Theorem 3. [7] *In the Banach space considered above, let the operators $\mathcal{A}\colon X \to X$, $A\colon X \to X$ be given with the following properties:*

(i) *A is a linear bounded operator with the spectral radius $r(A)$ less than 1,*

(ii) *if $\Theta \prec x \prec y$, then $Ax \prec Ay$,*

(iii) *$m(\mathcal{A}x - \mathcal{A}y) \prec Am(x-y)$ for all $x, y \in X$.*

Then the equation $\mathcal{A}x = x$ has a unique solution in the space $(X, \|\cdot\|, \prec, m)$.

Let us return to the proof of Theorem 2. It is easy to see that the problem (6) is equivalent to the functional integral equation:

$$z(t) = f\left(t, \int_0^{h(t)} z(s)ds, z(H(t))\right), \qquad t \in [0, T],$$

where $x(t) = \int_0^t z(s)ds$ for $t \in [0, T]$.

Let us consider the following operator:

$$(\mathcal{A}z)(t) = f\left(t, \int_0^{h(t)} z(s)ds, z(H(t))\right), \qquad t \in [0, T]. \tag{7}$$

To prove our theorem it is sufficient to show that under the assumptions 6° - 9° the operator (7) has a unique fixed point in the space of continuous functions on the interval $[0, T]$. In view of 6° we have:

$$\begin{aligned}|(\mathcal{A}z)(t) - (\mathcal{A}w)(t)| &= \left| f\left(t, \int_0^{h(t)} z(s)ds, z(H(t))\right)\right.\\ &\qquad \left. - f\left(t, \int_0^{h(t)} w(s)ds, w(H(t))\right)\right| \\ &\leqslant L_1(t)\int_0^{h(t)} |z(s) - w(s)|\, ds + L_2(t)\, |z(H(t)) - w(H(t))|.\end{aligned}$$

Let $L_i = \max\limits_{[0,T]} L_i(t)$, $i = 1, 2$. Then

$$\begin{aligned}|(\mathcal{A}z)(t) - (\mathcal{A}w)(t)| &\leqslant L_1 \int_0^{h(t)} |z(s) - w(s)|\, ds + L_2\, |z(H(t)) - w(H(t))| \\ &= ((A_1 + A_2)\,|z - w|)(t),\end{aligned} \tag{8}$$

where

$$(A_1u)(t) = L_1 \int_0^{h(t)} u(s)ds, \quad (A_2u)(t) = L_2 u(H(t)).$$

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We want to show now that the operators $A_1 + A_2$ and $\mathcal{A}$ satisfy the assumptions of Theorem 3. At first we prove that the spectral radius of the operator $A_1 + A_2$ is less than 1. As we showed in Section 3, the following inequality holds:

$$r(A_1 + A_2) \leqslant r(A_1) + r(A_2).$$

Since the operator A_1 - in view of the inequality $h(t) \leqslant t$ - is of Volterra type, then $r(A_1) = 0$. It is easy to verify that $r(A_2) = L_2$. By condition 7° $L_2 < 1$, so

$$r(A_1 + A_2) \leqslant L_2 < 1,$$

which means that the assumption (i) of Theorem 3 is satisfied.

Finally, let $(m(x))(t) = |x(t)|$ for $t \in [0, T]$. It is easy to see that the condition $3'$ and the assumptions (ii) and (iii) are satisfied (particularly the assumption (iii) is satisfied in view of (8)). Hence, in virtue of Theorem 3, the operator (7) has exactly one fixed point in the space of continuous functions on the interval $[0, T]$. This ends the proof of Theorem 2. $\square$

REMARK. The problem (6) is similar to the initial value problems considered in the papers [**1**, **2**] and [**6**]. Banaś in [**2**] proved the theorem on the existence of a solution to problem (6) in terms of measures of noncompactness. In [**1**] and [**6**] the authors proved the theorems on the uniqueness of solution of problems similar to (6) under stronger assumptions than those given in our paper.

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