SPECTRAL DECOMPOSITION OF SPHERICAL IMMERSIONS WITH RESPECT TO THE JACOBI OPERATOR

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Abstract. We study the spectral decomposition with respect to the Jacobi operator, J, of spherical immersions and characterize those with a simple decomposition in terms of the Finite Chen-type submanifolds. As a consequence, we give an application to the inverse problem for J.

1. Introduction. The Jacobi operator J (or the second variation operator) was introduced by Simons in [**11**]. It appears in the study of the second variation formula of the area function for a compact minimal submanifold M of a Riemannian manifold $\tilde{M}$ and it is an elliptic operator acting on the normal bundle of $M, N(M)$. A cross-section V of $N(M)$ is a *Jacobi* field if $JV = 0$, [**11**]. This definition is a generalization of the Jacobi fields over geodesics. Spectral properties of this operator have been studied in [**7**], [**9**].

For another classical elliptic operator, the Laplacian Δ, the spectral behaviour of an isometric immersion of a Riemannian manifold M in an Euclidean space has been widely studied: *finite type* submanifolds (a generalization of minimal submanifolds) were introduced by B.-Y. Chen in the late seventies and can be characterized by a variational minimal principle. For a recent survey on this subject, see [**6**].

We have found in [**2**] a relation between both elliptic operators (one can see also [**10**]). This suggests studying the spectral behaviour with respect to J of the position vector of spherical submanifolds and relate it to the Chen-type of the submanifold.

In this paper, we first define the notion of *finite J-type* for spherical submanifolds. Then, we analyse those spherical submanifolds with the easiest spectral decomposition with respect to J. Thus, we find that J-type 1 characterizes minimality in the sphere and therefore is equivalent to the Chen-type 1. The following step is to study J-type 2 spherical immersions. We find that, analogously to the previous case, this family coincides with that of Chen-type 2 for hypersurfaces. Finally, we use this to see that there exists a spectral condition for a spherical hypersurface with constant mean and scalar curvatures to be totally geodesic.

2. Preliminaries. Let M^p be a Riemannian manifold of dimension p, isometrically immersed in a Riemannian manifold $\tilde{M}^n$ of dimension $n = p + q$. The normal bundle of $M^p, N(M)$ is then a real q-dimensional vector bundle with inner product induced by the metric of $\tilde{M}^n$.

Let J be the *Jacobi* operator (or the second variation operator) acting on cross-sections ξ of $N(M)$. It is a second order differential operator which is defined by

$$J : N(M) \to N(M),$$
$$J\xi = (\Delta^D - \tilde{A} + \tilde{R})\xi, \tag{1}$$

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where Δ^D, denotes the Laplacian relative to the normal connection D in $N(M)$, $\tilde{A}$ is the *Simons* operator **[11]**, defined on $N(M)$ by

$$< \tilde{A}\xi, \eta >= \operatorname{trace}(A_\xi \circ A_\eta), \quad \xi, \eta \in N(M), \tag{2}$$

with A denoting the Weingarten map, and

$$\tilde{R}\xi = \sum_{i=1}^{p} (\overline{R}(E_i, \xi, E_i))^\perp, \tag{3}$$

where $\perp$ denotes the normal component and $\bar{R}$ is the Riemannian curvature of $\tilde{M}^n$. A cross-section $\xi \in N(M)$is called a *Jacobi field* **[11]**, if $J\xi = 0$.

Suppose that M^p is compact. In this case we define an inner product in $N(M)$ by

$$(\xi, \eta) = \int_M < \xi, \eta > dv. \tag{4}$$

Then J is self-adjoint with respect to (4). Moreover J is a strongly elliptic operator and it has an indefinite sequence of distinct eigenvalues

$$\operatorname{Spec}(M, J) = \{\mu_1 < \mu_2, .. < \mu_k < ... \to \infty\}.$$

Let $\Gamma_k = \{\xi \in N(M)/J\xi = \mu_k\xi\}$ be the eigenspace of J associated with μ_k. Then the dimension of Γ_k, that is the multiplicity of μ_k, is finite. Finally we know that the decomposition $\sum_{k=1}^{\infty} \Gamma_k$ is orthogonal with respect to (4) and it is dense in $N(M)$ in the L_2-sense.

For each $\xi \in N(M)$, let ξ_k be the projection of ξ onto Γ_k. Then one has the following spectral decomposition

$$\xi = \sum_{k=1}^{\infty} \xi_k, \quad \text{(in the } L_2\text{-sense)}. \tag{5}$$

3. The *J*-type of spherical immersions. Assume that $x : M^p \to S^n(1) \subset E^{n+1}$ is an isometric immersion of a compact Riemannian manifold in the unit n-dimensional sphere, $S^n(1)$, centered at the origin of the Euclidean space E^{n+1}. The immersion (M^p, x) is said to be of *Chen-type* k **[4]**, if the position vector x can be decomposed in the following way:

$$x = v + \sum_{i=1}^{k} x_{t_i},$$

where $v \in E^{n+1}$ is a constant vector and $\Delta x_{t_i} = \lambda_{t_i} x_{t_i}$, Δ being the Laplacian of M^p and $\lambda_{t_i} \in R$. Moreover, if the centre of mass of (M^p, x) coincides with that of $S^n(1)$, then the immersion is called *mass-symmetric* **[4]**.

For an isometric immersion $x : M^p \to S^n(1) \subset E^{n+1}$ the Jacobi operator J on the normal bundle $N(M)$ is nothing but

$$J = \Delta^D - \tilde{A}. \tag{6}$$

On the other hand, since $x \in N(M)$ it admits a spectral decomposition with respect to J as in (5), given by

$$x = \sum_{i=1}^{\infty} x_i, \quad x_i \in \Gamma_i. \tag{7}$$

We say that (M^p, x) is of *J-type k* if only k non-zero members appear in (7). In particular, it is of J-type 2 if

$$x = x_s + x_r \tag{8}$$

with $x_s, x_r \in N(M)$ and $Jx_s = \mu_s x_s, Jx_r = \mu_r x_r, \mu_s, \mu_r \in R$. First we prove the following result.

PROPOSITION 1. *Let $x : M^p \to S^n(1) \subset E^{n+1}$ be an isometric immersion of a Riemannian manifold in the unit sphere. Then*

$$Jx = -\Delta x = pH, \tag{9}$$

H being the mean curvature vector of M^p in E^{n+1}. In particular,

(i) *$Jx = \mu x, \mu \in R$, if and only if, (M^p, x) is minimal in $S^n(1)$. Moreover, in this case, $-p \in$ Spec(M^p, J).*

(ii) *$\mu_1 \leq -p$. Equality holds, if and only if, (M^p, x) is minimal in $S^n(1)$.*

Proof. From (6) we get

$$Jx = \Delta^D x - \tilde{A}x = -\tilde{A}x. \tag{10}$$

Now, choose an orthonormal basis $\{\eta_{p+1}, ..., \eta_n, -x\}$ normal to M^p in E^{n+1}. Since each element of $\{\eta_{p+1}, ..., \eta_n\}$ is perpendicular to M^p in $S^n(1)$, we have $A'_{\eta_i} = A_{\eta i}$, where A' and A are the Weingarten maps of M^p in $S^n(1)$ and in E^{n+1} respectively. Therefore we have from (10) and (2),

$$\begin{aligned} Jx &= -\left\{ \sum_{r=p+1}^{n} \text{trace}\,(A_x \circ A_{\eta r})\eta_r \right\} - \text{trace}\,(A_x \circ A_{-x})(-x) \\ &= \left\{ \sum_{r=p+1}^{n} \text{trace}\,A'_{\eta r}\eta_r \right\} - px = pH' - px = pH, \end{aligned} \tag{11}$$

where H' denotes the mean curvature vector of M^p in $S^n(1)$. Thus (11) and the Beltrami equation, $\Delta x = -pH$ end the proof of (9). Part (i) is precisely Takahashis's theorem, (see [**4**], p. 136). To prove part (ii) one can apply J to (7) and use (9) to obtain

$$pH = \sum_{k=1}^{\infty} \mu_k x_k. \tag{12}$$

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But for compact submanifolds of E^m we have $\int_M \{1+ < x, H >\} dv = 0$. Then the scalar product with respect to (4) of (12) with x gives

$$-p \cdot \mathrm{Vol}(M) = \sum_{k=1}^{\infty} \mu_k \|x_k\|^2.$$

Thus, there must be a negative eigenvalue μ_j. Then

$$p \cdot \mathrm{Vol}(M) = \sum_{k=1}^{\infty}(-\mu_k)\|x_k\|^2 \leq -\mu_1 \sum_{k=1}^{\infty} \|x_k\|^2 = -\mu_1 \cdot \mathrm{Vol}(M) \tag{13}$$

Hence $\mu_1 \leq -p$. Equality in (13) holds if and only if $\mu_1 = \mu_k$ $\forall k$.

This shows the equivalence of spherical submanifolds of Chen-type 1 with those of J-type 1. Now, we study the spherical immersions whose immersions are constructed with two eigenvectors of J. Given $x : M^p \to S^{p+1}(1) \subset E^{p+2}$ an isometric immersion of a compact hypersurface in $S^{p+1}(1)$, we denote by H, α, σ the mean curvature vector, mean curvature function and scalar curvature of the immersion in E^{p+2} respectively and by H', α', σ' the corresponding elements in $S^{p+1}(1)$. First, we establish the following result.

Lemma 2. *Let $x : M^p \to S^{p+1}(1) \subset E^{p+2}$ be an isometric immersion of a compact hypersurface in $S^{p+1}(1)$. Then (M^p, x) has constant mean curvature if an only if $JH = -\Delta H$.*

Proof. We shall use the following formula which was proved in **[2]** (see also **[18]**)

$$\Delta H = JH + 2\tilde{A}H + (\Delta H)^T, \tag{14}$$

where $()^T$ denotes the tangential part to M^p. Thus combining (14) and (6) we have

$$JH = -\Delta H + 2\Delta^D H + (\Delta H)^T, \tag{15}$$

where $(\Delta H)^T$ is given by **[3]**

$$(\Delta H)^T = \frac{p}{2} \nabla \alpha^2 + 2\,\mathrm{trace}\, A_{DH}. \tag{16}$$

Then it is enough to observe that $(\Delta H)^T$ is tangential, $\Delta^D H$ is normal and that they are both zero if and only if α is constant.

The following Proposition relates the J-type to the Chen-type for spherical hypersurfaces

Proposition 3. *Let $x : M^p \to S^{p+1}(1) \subset E^{p+2}$ be an isometric immersion of a compact hypersurface in $S^{p+1}(1)$. Then (M^p, x) is of J-type 2 if and only if it is of Chen-type 2 or it is a small hypersphere.*

Proof. We shall need the following Chen's formula (See Lemma 4.2 of **[4]**).

$$\Delta H = \Delta^D H + (\Delta H)^T + \|\sigma\|^2 H' - p\alpha^2 x. \tag{17}$$

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Assume first that (M^p, x) is of J-type 2. Then $x = x_r + x_s$ with $Jx_s = \mu_s x_s$, $Jx_r = \mu_r x_r$. Hence $Jx = \mu_r x_r + \mu_s x_s$ and $J^2 x = \mu_r^2 x_r + \mu_s^2 x_s$. From these two equations we have

$$J^2 x = bJx + cpx, \tag{18}$$

where $b = \mu_r + \mu_s$ and $c = -\frac{\mu_r \mu_s}{p}$. Using (9) in (18) one has

$$JH = bH + cx. \tag{19}$$

Combining (15), (17) and (19), we find that M^p must have constant mean curvature and constant length of the second fundamental form given by

$$\alpha^2 = \frac{c-b}{p}; \ \|\sigma\|^2 = -b. \tag{20}$$

Then, by using (9), Lemma 2 and (18), we have

$$\Delta^2 x - b\Delta x + cx = 0,$$

and so it must be of Chen-type 2 if $c \neq 0$ and a small hypersphere if $c = 0$, **[4]**.

Conversely, assume that M^p is a small hypersphere with centre x_o (which is perpendicular to M^p); then it is known that $\Delta(x - x_o) = \lambda(x - x_o)$, and so, by using Proposition 1, we have

$$J(x - x_o) = Jx - Jx_0 = -\Delta(x - x_o) = -\Delta x = Jx.$$

Thus $x = (x - x_0) + x_o$ with $J(x - x_o) = -\lambda(x - x_o)$ and $Jx_o = 0$. Therefore it is of J-type 2. Now we suppose that (M^p, x) is of Chen-type 2. Then it is mass-symmetric and has constant mean and scalar curvatures **[1]**, **[8]**. Thus we have by **[4]**

$$\Delta^2 x + b_1 \Delta x + c_1 x = 0. \tag{21}$$

Since α is constant, we use Lemma 2 and (9) in (21) to obtain

$$J^2 x - b_1 Jx + c_1 x = 0. \tag{22}$$

Since the position vector x lies in $N(M)$, we have according to (7)

$$x = \sum_{k=1}^{\infty} x_k, \tag{23}$$

with $Jx_k = \mu_k x_k$. But (22) and (23) give

$$\sum_{k=1}^{\infty} (\mu_k^2 - b_1 \mu_k + c_1) x_k = 0. \tag{24}$$

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Scalar multiplication with x_k (with respect to (4)) gives

$$(\mu_k^2 - b_1\mu_k + c_1)\|x_k\|^2 = 0,$$

which proves that all the x_k must be zero except at most two of them. But if only one were different from zero, then Proposition 1 would show that (M^p, x) is of Chen-type 1, and this is a contradiction. Hence, there must be exactly two x_k different of zero and (M^p, x) is of J-type 2.

Remark 1. It is not difficult to see that if $x : M^p \to S^{p+1}(1)$ is an immersion of Chen-type 2 and $x = x_t + x_q$ is the spectral decomposition with respect to Δ with eigenvalues λ_t and λ_q, then $x = x_t + x_q$ is also the spectral decomposition with respect to J with eigenvalues $\mu_t = -\lambda_t$, $\mu_q = -\lambda_q$. Since M^p is compact, the eigenvalues of J in the J-type 2 decomposition of x must be non-positive.

The following two results are a reformulation, using the above Proposition, of known facts on Chen-type 2 immersions [6].

Corollary 4. *Let $x : M^p \to S^{p+1}(1)$ be an isometric immersion of a compact Riemannian manifold in the unit sphere which is not a small hypersphere. Then (M^p, x) is of J-type 2 if and only if it has constant mean and scalar curvatures.*

Corollary 5. *Let $x : M^2 \to S^3(1)$ be an isometric immersion of a closed surface in the unit 3-sphere. Then (M^2, x) is of J-type 2 if and only if M^2 is the standard immersion of the flat torus $S^1(a) \times S^1(b)$ with $a \neq b$.*

4. An application. Now let J' be the Jacobi operator defined on the normal bundle of M^p in $S^{p+1}(1)$. If η' is a normal vector field to M^p in $S^{p+1}(1)$, then

$$J'\eta' = (\Delta^{D'} - \tilde{A}' + \tilde{R}')\eta', \tag{25}$$

where $D^{D'}$, $\tilde{A}'$, $\tilde{R}'$ are the operators defined through (1) to (3) relative to the normal bundle of M^p in $S^{p+1}(1)$.

By a straightforward computation, the following relation is obtained

$$JH' = J'H' + pH' + p\alpha'^2 x, \tag{26}$$

H' being the mean curvature vector of M^p in S^{p+1}. Therefore we have

Lemma 6. *Let $x : M^p \to S^{p+1}(1) \subset E^{p+2}$ be an isometric immersion of a compact Riemannian manifold in the unit sphere. Then the mean curvature vector field H' in M^p in S^{p+1}, is a Jacobi vector field for the Jacobi operator of the normal bundle of M^p in E^{p+2}, if and only if M^p is minimal in $S^{p+1}(1)$.*

As an application we obtain the following result:

PROPOSITION 7. *Let M^p be a closed hypersurface of the unit hypersphere $S^{p+1}(1)$. Suppose that it has constant mean and scalar curvatures and that M^p and $S^p(1)$ are isospectral with respect to J, then $M^p = S^p(1)$.*

Proof. Using Corollary 4, we see that M^p is either a small hypersphere or of J-type 2. Isospectrality implies in the first case that $M^p = S^p(1)$. Assume then that it is of J-type 2. Thus its position vector decomposes $x = x_r + x_s$ with $Jx_r = \mu_r x_r$; $Jx_s = \mu_s x_s$ and we assume that $\mu_r \leq \mu_s \leq 0$. Then, from (20), one has

$$\alpha'^2 = -\frac{b}{p} + \frac{c}{p} - 1; \; \|\sigma'\|^2 = -p - b, \tag{27}$$

with $b = \mu_r + \mu_s$ and $c = -\frac{\mu_r\mu_s}{p}$, and α', σ' being the mean curvature function and second fundamental form of M^p in $S^{p+1}(1)$ respectively. By using the asymptotic expansion of the heat kernel of J [7], we get in our case that

$$\frac{7}{6}\|\sigma'\|^2 - \frac{1}{6}\alpha'^2 \tag{28}$$

is a spectral invariant. In particular, since $S^p(1)$ is totally geodesic, we have from (28)

$$b = -p - \frac{c}{7p-1}, \tag{29}$$

but $-c \geq 0$ gives

$$b \geq -p. \tag{30}$$

Now, we wish to prove that $\mu_r \leq -p$. Using Proposition 1,

$$(x, Jx) = p \cdot (x, H) = p \int_M < x, H >= \mu_r\|x_r\|^2 + \mu_s\|x_s\|^2, \tag{31}$$

where $\|\|$ denotes the norm with respect to the inner product (4). Since $\int_M (1+ < x, H >) = 0$, one has from (31)

$$p \cdot \mathrm{Vol}(M) = -\mu_r\|x_r\|^2 - \mu_s\|x_s\|^2. \tag{32}$$

Therefore, combining (32) and $\|x_r\|^2 + \|x_s\|^2 = \|x\|^2 = \mathrm{Vol}(M)$, one obtains $\mu_r \leq -p \leq \mu_s \leq 0$. Hence

$$b = \mu_r + \mu_s \leq -p \tag{33}$$

From (30) and (33) we have

$$b = -p, \tag{34}$$

which together with (29) gives

$$c = 0. \tag{35}$$

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Consequently $\mu_r = -p$ and $\mu_s = 0$ which means that x_s is a Jacobi field for J. By Proposition 1, $H = -x_r$ and therefore, since $H = H' - x$, we have $H' = -x_s$. Thus H' is a Jacobi field for J and Lemma 6 gives $H' = 0$ (observe also that $H' = 0$ implies $x = x_r$ so that by Proposition 1, M^p would be minimal.) Then the result follows from Corollary 3.3 of [7].

Corollary 8. *No spherical hypersurface $M^p \subset S^{p+1}(1)$ of Chen-type 2 is isospectral (with respect to J) with $S^p(1)$.*

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