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ON THE CLASSIFICATION OF FOUR-DIMENSIONAL MÖBIUS TRANSFORMATIONS

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Abstract Based on the identification of four-dimensional Möbius transformations

$$g(x) = (ax + b)(cx + d)^{-1}$$

by the matrix group $\mathrm{PS}_{\triangle} L(2, \mathbb{H})$ of quaternionic 2×2 matrices with Dieudonné determinant equal to 1, we give an explicit expression for the classification of g in terms of a, b, c and d.

Keywords: quaternion; four-dimensional Möbius transformations; classification

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1. Introduction

It is well known that the orientation preserving isometries of n-dimensional hyperbolic space, $\mathbf{H}^n$, are the Möbius transformations $M(\overline{\mathbb{R}^{n-1}}) \simeq \mathrm{iso}^+(\mathbf{H}^n)$. It is important to classify Möbius transformations by their dynamics and their fixed points. It is also generally important that these classes should be conjugacy invariants. When $n = 2$ or $n = 3$ this is very well known and completely standard.

For arbitrary dimensions, Ahlfors [**1**] related Möbius transformations to 2×2 matrices whose entries lie in a (suitably chosen part of a) Clifford algebra and discussed the classification of elements in $M(\overline{\mathbb{R}^{n-1}})$ for $n \geqslant 4$. Wada [**10**], Waterman [**11**], Cao and Waterman [**2**] and Kim and Parker [**9**] also had done some related research in this direction.

In the cases of $n = 4$ and $n = 5$ it is possible to relate Möbius transformations to 2×2 quaternionic matrices. Kellerhals has recently used quaternionic Möbius transformations to study isometries of hyperbolic 4-space [**7**] and hyperbolic 5-space [**8**] and the geometry of the corresponding hyperbolic manifolds. In [**3**], Cao *et al*. considered quaternionic Möbius transformations preserving the unit ball in $\mathbb{H}$, which is identified with $\mathrm{U}(1, 1; \mathbb{H})$. They gave an explicit expression for the classification of g in terms of a, b, c and d.

One may regard these quaternionic Möbius transformations as three-dimensional Möbius transformations.

In this paper we follow the choice of the hyperbolic 5-space model as in [**8**]. Take the hyperbolic 5-space $\mathbf{H}^5$ with its canonical orientation and parametrize the space with the aid of $\mathbb{H}$ by writing $E^5_+ = \mathbb{H} \times \mathbb{R}_+$ so that $\partial \mathbf{H}^5 = \bar{\mathbb{H}}$. Let $S_\triangle L(2;\mathbb{H})$ denote the collection of all quaternionic 2×2 matrices with *Dieudonné determinant* $\det_\triangle = 1$, which is expressed as

$$\det{}_\triangle(g) =_+ \sqrt{|ad|^2 + |bc|^2 - 2\operatorname{Re}(a\bar{c}d\bar{b})}. \tag{1.1}$$

Proposition 1.1 (see [8, 12]). *Let*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S_\triangle L(2;\mathbb{H}).$$

Then

$$g^{-1} = \begin{pmatrix} d_\sim & -b_\sim \\ -c_\sim & a_\sim \end{pmatrix} = \begin{pmatrix} {}^\sim d & -{}^\sim b \\ -{}^\sim c & {}^\sim a \end{pmatrix},$$

where

$$\begin{aligned} d_\sim &= d(ad - bd^{-1}cd)^{-1} := dr_{11}^{-1}, & b_\sim &= b(db^{-1}ab - cb)^{-1} := br_{12}^{-1}; \\ c_\sim &= c(ac^{-1}dc - bc)^{-1} := cr_{21}^{-1}, & a_\sim &= a(da - ca^{-1}ba)^{-1} := ar_{22}^{-1}; \\ {}^\sim d &= (da - dbd^{-1}c)^{-1}d := l_{11}^{-1}d, & {}^\sim b &= (bdb^{-1}a - bc)^{-1}b := l_{12}^{-1}b; \\ {}^\sim c &= (cac^{-1}d - cb)^{-1}c := l_{21}^{-1}c, & {}^\sim a &= (ad - aca^{-1}b)^{-1}a := l_{22}^{-1}a. \end{aligned}$$

By coefficient comparison of $gg^{-1} = I = g^{-1}g$ and the condition $\det_\triangle = 1$, one obtains

$$a_\sim c = c_\sim a, \qquad |l_{ij}| = |r_{ij}| = 1. \tag{1.2}$$

The group $S_\triangle L(2;\mathbb{H})$ acts on $\bar{\mathbb{H}}$ by linear fractional transformations

$$g(x) = (ax + b)(cx + d)^{-1} \tag{1.3}$$

with $g(\infty) = \infty$ for $c = 0$, and with $g(\infty) = ac^{-1}$ and $g(-c^{-1}d) = \infty$ for $c \neq 0$.

The fixed point set of a transformation $g \in \mathrm{PS}_\triangle\, L(2;\mathbb{H})$ is defined by

$$\operatorname{fix}(g) = \{v \in \bar{\mathbb{H}} : g(v) = v\}.$$

By passing to the projectivized group $\mathrm{PS}_\triangle\, L(2;\mathbb{H}) := S_\triangle L(2;\mathbb{H}) / \{\pm I\}$, one gets the isomorphism

$$\mathrm{PS}_\triangle\, L(2;\mathbb{H}) \simeq \operatorname{iso}^+(\mathbf{H}^5).$$

In the following, we will conceive an element of $\mathrm{PS}_\triangle\, L(2;\mathbb{H})$ as a four-dimensional Möbius transformation by the correspondence (1.3).

In this paper, we give a classification involving the norm of its right eigenvalues (cf. [**5**]) and its fixed point(s). We say that a non-trivial element g is

(i) *parabolic* if the norms of its right eigenvalues are 1 and it has exactly one fixed point in $\bar{\mathbb{H}}$,

(ii) *loxodromic* if the norm of one of its right eigenvalues is bigger than 1,

(iii) *elliptic* if the norms of its right eigenvalues are 1 and it has at least two fixed points in $\bar{\mathbb{H}}$.

For a non-trivial element g, since the cardinality of its fixed point(s) and the norms of its right eigenvalues are conjugate invariant [**5**, **6**], the above classification is conjugate invariant and complete.

In order to state our result conveniently, we give here a brief background of the real quaternion division ring $\mathbb{H}$, whose elements are of the form $z = z_1 + z_2\mathbf{i} + z_3\mathbf{j} + z_4\mathbf{k} \in \mathbb{H}$, where $z_i \in \mathbb{R}$ and $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. Let $\bar{z} = z_1 - z_2\mathbf{i} - z_3\mathbf{j} - z_4\mathbf{k}$ be the *conjugate* of z, and

$$|z| = \sqrt{\bar{z}z} = \sqrt{z_1^2 + z_2^2 + z_3^2 + z_4^2}$$

be the *modulus* of z. We define $\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z})$ to be the *real part* of z, and $\mathrm{Im}(z) = \frac{1}{2}(z - \bar{z})$ to be the *imaginary part* of z. Also $z^{-1} = \bar{z}|z|^{-2}$ is the *inverse* of z. Two quaternions z and w are *similar*, which is denoted by $z \sim w$, if there exists non-zero $q \in \mathbb{H}$ such that $z = qwq^{-1}$. For a unit quaternion q, we can write

$$q = \exp(\alpha\mathbf{J}) := \cos\alpha + \mathbf{J}\sin\alpha \quad \text{for some } \alpha \in [0, 2\pi),$$

where $\mathbf{J} = \mathrm{Im}(q)/|\mathrm{Im}(q)|$ is a pure unit quaternion and $\mathbf{J}^2 = -1$.

As in [**4**], by identifying a quaternion $a = a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k}$ with a real vector $\boldsymbol{a} = (a_1, a_2, a_3, a_4)^{\mathrm{T}} \in \mathbb{R}^4$, the addition in $\mathbb{H}$ becomes the componentwise addition of vectors in $\mathbb{R}^4$ and multiplication can be expressed as $\boldsymbol{ab} = L_a\boldsymbol{b}$ and $\boldsymbol{ba} = R_a\boldsymbol{b}$, where

$$L_a = \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & -a_4 & a_3 \\ a_3 & a_4 & a_1 & -a_2 \\ a_4 & -a_3 & a_2 & a_1 \end{pmatrix}, \qquad R_a = CL_a^{\mathrm{T}}C = \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & a_4 & -a_3 \\ a_3 & -a_4 & a_1 & a_2 \\ a_4 & a_3 & -a_2 & a_1 \end{pmatrix},$$

$C = \mathrm{diag}(1, -1, -1, -1)$ and M^{T} denotes the transpose of matrix M.

The inner product of p and q is

$$p \cdot q := (\boldsymbol{p}, \boldsymbol{q}) = \tfrac{1}{2}(p\bar{q} + q\bar{p}). \tag{1.4}$$

When $c \neq 0$,

$$g(x) = (ax + b)(cx + d)^{-1} = ac^{-1} + \frac{|c|^{-2}(|c|^{-2}(-\bar{c}_{\sim}))\overline{(x + c^{-1}d)}\bar{c}}{|x + c^{-1}d|^2}. \tag{1.5}$$

Thus,

$$\boldsymbol{g}(x) = \boldsymbol{a}\boldsymbol{c}^{-1} + |c|^{-2} A(g) \frac{(\boldsymbol{x} + \boldsymbol{c}^{-1}\boldsymbol{d})}{|x + c^{-1}d|^2}, \tag{1.6}$$

where $A(g) = |c|^{-2} L_{(-\bar{c}\sim)} R_{\bar{c}} C$ is a real orthogonal matrix with determinant -1.

For

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PS}_{\triangle}\, L(2; \mathbb{H})$$

with $c \neq 0$, let

$$\sigma := \tfrac{1}{2}(ac^{-1} + c^{-1}d), \qquad p := \frac{\mathrm{Im}(\sigma c)}{|\mathrm{Im}(\sigma c)|}, \qquad q := \frac{\mathrm{Im}(\bar{\sigma} r_{21} \bar{c})}{|\mathrm{Im}(\sigma c)|}. \tag{1.7}$$

For the case $\sigma c \sim \bar{\sigma} r_{21} \bar{c} \notin \mathbb{R}$, we define

$$\cos\phi = \frac{|\sigma_p|}{|\sigma|},$$

where

$$\sigma_p = \begin{cases} -\frac{1}{2}(\bar{\sigma} + p\bar{\sigma}p) & \text{provided } p = \bar{q}, \\ \left[1 + \dfrac{((p - \bar{q})\bar{\sigma})^2 + ((1 + p\bar{q})\bar{\sigma})^2}{2|\sigma|^2 |p - \bar{q}|^2}\right]\sigma & \text{provided } p \neq \bar{q}. \end{cases} \tag{1.8}$$

For the case $\sigma c = \bar{\sigma} r_{21} \bar{c} \in \mathbb{R}$, we define $\cos\phi = 1$.

We will show in § 3 that the following lemma holds.

Lemma 1.2. *If $g \in \mathrm{PS}_{\triangle}\, L(2, \mathbb{H})$ with $c \neq 0$ and $\sigma c = \bar{\sigma} r_{21} \bar{c} \in \mathbb{R}$, then* $\mathrm{rank}(I - A(g)) = \mathrm{rank}(I - A(g), \boldsymbol{\sigma})$.

Our main theorem is as follows.

Theorem 1.3. *Let*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PS}_{\triangle}\, L(2, \mathbb{H}) \quad \text{with } c \neq 0.$$

(a) *If $\sigma c \sim \bar{\sigma} r_{21} \bar{c}$,*

  (I) *and if* $\mathrm{rank}(I - A(g)) = \mathrm{rank}(I - A(g), \boldsymbol{\sigma})$,

    (i) *and if $|\sigma c| < \cos\phi$, then g is elliptic,*

    (ii) *and if $|\sigma c| = \cos\phi$, then g is parabolic,*

    (iii) *and if $|\sigma c| > \cos\phi$, then g is loxodromic;*

  (II) *and if* $\mathrm{rank}(I - A(g)) \neq \mathrm{rank}(I - A(g), \boldsymbol{\sigma})$, *then g is loxodromic.*

(b) *If $\sigma c \nsim \bar{\sigma} r_{21} \bar{c}$, then g is loxodromic.*

It is immediate from Theorems 1.3 and 2.1 (cf. § 2) that we can state the following corollaries.

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Corollary 1.4. *A non-trivial element g is elliptic if and only if one of the following holds:*

(a) $c = 0$,

(i) $d \nsim a$ *and* $|a| = |d|$ *or*

(ii) $d \sim a \notin \mathbb{R}$ *and* $b\bar{d} = ab$;

(b) $c \neq 0$, $\sigma c \sim \bar{\sigma} r_{21} \bar{c}$, $\operatorname{rank}(I - A(g)) = \operatorname{rank}(I - A(g), \boldsymbol{\sigma})$ *and* $|\sigma c| < \cos\phi$.

Corollary 1.5. *A non-trivial element g is parabolic if and only if one of the following holds:*

(a) $c = 0$,

(i) $d \sim a \notin \mathbb{R}$ *and* $b\bar{d} \neq ab$ *or*

(ii) $d = a \in \mathbb{R}$ *and* $b \neq 0$;

(b) $c \neq 0$, $\sigma c \sim \bar{\sigma} r_{21} \bar{c}$, $\operatorname{rank}(I - A(g)) = \operatorname{rank}(I - A(g), \boldsymbol{\sigma})$ *and* $|\sigma c| = \cos\phi$.

Corollary 1.6. *A non-trivial element g is loxodromic if and only if one of the following holds:*

(a) $c = 0, d \nsim a$ *and* $|a| \neq |d|$;

(b) $c \neq 0$,

(i) $\sigma c \sim \bar{\sigma} r_{21} \bar{c}, \operatorname{rank}(I - A(g)) = \operatorname{rank}(I - A(g), \boldsymbol{\sigma})$ *and* $|\sigma c| > \cos\phi$, *or*

(ii) $\operatorname{rank}(I - A(g)) \neq \operatorname{rank}(I - A(g), \boldsymbol{\sigma})$, *or*

(iii) $\sigma c \nsim \bar{\sigma} r_{21} \bar{c}$.

We now outline the layout of the paper. In § 2, we discuss the classification of Möbius transformation g with $c = 0$. By showing that transformation g with $c \neq 0$ can be $\mathrm{PS}_\triangle L(2, \mathbb{H})$ conjugate to an upper triangular matrix, we also obtain the standard forms of elements in $\mathrm{PS}_\triangle L(2, \mathbb{H})$. With the aid of the isometric sphere, we prove our main theorem in § 3 by a similar method to that employed by Ahlfors in [**1**]. In § 4, we point out that some earlier results in dimensions 2 and 3 can be deduced from the results in this paper and thus show that all the classes we indicated are non-empty.

2. The case $c = 0$ and the standard forms

By our trichotomy classification and the application of [**4**, Theorem 3] to $xd - ax = b$, we have the following theorem.

Theorem 2.1. *Let v be the finite fixed point of the non-trivial four-dimensional Möbius transformation $g(x) = (ax + b)d^{-1}$.*

(a) *If $d \nsim a$, then*

$$v = q^{-1}(b\bar{d} - ab), \quad q = 2(\mathrm{Re}(a) - \mathrm{Re}(d))a + |d|^2 - |a|^2,$$

(i) *if $|a| = |d|$, then g is elliptic with two fixed points;*

(ii) *if $|a| \neq |d|$, then g is loxodromic.*

(b) *If $d \sim a \notin \mathbb{R}$,*

(i) *and if $b\bar{d} = ab$, then*

$$v = \frac{1}{4|\mathrm{Im}(d)|^2}(ab - bd) + h - \frac{1}{|\mathrm{Im}(d)|^2}\,\mathrm{Im}(a)h\,\mathrm{Im}(d), \quad \forall h \in \mathbb{H},$$

and in this case g is elliptic with fixed points forming a two-dimensional affine subspace of $\mathbb{R}^4$;

(ii) *and if $b\bar{d} \neq ab$, then g is parabolic.*

(c) *If $d = a \in \mathbb{R}$, then g is parabolic provided $b \neq 0$.*

Point $v \in \mathrm{fix}(g)$ if and only if

$$v = g(v) = (av + b)(cv + d)^{-1}. \tag{2.1}$$

Multiplying both sides from the right by $cv + d$, we see that v satisfies the equation

$$vcv + vd - av - b = 0. \tag{2.2}$$

If $c \neq 0$, then the above equation is

$$t^2 - (cac^{-1} + d)t + cac^{-1}d - cb = 0, \quad t = cv + d,$$

that is,

$$t^2 - 2c\sigma t + l_{21} = 0, \quad t = cv + d. \tag{2.3}$$

By (2.2), we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} v \\ 1 \end{pmatrix} = \begin{pmatrix} av + b \\ cv + d \end{pmatrix} = \begin{pmatrix} vcv + vd \\ cv + d \end{pmatrix} = \begin{pmatrix} v \\ 1 \end{pmatrix}(cv + d). \tag{2.4}$$

Lemma 2.2. *Let*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PS}_{\triangle}\, L(2, \mathbb{H}).$$

Then g is $\mathrm{PS}_{\triangle}\, L(2, \mathbb{H})$ conjugate to an upper triangular matrix.

Proof. The case $c = 0$ or $b = 0$ is obvious. For the cases $b \neq 0$ and $c \neq 0$, letting v be a fixed point of g, we have

$$(a - vc)|v|^2 = vd\bar{v} - b\bar{v} \tag{2.5}$$

and

$$(-\bar{v}a + d\bar{v})|v|^2 = \bar{v}b\bar{v} - |v|^4 c. \tag{2.6}$$

Let

$$U = \frac{1}{\sqrt{1 + |v|^2}} \begin{pmatrix} v & -1 \\ 1 & \bar{v} \end{pmatrix}.$$

Then $\det_{\triangle}(U) = 1$ and

$$U^{-1}gU = \begin{pmatrix} \dfrac{cv + d + \bar{v}av + \bar{v}b}{1 + |v|^2} & \dfrac{-\bar{v}a + d\bar{v} + \bar{v}b\bar{v} - c}{1 + |v|^2} \\ 0 & \dfrac{(a - vc) + vd\bar{v} - b\bar{v}}{1 + |v|^2} \end{pmatrix} = \begin{pmatrix} cv + d & \dfrac{\bar{v}b\bar{v}}{|v|^2} - c \\ 0 & a - vc \end{pmatrix}. \tag{2.7}$$

The proof is completed. □

If g with $c = 0$ has a finite fixed point v, then conjugation by $f(x) = x - v$ sends the fixed point of g to ∞ and results in

$$fgf^{-1}(x) = axd^{-1}. \tag{2.8}$$

The above observation, together with the fact that any quaternion is conjugate to a complex number and using Theorem 2.1 and Lemma 2.2, implies that the standard forms of four-dimensional Möbius transformation are as follows.

Theorem 2.3. *Let*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PS}_{\triangle} L(2; \mathbb{H})$$

and let $\alpha, \beta, \theta \in [0, \pi)$.

(a) *g is loxodromic if g is conjugate to*

$$\begin{pmatrix} t\mathrm{e}^{\mathbf{i}\alpha} & 0 \\ 0 & t^{-1}\mathrm{e}^{-\mathbf{i}\beta} \end{pmatrix},$$

where $t > 0$, $t \neq 1$.

(b) *g is elliptic if g is conjugate to*

$$\begin{pmatrix} \mathrm{e}^{\mathbf{i}\alpha} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\beta} \end{pmatrix}.$$

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(c) *g is parabolic if g is conjugate to*

$$\begin{pmatrix} e^{\mathbf{i}\alpha} & \exp(\theta\mathbf{J}) \\ 0 & e^{-\mathbf{i}\alpha} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

where $\exp(\theta\mathbf{J})e^{\mathbf{i}\alpha} \neq e^{\mathbf{i}\alpha}\exp(\theta\mathbf{J})$.

3. The proof of Theorem 1.3

We define the *isometric sphere* $I(g)$ of Möbius transformation $g(x) = (ax+b)(cx+d)^{-1}$ with $c \neq 0$ by

$$I(g) = \{x \in \mathbb{H} : |x + c^{-1}d| = |c|^{-1}\}.$$

By Proposition 1.1, the isometric sphere of g^{-1} is

$$I(g^{-1}) = \{x \in \mathbb{H} : |x - ac^{-1}| = |c|^{-1}\}.$$

Let $\operatorname{ext}(I(g))$ and $\operatorname{int}(I(g))$ be the exterior and interior of isometric sphere $I(g)$, respectively. By exploiting (2.7) and Theorem 2.1, the location of its fixed point can be stated as follows.

Theorem 3.1. *Let v be a fixed point of $g(x) = (ax+b)(cx+d)^{-1}$ with $c \neq 0$. Then*

(a) *$v \in I(g) \cap I(g^{-1})$ provided that g is elliptic or parabolic,*

(b) *$v \in \operatorname{ext}(I(g)) \cap \operatorname{int}(I(g^{-1}))$ or $v \in \operatorname{int}(I(g)) \cap \operatorname{ext}(I(g^{-1}))$ provided that g is loxodromic.*

The *normalized form* of

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with } c \neq 0$$

is the transformation

$$g_0 = fgf^{-1} = \begin{pmatrix} \sigma c & \sigma c\sigma + b - ac^{-1}d \\ c & c\sigma \end{pmatrix}, \tag{3.1}$$

where

$$f = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \beta = \tfrac{1}{2}(c^{-1}d - ac^{-1}).$$

If g_0 has a fixed point v_0, then the original form has the fixed point $v_0 + \frac{1}{2}(ac^{-1} - c^{-1}d)$. As far as the fixed points or type criterion are concerned it is therefore sufficient to consider the normalized form. Recall that g and its normalized form g_0 share the same values of c, σ, l_{21}, r_{21} and $A(g)$. Thus, it suffices to prove Theorem 1.3 in its normalized form. We divide our proof into several lemmas.

Lemma 3.2. *If $\sigma = 0$, then g is elliptic.*

Proof. By (2.3), the fixed point v of g is a solution to the equation $(cv)^2 = -l_{21}$. By [**6**, Theorem 2.3], g has at least two fixed points in $\mathbb{H}$. It follows from (2.4) and $|l_{21}| = 1$ that all the right eigenvalues of g have norm 1. Thus, g is elliptic. The proof is completed. $\square$

In what follows we assume that $\sigma \neq 0$.

We denote by $V(g)$ all the solutions to the following equation

$$\sigma c x = x \bar{\sigma} r_{21} \bar{c}.$$

More explicitly,

$$V(g) = \begin{cases} \{0\} & \text{if } \sigma c \nsim \bar{\sigma} r_{21} \bar{c}, \\ \left\{ h - \dfrac{\operatorname{Im}(\sigma c) h \operatorname{Im}(\bar{\sigma} r_{21} \bar{c})}{|\operatorname{Im}(\sigma c)|^2},\ \forall h \in \mathbb{H} \right\} & \text{if } \sigma c \sim \bar{\sigma} r_{21} \bar{c} \notin \mathbb{R}, \\ \mathbb{H} & \text{if } \sigma c = \bar{\sigma} r_{21} \bar{c} \in \mathbb{R}. \end{cases} \tag{3.2}$$

By the above expression, $\dim V(g) = 0, 2$ or 4.

Let ϕ be the angle between σ and its orthogonal projection σ_p on $V(g)$ if $\dim V(g) \geqslant 2$.

Lemma 3.3. *If $\sigma c \sim \bar{\sigma} r_{21} \bar{c} \notin \mathbb{R}$, letting p and q be as in (1.7), then*

$$\cos \phi = \frac{|\sigma_p|}{|\sigma|}, \tag{3.3}$$

where σ_p is as in (1.8).

Proof. By (1.7), the element of $V(g)$ can be expressed as

$$x = h - phq, \quad \forall h \in \mathbb{H}. \tag{3.4}$$

For the first case, $p = \bar{q}$, (3.4) can be viewed as

$$\boldsymbol{x} = (I - L_p R_{\bar{p}}) \boldsymbol{h}, \quad \forall h \in \mathbb{H}. \tag{3.5}$$

Since $I - L_p R_{\bar{p}}$ is a real symmetric matrix and 1 and p are two orthogonal vectors in the null space of $I - L_p R_{\bar{p}}$, 1 and p are thus two orthogonal basis in the orthogonal complement $V(g)^{\perp} = \operatorname{span}\{1, p\}$. Hence, the orthogonal projection $\sigma_p^{\perp}$ on $V(g)^{\perp}$ is

$$\sigma_p^{\perp} = (\sigma \cdot 1)1 + (\sigma \cdot p)p = \sigma + \tfrac{1}{2}(\bar{\sigma} + p\bar{\sigma}p).$$

Thus,

$$\sigma_p = \sigma - \sigma_p^{\perp} = -\tfrac{1}{2}(\bar{\sigma} + p\bar{\sigma}p).$$

For the second case $p \neq \bar{q}$, since $p - \bar{q}$ and $1 + p\bar{q}$ are two orthogonal vectors of $V(g)$,

$$x_1 = \frac{p - \bar{q}}{|p - \bar{q}|} \quad \text{and} \quad x_2 = \frac{1 + p\bar{q}}{|1 + p\bar{q}|} = \frac{1 + p\bar{q}}{|p - \bar{q}|}$$

are the standard orthogonal basis of $V(g)$. Hence, the orthogonal projection σ_p of σ on $V(g)$ is

$$\sigma_p = (\sigma \cdot x_1)x_1 + (\sigma \cdot x_2)x_2 = \sigma + \frac{1}{2|p-\bar{q}|^2}[(p-\bar{q})\bar{\sigma}(p-\bar{q}) + (1+p\bar{q})\bar{\sigma}(1+p\bar{q})].$$

The proof is completed. □

By (1.6), the equation

$$ac^{-1} - |c|^{-2}\bar{c}_{\sim}\overline{(x + c^{-1}d)}\bar{c} = x \tag{3.6}$$

is solvable if and only if

$$\operatorname{rank}(I - A(g)) = \operatorname{rank}(I - A(g), \boldsymbol{\sigma}). \tag{3.7}$$

Let v be a fixed point of g. Then

$$\sigma c v + \sigma c \sigma + b - ac^{-1}d = v(cv + c\sigma). \tag{3.8}$$

Multiplying both sides from the right by c, we have

$$(v - \sigma)c(v + \sigma)c = bc - ac^{-1}dc = -r_{21}. \tag{3.9}$$

Hence,

$$|(v-\sigma)|\,|(v+\sigma)| = |c|^{-2}. \tag{3.10}$$

The following lemma is strongly motivated by the method of Ahlfors [**1**].

Lemma 3.4. *If $\sigma c \sim \bar{\sigma} r_{21}\bar{c} \notin \mathbb{R}$, then*

(a) *g is elliptic if and only if $|\sigma c| < \cos\phi$ and $\operatorname{rank}(I - A(g)) = \operatorname{rank}(I - A(g), \boldsymbol{\sigma})$,*

(b) *g is parabolic if and only if $|\sigma c| = \cos\phi$ and $\operatorname{rank}(I - A(g)) = \operatorname{rank}(I - A(g), \boldsymbol{\sigma})$,*

(c) *g is loxodromic if and only if one of the following holds:*

  (i) *$|\sigma c| > \cos\phi$ and $\operatorname{rank}(I - A(g)) = \operatorname{rank}(I - A(g), \boldsymbol{\sigma})$;*

  (ii) *$\operatorname{rank}(I - A(g)) \neq \operatorname{rank}(I - A(g), \boldsymbol{\sigma})$.*

Proof. We begin with the elliptic case. Let v be a fixed point of a elliptic element g. By Theorem 3.1 and (1.6), we have

$$(I - A(g))(\boldsymbol{v} + \boldsymbol{c}^{-1}\boldsymbol{d}) = 2\boldsymbol{\sigma}. \tag{3.11}$$

Thus, if g is elliptic then $\operatorname{rank}(I - A(g)) = \operatorname{rank}(I - A(g), \boldsymbol{\sigma})$.

As g is elliptic, the intersection isometric spheres of g and its inverse is a two-dimensional sphere $S^2(g)$, situated in the hyperplane through the origin perpendicular

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to σ with centre 0 and radius $r = \sqrt{|c|^{-2} - |\sigma|^2}$. The fact that v is perpendicular to σ implies that $\bar{v} = -\sigma^{-1}v\bar{\sigma}$, which leads to

$$|c|^{-2}(v-\sigma)^{-1} = |v-\sigma|^2(v-\sigma)^{-1} = \overline{v-\sigma} = -\sigma^{-1}(v+\sigma)\bar{\sigma}.$$

Substituting this into (3.9), we get

$$\sigma c(v+\sigma) = (v+\sigma)\bar{\sigma}r_{21}\bar{c}. \tag{3.12}$$

By [**4**, Theorem 3], we get the expression for $V(g)$ as in (3.2). Since g has at least two fixed points, $-\sigma+V(g)$ intersects $S^2(g)$, which implies that $|\sigma|\tan\phi < r$, i.e. $|\sigma c| < \cos\phi$.

For the converse, the conditions $\dim V(g) \geqslant 2$ and $|\sigma c| < \cos\phi$ guarantee the existence of two points u and v that lie on $S^2(g)$ and satisfy (3.12). It follows from the locations of u, v and (3.12) that u and v are two solutions to (3.9), which implies that u and v are two fixed points of g. By Lemma 2.2, we can conjugate g to the upper triangular matrix

$$\begin{pmatrix} cv + c\sigma & \dfrac{\bar{v}(\sigma c\sigma + b - ac^{-1}d)\bar{v}}{|v|^2} - c \\ 0 & \sigma c - vc \end{pmatrix}.$$

By the definition of $S^2(g)$, we know that $|v \pm \sigma| = |c|^{-1}$, which implies that the norm of its right eigenvalues is 1. By Theorem 2.1, g is elliptic. This completes the proof of (a).

If g is parabolic, as in the proof of (a), we have $\operatorname{rank}(I - A(g)) = \operatorname{rank}(I - A(g), \boldsymbol{\sigma})$. Exactly as in the proof of (a), a point $v \in S^2(g)$ is a fixed point if and only if $v \in -\sigma + V(g)$. Therefore, if g is parabolic, then $S^2(g)$ and $-\sigma+V(g)$ meet in a single point, which implies that $|\sigma c| = \cos\phi$.

For the converse, the conditions $\dim V(g) \geqslant 2$ and $|\sigma c| = \cos\phi$ guarantee the existence of a single point v that lies on $S^2(g)$ and satisfies (3.12). It follows that v is a fixed point of g. It follows from the proof of Lemma 2.2 that any other fixed point u would satisfy $|cu + d| = 1$ and hence would also lie on $S^2(g)$ and on $-\sigma + V(g)$. This contradiction implies that g is parabolic. This completes the proof of (b).

The proof of (c) follows from the trichotomy classification and the proof of (a) and (b). The proof is completed. □

Proof of Lemma 1.2. If $\sigma = 0$, our result follows automatically. For the case $\sigma \neq 0$, we have $r_{21} = 1$ and $c_{\sim} = cr_{21}^{-1} = c$. It is easy to check that $x = \frac{1}{2}(ac^{-1} - c^{-1}d)$ is a solution to (3.6). The proof is completed. □

By Lemma 1.2 and the proof of Lemma 3.4, we have the following.

Lemma 3.5. *If $\sigma c = \bar{\sigma}r_{21}\bar{c} \in \mathbb{R}$, then*

(a) *g is elliptic if and only if $|\sigma c| < 1$,*

(b) *g is parabolic if and only if $|\sigma c| = 1$,*

(c) *g is loxodromic if and only if $|\sigma c| > 1$.*

Lemma 3.6. *If $\sigma c \nsim \bar{\sigma} r_{21} \bar{c}$, then g is loxodromic.*

Proof. Suppose that g is elliptic or parabolic. Then, by Theorem 3.1, the fixed point v of g satisfies equation (3.12). Thus, $\sigma c \nsim \bar{\sigma} r_{21} \bar{c}$ implies that $v = -\sigma$. Conjugating by

$$h = \begin{pmatrix} 0 & 1 \\ -1 & v \end{pmatrix} \in \mathrm{PS}_{\triangle} L(2, \mathbb{H}),$$

we have

$$hgh^{-1} = \begin{pmatrix} c(v+\sigma) & -c \\ 0 & (\sigma - v)c \end{pmatrix} = \begin{pmatrix} 0 & -c \\ 0 & (\sigma - v)c \end{pmatrix}. \tag{3.13}$$

This is a contradiction, which implies that g is loxodromic.

The proof is completed. □

Proof of Theorem 1.3. This follows from the proofs of Lemmas 1.2, 3.2–3.6. □

4. Examples

It is evident that $\mathrm{PSL}(2, \mathbb{C})$ and $\mathrm{U}(1,1;\mathbb{H})$ are subgroups of $\mathrm{PS}_{\triangle} L(2;\mathbb{H})$. In this section, we indicate that the classifications obtained of these groups can be derived by the result in this paper and thus show that all the classes we indicated are non-empty.

For the case of $\mathrm{PSL}(2, \mathbb{C})$, Proposition 4.1 and Theorems 1.3 and 2.1 imply that our classification is compatible with the classical complex case.

Proposition 4.1. *Let*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{C}).$$

Then

(i) $\operatorname{rank}(I - A(g)) = \operatorname{rank}(I - A(g), \boldsymbol{\sigma})$ *provided* $a + d \in \mathbb{R}$,

(ii) $\operatorname{rank}(I - A(g)) \neq \operatorname{rank}(I - A(g), \boldsymbol{\sigma})$ *provided* $a + d \notin \mathbb{R}$.

Proof. Let $y = x + c^{-1}d$. In the complex case $r_{21} = 1$ and $2\sigma c = a + d$. Equation (3.6) has a solution if and only if

$$2\sigma c = yc + \overline{yc}$$

has a solution. The proof is completed. □

For the case of $\mathrm{U}(1,1;\mathbb{H})$, we recall that the basic relations of the entries of

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

are as follows:

$$|a| = |d|, \qquad |b| = |c|, \qquad |a|^2 - |c|^2 = 1, \qquad \bar{a}b = \bar{c}d, \qquad a\bar{c} = b\bar{d}. \tag{4.1}$$

For $g \in \mathrm{U}(1,1;\mathbb{H})$, it is easy to check that $x = 0$ is a solution to (3.6), which implies that the condition $\operatorname{rank}(I - A(g)) = \operatorname{rank}(I - A(g), \boldsymbol{\sigma})$ always holds.

For $g \in \mathrm{U}(1,1;\mathbb{H})$, we have

$$l_{21} = \bar{c}^{-1}b, \qquad \sigma c = \operatorname{Re}(a) + \tfrac{1}{2}\bar{a}(r_{21} - 1), \qquad \bar{\sigma} r_{21}\bar{c} = \bar{\sigma} b = \operatorname{Re}(d) + \tfrac{1}{2}\bar{d}(l_{21} - 1). \quad (4.2)$$

Hence,

$$\sigma c \sim \bar{\sigma} r_{21}\bar{c}, \quad |\operatorname{Im}(\sigma c)| = |\operatorname{Im}(\bar{\sigma} r_{21}\bar{c})| = \tfrac{1}{2}|\operatorname{Im}((l_{21} - 1)\bar{d})|. \quad (4.3)$$

If $b = \bar{c} \neq 0$, then $r_{21} = l_{21} = 1$ and $\sigma c = \bar{\sigma} b = \operatorname{Re}(a) = \operatorname{Re}(d)$. This observation shows that the case (b) of Theorem 1.3 in [**3**] is the same as case $\sigma c = \bar{\sigma} r_{21}\bar{c}$ of Theorem 1.3 applied in $\mathrm{U}(1,1;\mathbb{H})$.

For $g \in \mathrm{U}(1,1;\mathbb{H})$ with $b \neq \bar{c} \neq 0$, we define

$$\Delta = |\operatorname{Im}((l_{21} - 1)\bar{d})|^2 - |l_{21} - 1|^2 = 4|\operatorname{Im}(\sigma c)|^2 - |l_{21} - 1|^2, \quad (4.4)$$

which is the same as in [**3**, Theorem 1.3] by (4.3).

If $\sigma = 0$, then $\Delta = -|l_{21} - 1|^2 < 0$ for $l_{21} \neq 1$. By Lemma 3.2, g is elliptic, which is the same as the classification of [**3**, Theorem 1.3].

For the remaining case $\sigma \neq 0$, we have the following Proposition 4.2, whose proof can be achieved by the relations of (4.2)–(4.4), (1.7) and (1.8).

Proposition 4.2. *Let*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{U}(1,1;\mathbb{H}).$$

Then

$$\begin{aligned} |\sigma c| &> \cos\phi, && \textit{if and only if } \Delta > 0, \\ |\sigma c| &= \cos\phi, && \textit{if and only if } \Delta = 0, \\ |\sigma c| &< \cos\phi && \textit{if and only if } \Delta < 0. \end{aligned}$$

The above observations indicates our classification restricted in $\mathrm{U}(1,1;\mathbb{H})$ is equivalent to the main result in [**3**].

We remark that the classes we indicated are non-empty. The examples of (I) of case (a) in Theorem 1.3 have been given in [**3**, § 4]. Here we give some other examples for the remaining cases.

Example 4.3.

$$g = \begin{pmatrix} \frac{1}{2}\mathbf{i} & 1 \\ -\frac{5}{4} & \frac{1}{2}\mathbf{i} \end{pmatrix} \in SL(2,\mathbb{C}),$$

which is an element of (II) in case (a) in Theorem 1.3.

Example 4.4. Let

$$g = \begin{pmatrix} 0 & -\mathbf{k} \\ 1 & 1 \end{pmatrix}.$$

Then $\sigma c \nsim \bar{\sigma} r_{21}\bar{c}$. Thus, g is loxodromic, which also can be seen by (2.3) and [**6**, Example 2.11]. This transformation is an element of Theorem 1.3 (b).

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