A stochastic ergodic theorem for superadditive processes

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Abstract. An elementary proof is given of Krengel's stochastic ergodic theorem in the setting of multiparameter superadditive processes.

1. *Introduction*

The stochastic ergodic theorem due to U. Krengel [4] asserts that if T is a linear contraction of the L_1 space of a probability space and if $f \in L_1$ then the Cesaro averages

$$n^{-1}(f + Tf + \cdots + T^{n-1}f)$$

converge in probability. Our purpose in this paper is to give an extension of this result to multiparameter superadditive processes (theorem (3.12)). That the existence of 'exact dominants' [1] implies the one-parameter extension to the superadditive case was shown by Fong [3]. The multiparameter additive case was also proved recently by Krengel [5]. Our proof is by a truncation argument already used by us earlier [1], depending only on certain properties of L_1 as a Banach lattice. The extension to Banach lattices is however less elementary, and we hope to present it in a separate article.

Note that the definition of superadditivity used in this article is the one given by Smythe [6].

An apparently new result of some independent interest is theorem 2.4, which gives the decomposition of the space into a 'positive' and a 'null' part for an arbitrary family of positive contractions on L_1.

2. *Basic Results*

Consider the real L_1 space of a measure space $(X, \mathcal{F}, \mu)$. Functional relations below are often to be understood modulo sets of measure zero. We will assume that μ is σ-finite. This is no loss of generality, as we are going to deal only with countable classes of functions. The characteristic function of a set $F \in \mathcal{F}$ is χ_F. The cone of non-negative functions in L_1 is denoted by L_1^+. By a contraction on L_1 we mean a linear operator $T: L_1 \to L_1$ with $\|T\| \le 1$. An operator $T: L_1 \to L_1$ is called positive if $TL_1^+ \subset L_1$.

Definition (2.1). A sequence f_n in L_1 is said to *converge stochastically* to $f \in L_1$ if

$$\lim_n \mu(A \cap \{x \,|\, |f_n(x) - f(x)| > \alpha\}) = 0$$

whenever $\alpha > 0$ and whenever $A \in \mathcal{F}$ has finite measure. If μ is a finite measure then this is, of course, just convergence in measure.

Although this is the usual definition of stochastic convergence (c.f. [5]), we will use a different formulation given in the following lemma.

LEMMA (2.2). *Let f_n be a sequence in L_1 and let $f \in L_1$. Then the following are equivalent*:

    (i) f_n *converges to f stochastically*;
    (ii) $\lim_n \||f_n - f| \wedge \phi\| = 0$ *whenever $\phi \in L_1^+$.*

Proof. Let $g_n = |f_n - f|$. Assume (i) and let $\phi \in L_1^+$, $\varepsilon > 0$ be given. Choose $A \in \mathcal{F}$ with $\mu(A) < \infty$ such that

$$\|\chi_{A^c} \phi\| < \varepsilon.$$

Choose $\delta > 0$ such that $\|\chi_E \phi\| < \varepsilon$ whenever $\mu(E) < \delta$. Choose n_0 such that $\mu(A \cap B_n) < \delta$ whenever $n \geq n_0$, where

$$B_n = \{x \,|\, g_n(x) > \varepsilon(\mu(A))^{-1}\}.$$

Then

$$\|g_n \wedge \phi\| \leq \|\chi_{A^c} \phi\| + \|\chi_{A \cap B_n} \phi\| + \|\chi_{A \cap B_n^c} g_n\| \leq 3\varepsilon \qquad \text{whenever } n \geq n_0.$$

Conversely, assume (ii) and let $A \in \mathcal{F}$, with $\mu(A) < \infty$, and $\alpha > 0$ be given. If

$$B_n = \{x \,|\, g_n(x) > \alpha\} \quad \text{and} \quad \phi = \alpha \chi_A$$

then

$$\alpha \mu(A \cap B_n) \leq \|g_n \wedge \phi\|$$

which shows that

$$\lim_n \mu(A \cap B_n) = 0. \qquad \qquad \square$$

If f_n is a sequence in L_1^+ dominated by a fixed $\phi \in L_1^+$, then f_n has a subsequence that converges weakly in L_1. If, in addition,

$$\lim \sup_n \|f_n\| > 0,$$

then this sequence can be chosen to have a non-zero weak limit. If f_n and ϕ_k are two arbitrary sequences in L_1^+, then f_n has a subsequence f_{n_i} such that

$$\text{w-}\lim_i f_{n_i} \wedge \phi_k = g_k$$

exists for each k, where w-lim denotes the weak limit in L_1. We omit the easy and familiar proofs of these facts.

THEOREM (2.3). *Let f_n be a sequence in L_1^+ such that*

$$\sup_n \|f_n\| = M < \infty.$$

Then either f_n converges stochastically to zero, or there is a non-zero $g \in L_1^+$ such that $Tg = g$, whenever $T: L_1 \to L_1$ is a positive contraction such that

$$\lim_n \|f_n - Tf_n\| = 0.$$

Furthermore, if the support of each f_n is contained in a set $E \in \mathscr{F}$, then the support of g is also contained in E.

Proof. Assume that f_n does not converge stochastically to zero. Let $\phi \in L_1^+$ be such that

$$\limsup_n \|f_n \wedge \phi\| > 0.$$

We will assume, without loss of generality, that $\phi > 0$ a.e. on X. By passing to a subsequence, if necessary, we may assume that

$$\text{w-}\lim_n f_n \wedge (k\phi) = g_k \quad \text{and} \quad \lim_n \|f_n - f_n \wedge (k\phi)\| = \alpha_k$$

exist for each integer $k \geq 1$, and that $g_1 \neq 0$. Then $\lim_k g_k = g$ exists a.e. and in L_1-norm, as g_k is a non-decreasing bounded sequence in L_1^+. Hence g is a non-zero function in L_1^+. It is also clear that if each f_n has support contained in E then the support of g is also contained in E. We will now show that if $T: L_1 \to L_1$ is a positive contraction and if

$$\lim_n \|f_n - Tf_n\| = 0,$$

then $Tg = g$. Let

$$r_n^k = f_n \wedge (k\phi), \qquad u_n^k = (Tf_n) \wedge (k\phi)$$

and

$$f_n = r_n^k + s_n^k, \qquad Tf_n = u_n^k + v_n^k.$$

Since $\|f_n - Tf_n\| \to 0$, the uniform integrability of $(r_n^k - u_n^k)$ implies that of $(s_n^k - v_n^k)$, hence we have

$$\|r_n^k - u_n^k\| \to 0, \quad \|s_n^k - v_n^k\| \to 0 \qquad \text{as } n \to \infty,$$

for each fixed k. Note that

$$\alpha_k = \lim_n \|s_n^k\| = \lim_n \|v_n^k\|$$

is non-increasing in k and

$$\alpha = \lim_k \alpha_k \geq 0$$

exists. Given $\varepsilon > 0$, find k such that

$$\alpha_k - \alpha < \varepsilon \quad \text{and} \quad \|g - g_k\| < \varepsilon.$$

Then find $l > k$ such that

$$T(k\phi) \leq l\phi + h$$

for some $h \in L_1^+$ with $\|h\| < \varepsilon$. Since $r_n^k \leq k\phi$, then

$$Tr_n^k \leq l\phi + h.$$

Now
$$Tf_n = Tr_n^k + Ts_n^k = u_n^l + v_n^l$$
shows that we can write
$$u_n^l = (Tr_n^k - h_n) + w_n \quad \text{and} \quad v_n^l = (Ts_n^k - w_n) + h_n$$
where
$$0 \le h_n = Tr_n^k - (u_n^l \wedge Tr_n^k) \le Tr_n^k - (l\phi \wedge Tr_n^k) \le h$$
and
$$0 \le w_n = Ts_n^k \wedge [u_n^l - (u_n^l \wedge Tr_n^k)] \le Ts_n^k.$$
Hence
$$\|v_n^l\| = \|Ts_n^k\| - \|w_n\| + \|h_n\| \to \alpha_l \qquad \text{as } n \to \infty,$$
which implies that
$$\limsup_n \|w_n\| = \limsup_n (\|Ts_n^k\| + \|h_n\|) - \alpha_l$$
$$\le \limsup_n (\|s_n^k\| + \|h_n\|) - \alpha_l$$
$$\le \alpha_k + \varepsilon - \alpha_l < 2\varepsilon.$$
Therefore
$$\limsup_n \|u_n^l - Tr_n^k\| \le \limsup_n (\|h_n\| + \|w_n\|) \le 3\varepsilon$$
which shows that $\|g_l - Tg_k\| < 3\varepsilon$ so that $\|g - Tg\| < 5\varepsilon$. $\square$

THEOREM (2.4). *Let* $\{T_i\}$, $i \in I$, *be an arbitrary family of positive* L_1 *contractions. Then there exists a unique (modulo sets of measure zero) partition of X into two sets P and N such that*

(i) *there is a* $g \in L_1^+$ *such that* $T_i g = g$ *for all* $i \in I$ *and such that P is the support of* g: $P = \{x \mid g(x) > 0\}$;

(ii) *if* f_n *is a bounded sequence in* L_1^+ *such that*
$$\lim_n \|f_n - T_i f_n\| = 0 \qquad \text{for each } i \in I$$
then $\chi_N f_n$ *converges stochastically to zero.*

Proof. We will obtain P as the largest set that supports an invariant function. Let $\mathcal{G}$ be the collection of all sets $G \in \mathcal{F}$ that can be obtained as
$$G = \{x \mid g(x) > 0\},$$
where $g \in L_1^+$ and $T_i g = g$ for all $i \in I$. Let ν be any finite measure on $(X, \mathcal{F})$, equivalent to μ, and let
$$\alpha = \sup\{\nu(G) \mid G \in \mathcal{G}\}.$$
Let G_n be a sequence in $\mathcal{G}$ such that $\nu(G_n) \to \alpha$. For each n, choose $g_n \in L_1^+$ such that
$$G_n = \{x \mid g_n(x) > 0\} \quad \text{and} \quad T_i g_n = g_n \qquad \text{for all } i \in I.$$
We may also assume that $\|g_n\| = 1$. Then
$$g = \sum_{n=1}^{\infty} (1/2^n) g_n$$

is another function in L_1^+ such that $T_i g = g$ for all $i \in I$ and such that $P = \{x \mid g(x) > 0\} \in \mathcal{G}$ satisfies $\nu(P) = \alpha$. Let $N = X - P$. Then (i) is satisfied.

To see that (ii) is also satisfied, let f_n be a bounded sequence L_1^+ such that

$$\lim_n \|f_n - T_i f_n\| = 0 \qquad \text{for all } i \in I.$$

Let $h_n = \chi_N f_n$ and let $T_i' : L_1 \to L_1$ be defined as

$$T_i' f = \chi_N T_i f, \qquad f \in L_1,$$

which is a positive contraction for each $i \in I$. We claim that

$$\|h_n - T_i' h_n\| \to 0 \qquad \text{as } n \to \infty.$$

First observe that

$$\chi_N T_i \chi_P f = T_i' \chi_P f = 0 \qquad \text{for any } f \in L_1,$$

which follows easily from the fact that

$$\|\chi_P f - (\chi_P f \wedge kg)\| \to 0 \qquad \text{as } k \to \infty$$

and that

$$T_i(\chi_P f \wedge kg) \le kg.$$

Hence

$$\|h_n - T_i' h_n\| = \|h_n - T_i' h_n - T_i' \chi_P f_n\| = \|\chi_N f_n - \chi_N T_i f_n\| \le \|f_n - T_i f_n\| \to 0$$

as $n \to \infty$. Therefore, if h_n does not converge to zero stochastically, then by the previous theorem there is a non-zero $h \in L_1^+$ such that $T_i' h = h$ for all $i \in I$. Since each h_n has support in N, the support of h is also in N. Now, if $\chi_P T_i h = r_i$, then

$$T_i h = \chi_N T_i h + \chi_P T_i h = T_i' h + r_i = h + r_i$$

implies that $r_i = 0$, since T_i is a contraction. Hence $T_i h = h$ for all $i \in I$. This a contradiction because now the support of $g + h$ also belongs to $\mathcal{G}$ and its ν-measure is strictly larger than the ν-measure of P.

To see the uniqueness, note that if $\phi \in L_1^+$ is a function such that $T_i \phi = \phi$ for all $i \in I$, then the support of ϕ is contained in any set P such that $N = X - P$ satisfies (ii). In fact, in this case the constant sequence $f_n = \phi$ satisfies the hypothesis in (ii) and consequently $\chi_N f_n = \chi_N \phi$ must converge to zero stochastically. $\qquad \square$

Definition (2.5). The sets P and N obtained in theorem 2.4 will be called, respectively, the *positive* and the *null* parts of the family $\{T_i\}$.

3. *Superadditive Processes*

Let V be the set consisting of K-dimensional vectors $v = (v_1, \ldots, v_K)$ with non-negative integer coordinates, and let $e = (1, \ldots, 1)$ be the vector with all coordinates equal to 1. If $u, v \in V$, then $[u, v)$ denotes the set

$$\{w \mid w \in V, \quad u_k \le w_k < v_k, \quad k = 1, \ldots, K\}$$

and $\pi(v)$ denotes the number of points in $[0, v)$. If $u, v \in V$, then uv will be defined as the vector with coordinates

$$(uv)_k = u_k v_k.$$

Finally, $\lim_v$ denotes the limit as $v_k \to \infty$ for each k, independently of each other.

For each $v \in V$, let T_v be a positive L_1 contraction and assume that $\{T_v\}$ is a semi-group:

$$T_u T_v = T_{u+v} \qquad \text{for all } u, v \in V.$$

If, in addition,

$$\int T_v f \, d\mu = \int f \, d\mu \qquad \text{for all } v \in V, \quad \text{for all } f \in L_1,$$

then $\{T_v\}$ will be called a *Markovian* semi-group. For $u, v \in V$, let

$$S_u^v = \sum_{w \in [0,u)} T_{wv}$$

and

$$
\begin{aligned}
A_u^v &= [\pi(u)]^{-1} S_u^v, &&\text{if } \pi(u) > 0 \\
A_u^v &= 0 &&\text{if } \pi(u) = 0.
\end{aligned}
$$

If $v = e$ then these are, respectively, the ordinary sums and the averages of the semi-group over $[0, u)$, and will be denoted as S_u and A_u. Observe that $A_v A_u^v = A_{uv}$.

Let P and N be the positive and null parts of the family $\{T_v\}$, as obtained in § 2. Let $L_1(P)$ be the set of L_1 functions with supports contained in P. Note that T_v maps $L_1(P)$ into $L_1(P)$ and the restriction of $\{T_v\}$ to $L_1(P)$ is a Markovian semi-group. We need the mean ergodic theorem in the following K-dimensional form.

THEOREM (3.1). *If $f \in L_1(P)$ then $\lim_v A_v f$ exists in the L_1-norm.*

This is a direct consequence of the corresponding 1-dimensional form, which, in turn, follows from the Kakutani-Yosida mean ergodic theorem.

Finally note that if $f \in L_1$, then $\lim_u \chi_N A_u f = 0$ stochastically, in the sense that

$$\lim_u \| |\chi_N A_u f| \wedge \phi \| = 0 \qquad \text{whenever } \phi \in L_1^+.$$

This follows from theorem (2.4), since

$$\lim_u \| T_v A_u f - A_u f \| = 0 \qquad \text{for all } v \in V.$$

Definition (3.2). A partition of a set $E \subset V$ is called an *admissible* partition if it is the common refinement of finitely many partitions of the form

$$(E \cap \{w \mid w_k < n\}, E \cap \{w \mid w_k > n\}),$$

where k and n are integers and $1 \leq k \leq K$.

Definition (3.3). A function $F: V \to L_1$ is called a *superadditive process* (with respect to $\{T_v\}$) if

$$F_v \geq \sum_{i=1}^{n} T_{u^i} F_{v^i - u^i}$$

whenever $\{[u^i, v^i)\}_{i=1}^{n}$ is an admissible partition of $[0, v)$. The function $f: V \to L_1$ defined as

$$
\begin{aligned}
f_v &= [\pi(v)]^{-1} F_v &&\text{if } \pi(v) > 0 \\
f_v &= 0 &&\text{if } \pi(v) = 0
\end{aligned}
$$

is called the average of F. If $\sup_v \|f_v\| < \infty$, then F is called a *bounded* superadditive process. If $F_v \in L_1^+$ for each $v \in V$, then F is called *positive*.

If both F and $-F$ are superadditive then F is called *additive*. Note that F is additive if and only if $F_v = S_v h$, where $h = F_e \in L_1$. An additive process is always bounded, and if $h \in L_1^+$ then $S_v h$ is also positive. Also, the sum of two superadditive processes is again a superadditive process.

LEMMA (3.4). *Any superadditive process is the difference of two positive superadditive processes. If the original process is bounded then both of these positive processes are also bounded.*

Proof. Since $\{[u, u+e)\}_{u \in [0,v)}$ is an admissible partition of $[0, v)$, we have that

$$F_v \geq S_v F_e.$$

Hence $F_v - S_v F_e$ is a positive superadditive process. Then both

$$F'_v = (F_v - S_v F_e) + S_v F_e^- \quad \text{and} \quad F''_v = S_v F_e^+$$

are positive superadditive processes and

$$F = F' - F''.$$

The boundedness is obvious. □

LEMMA (3.5). *Let f be the average of a positive bounded superadditive process with respect to a Markovian semi-group $\{T_v\}$ and let $\gamma = \sup_v \|f_v\|$. Let $0 < \delta < \gamma$, and u, $v \in V$ be such that*

$$\|f_v\| \geq \gamma - \delta \quad \text{and} \quad \lambda = \pi(v)[\pi(v+u)]^{-1} \geq 1 - (\delta/\gamma).$$

Then the L_1-distance between any two of the functions f_v, f_{v+u}, $T_u f_v$ is less than

$$\frac{4\delta\gamma}{\gamma - \delta}.$$

Proof. The set $[0, v+u)$ has in particular the following two admissible partitions: one containing $[0, v)$ as an atom and the other containing $[u, v+u)$ as an atom. Hence

$$0 \leq F_v \leq F_{v+u} \quad \text{and} \quad 0 \leq T_u F_v \leq F_{v+u},$$

because of the positivity of F. Therefore,

$$0 \leq \lambda f_v \leq f_{v+u} \quad \text{and} \quad 0 \leq \lambda T_u f_v \leq f_{v+u}.$$

But

$$\|\lambda f_v\| = \|\lambda T_u f_v\| \geq \lambda(\gamma - \delta) \geq \gamma - 2\delta \quad \text{and} \quad \|f_{v+u}\| \leq \gamma.$$

Hence $\|\lambda f_v - \lambda T_u f_v\| \leq 4\delta$ i.e.

$$\|f_v - T_u f_v\| \leq 4\delta/(1 - (\delta/\gamma)).$$

Also $\|f_{v+u} - \lambda f_v\| \leq 2\delta$ implies that

$$\|f_{v+u} - f_v\| \leq 2\delta + \|f_v - \lambda f_v\| \leq 2\delta + (\delta/\gamma)(\gamma - \delta) \leq 3\delta.$$

Similarly,

$$\|f_{v+n} - T_u f_v\| \leq \|f_{v+u} - \lambda T_u f_v\| + \|T_u f_v - \lambda T_u f_v\| \leq 2\delta + (\delta/\gamma)(\gamma - \delta) \leq 3\delta. \quad □$$

THEOREM (3.6). *Let f be the average of a positive bounded superadditive process with respect to a Markovian semi-group $\{T_v\}$. Then, for each $\varepsilon > 0$ there is a $z \in V$ such that*

$$\limsup_v \|A_v f_z - f_v\| < \varepsilon.$$

Proof. Let $\gamma = \sup_v \|f_v\| = 1$, and choose δ, $0 < \delta < 1$, such that $4\delta/(1-\delta) < \varepsilon/8$. Let $z \in V$ be such that $\pi(z) > 0$ and $\|f_z\| > 1 - \delta$. Finally choose $u^0 \in V$ such that

$$\pi(uz)[\pi(uz+w)]^{-1} > 1 - \delta \qquad \text{whenever } u \geq u^0, \quad w \in [0, z).$$

We claim that

$$\|A_v f_z - f_v\| < \varepsilon \qquad \text{whenever } v \geq u^0 z.$$

First note that any $v \geq u^0 z$ can be expressed as $v = uz + w$, where $u \geq u^0$ and $w \in [0, z)$. But

$$\|A_{uz+w} h - A_{uz} h\| \leq 2\delta \|h\| \qquad \text{for any } h \in L_1,$$

by a simple computation. Hence

$$\|A_{uz} f_z - A_{uz+w} f_z\| \leq 2\delta\gamma < \varepsilon/4.$$

To complete the proof it is now enough to show that

$$\|A_{uz} f_z - f_{uz+w}\| < 3\varepsilon/4$$

whenever $u \geq u^0$ and $w \in [0, z)$. Since $\{[rz, (r+e)z)\}_{r \in [0,u)}$ is an admissible partition of $[0, uz)$, we have that

$$0 \leq S_u^z F_z \leq F_{uz} \quad \text{or} \quad 0 \leq A_u^z f_z \leq f_{uz},$$

noticing that $\pi(uz) = \pi(u)\pi(z)$. Hence $1 - \delta < \|f_z\| = \|A_u^z f_z\| \leq \|f_{uz}\| \leq 1$ implies that

$$\|A_u^z f_z - f_{uz}\| < \delta < \varepsilon/8.$$

Since $\|f_{uz}\| > 1 - \delta$, we now apply lemma 3.5 to obtain

$$\|T_w f_{uz} - f_{uz}\| < \varepsilon/8.$$

Hence, together with

$$\|T_w A_u^z f_z - T_w f_{uz}\| < \varepsilon/8,$$

this gives

$$\|T_w A_u^z f_z - f_{uz}\| < \varepsilon/4.$$

Taking the average over $w \in [0, z)$, we then have

$$\|A_z A_u^z f_z - f_{uz}\| = \|A_{uz} f_z - f_{uz}\| < \varepsilon/4.$$

Then, again by lemma 3.5, we have that $\|f_{uz} - f_{uz+w}\| < \varepsilon/8$ and consequently,

$$\|A_{uz} f_z - f_{uz+w}\| < 3\varepsilon/8 < 3\varepsilon/4. \qquad \square$$

LEMMA (3.7). *Let F be a positive superadditive process with respect to $\{T_v\}$ and assume that P is the positive part of $\{T_v\}$. Then $\chi_P F$ is also a superadditive process.*

Proof. Let $\{[u^i, v^i)\}_{i=1}^n$ be an admissible partition of $[0, v)$. Then

$$\chi_P F_v \geq \chi_P \sum_{i=1}^n T_{u^i} F_{v^i - u^i} \geq \sum_{i=1}^n \chi_P T_{u^i} \chi_P F_{v^i - u^i} = \sum_1^n T_{u^i} \chi_P F_{v^i - u^i}.$$

Here the first inequality follows from the superadditivity of F, the second inequality from the positivity of F (and T_{u^i}) and the equality from the fact that $\chi_N T_u \chi_P h = 0$ for all $h \in L_1$ and $u \in V$. ☐

Remark. Call a set $A \in F$ an *absorbing* set for $\{T_v\}$, if

$$\chi_{X-A} T_v \chi_A h = 0 \qquad \text{for all } h \in L_1, \quad v \in V.$$

It is clear that the above proof shows that $\chi_A F$ is superadditive whenever A is absorbing and F is positive and superadditive.

Results close to the following were obtained by Derriennic and Krengel ([**2**, theorem 5.2]).

LEMMA (3.8). *Let f be the average of a positive and bounded superadditive process with respect to $\{T_v\}$ and let P be the positive part of $\{T_v\}$. Then $\lim_v \chi_P f_v$ exists in the L_1-norm.*

Proof. Let $f' = \chi_P f$ and let T'_v be the restriction of T_v to $L_1(P)$. Then $\{T'_v\}$ is a Markovian semi-group and f' is the average of a positive bounded process, superadditive with respect to this semi-group. Then theorem (3.6) shows that for each $\varepsilon > 0$ there is a $z \in V$ such that

$$\limsup_v \|A'_v f'_z - f'_v\| < \varepsilon,$$

where $A'_v f'_z$ are the averages with respect to T'_v, equal to $A_v f'_z$. Now $\lim_v A_v f'_z$ exists in the L_1 norm, for each $z \in V$, by the mean ergodic theorem (3.1), since $f'_z \in L_1(P)$. Hence $\lim_v f'_v$ must also exist in the L_1-norm. ☐

LEMMA (3.9). *Let f be the average of a positive superadditive process F with respect to $\{T_v\}$ and let N be the null part of $\{T_v\}$. Then $\lim_v \chi_N f_v = 0$ stochastically if at least one of the following conditions is satisfied:*

(3.10) $\{T_v\}$ *is a Markovian semi-group and F is a bounded process;*

(3.11) F *is dominated by an additive process; i.e. there is an $h \in L_1$ such that $F_v \le S_v h$ for all $v \in V$.*

Proof. If (3.10) is satisfied then for each $\varepsilon > 0$ there is a $z \in V$ such that

$$\limsup_v \|A_v f_z - f_v\| < \varepsilon.$$

But $\lim_v \chi_N A_v f_z = 0$ stochastically, i.e.

$$\lim_v \|(\chi_N A_v f_z) \wedge \phi\| = 0 \quad \text{for each } \phi \in L_1^+.$$

Hence we obtain

$$\limsup_v \|(\chi_N f_v) \wedge \phi\| < \varepsilon \qquad \text{for all } \phi \in L_1^+.$$

Therefore $\lim_v \chi_N f_v = 0$ stochastically. The proof under the assumption (3.11) is clear, since

$$0 \le \chi_N f_v \le \chi_N A_v h.$$
☐

The positivity assumption in lemmas (3.8) and (3.9) can be removed easily. First, since any bounded superadditive process is the difference of two positive and

bounded superadditive processes, it is clear that lemma (3.8) and a part of lemma (3.9), under the hypothesis (3.10), extend directly. To see that lemma (3.9) holds under the assumption (3.11), assume that $F_v \le S_v h$ with $h \in L_1$. Then

$$0 \le F_v - S_v F_e \le S_v(h - F_{\tilde{e}})$$

and lemma (3.9) applies to $F_v - S_v F_e$. But it also applies to $S_v F_e^+$ and $S_v F_e^-$ and consequently to $S_v F_e$. Collecting these results, we have the following theorem.

THEOREM (3.12). *Let f be the average of a superadditive process F with respect to a semi-group* $\{T_v\}$, $v \in V$, *of positive* L_1 *contractions. Let P be the positive part of* $\{T_v\}$. *Assume that at least one of the following two conditions is satisfied*:
    (3.10) $\{T_v\}$ *is a Markovian semi-group and F is a bounded process*;
    (3.11) *there is an* $h \in L_1$ *such that* $F_v \le S_v h$ *for all* $v \in V$.
Then $\lim_v f_v = \bar{f}$ *exists stochastically; i.e. there is an* $\bar{f} \in L_1$ *such that*

$$\lim_v \| \| f_v - \bar{f} | \Lambda \phi \| = 0 \qquad for \ all \ \phi \in L_1^+.$$

Furthermore, $\lim_v \chi_P f_v = \bar{f}$ *in the* L_1*-norm.*

There is an obvious extension of this result to additive processes with respect to semi-groups $\{\tilde{T}_v\}$ of general, not necessarily positive, L_1 contractions, assuming that $\{\tilde{T}_v\}$ is dominated by a semi-group $\{T_v\}$ of positive L_1 contractions, in the sense that

$$|\tilde{T}_v f| \le T_v |f| \qquad for \ all \ f \in L_1, \quad v \in V.$$

(Note that this is the case if the linear moduli of the generators of $\{\tilde{T}_v\}$ commute.)

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