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TOPOLOGICAL TRANSVERSALITY: APPLICATIONS TO DIFFERENTIAL EQUATIONS

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In this paper, existence results for both integro-differential and functional differential equations are discussed using topological transversality arguments. As applications, third and fourth order boundary value problems are considered. For third order problems, an example has been cited to show that our results cover a wider class of problems than Theorem 2.3 of D.J. O'Regan, Topological transversality: Applications to third order boundary value problems, SIAM J. Math. Anal. 18 (1987) 630–641.

1. Introduction

In this paper, we consider existence of solutions to the boundary value problems

$$\begin{aligned} &u^{(2)} = f\left(t,\, Su,\, u^{(1)}\right),\ t \in [0,\, 1] \\ &u \in B, \end{aligned} \tag{1.1}$$

where $f\colon [0,\, 1] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous, $\mathbb{B}$ is a suitable set of boundary conditions and S is one of the following forms:

(I) a Volterra integral operator of the type

$$[Su](t) = \int_c^t k(s,\, t)u(s)ds, \tag{1.2}$$

where $c \in [0,\, 1]$ and $k(s,\, t)$ is continuous for $0 \leqslant s \leqslant t \leqslant 1$;

(II) a Fredholm integral operator of the type

$$[Su](t) = \int_0^1 G(s,\, t,\, u(s))ds, \tag{1.3}$$

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where $G(s, t, r)$ is continuous for $0 \leqslant s,\ t \leqslant 1$ and $r \in \mathbb{R}$;

(III) a 'delay' operator of the type

$$[Su](t) = \begin{cases} u(g(t)), & \text{if } g(t) \in [0, 1] \\ \varphi_1(g(t)), & \text{if } g(t) \in [a, 0] \\ \varphi_2(g(t)), & \text{if } g(t) \in [1, b], \end{cases} \tag{1.4}$$

where $a \leqslant 0$, $b \geqslant 1$, $g\colon [0, 1] \to [a, b]$, $\varphi_1\colon [a, 0] \to \mathbb{R}$ and $\varphi_2\colon [1, b] \to \mathbb{R}$ are continuous functions with $\varphi_1(0) = \varphi_2(1) = 0$.

Equations of the type (1.1) with S as in (1.2) have recently been considered by Aftabizadeh and Leela [1] for Dirichlet boundary conditions and by Hu and Lakshmikanthan [6] for periodic boundary conditions, using monotone iterative techniques. Based on fixed point argument, Agarwal in [2] has discussed existence of solutions to the problem (1.1) with S as in (1.3).

Erbe and Krawcewicz [3] have considered the differential inclusions

$$u^{(2)} \in F\Big(t, u, S_1u, S_2u, \ldots, S_mu, u^{(1)}\Big)$$

with each S_i, $1 \leqslant i \leqslant m$, an operator of the type defined in equation (1.4). Boundary value problems for functional differential equations which are of the form (1.1) with S as in (1.4) are also discussed in [7], [8] and [9].

In the present paper, based on topological transversality arguments as in [5] we have derived existence results for the problem (1.1). Several applications of these results are also discussed. Finally, condition (a) of Theorem 3.1 generalises the condition (1) in Theorem 2.3 of [10, p.631], that is,

'there is a constant $M \geqslant 0$ such that $x_1 f(t, x_0, x_1, 0) > 0$, for $|x_1| > M$ and $(t, x_0) \in I \times \mathbb{R}$'.

Therefore, our results cover more classes of differential equations than Theorem 2.3 of O'Regan [10]. An example has been cited to illustrate this.

After presenting some preliminary material in Section 2, we prove our main results in Section 3. Section 4 deals with integro-differential equations. As applications, existence results for third and fourth order boundary value problems are discussed. In Section 5, existence of solutions to functional differential equations is examined. Finally in Section 6, an example of a non-linear boundary value problem satisfying the hypothesis of our main Theorem 3.1 is cited to compare our results with that of O'Regan [10].

2. Preliminaries, definitions and notations

Let $C^n[0, 1]$ denote the Banach space of n times continuously differentiable functions defined on $I = [0, 1]$ with the norm

$$\|u\|_n = \max\{\|u\|, \|u^{(1)}\|, \ldots, \|u^{(n)}\|\},$$

where $\|v\| = \sup_{t\in[0,1]} |v(t)|$, for $v \in C[0, 1]$.

Let $\mathbb{B}$ stand for various boundary conditions (see below) and further let $C^n_B[0, 1]$ denote the subset of $C^n[0, 1]$ consisting of functions satisfying the boundary conditions $\mathbb{B}$. $C_B[0, 1]$ is defined similarly.

Let E be a Banach space, $K \subseteq E$ be convex and $V \subseteq K$ be open in K. A compact map $T: \overline{V} \to K$ which is fixed point free on ∂V is said to be essential if every compact map $C: \overline{V} \to K$ with $C = T$ on ∂V, has a fixed point in V. For a fixed element u_0 in V, let $T(u) = u_0$, for all $u \in V$. Then, T is essential by an application of the Schauder fixed point theorem, [4, 5].

Let $C, T: \overline{V} \to K$ be compact maps. C and T are said to be homotopic if there is a compact map $H: \overline{V} \times [0, 1] \to K$ such that $H(\cdot, \lambda)$ is fixed point free on ∂V for each $\lambda \in [0, 1]$ and $H(\cdot, 0) = T$ with $H(\cdot, 1) = C$.

Now we state without proof the topological transversality theorem, for our subsequent use. For a proof, see Granas [4].

Theorem 2.1. *Let $C, T: \overline{V} \to K$ be compact homotopic maps without fixed points on ∂V. Then T is essential if and only if C is.*

3. Main results

Consider $\mathbb{B}$ as one of the following boundary conditions:

(i) $$u(0) = u(1) = 0;$$

(ii) $$u^{(1)}(0) = u^{(1)}(1) = 0.$$

Now we state the main result of this paper.

Theorem 3.1. *Let $\mathbb{B}$ denote either one of the boundary conditions (i) or (ii) and let Y be a subspace of $C[0, 1]$ containing $C_B[0, 1]$. Assume that $S: Y \to C[0, 1]$ is a continuous operator with*

$$\|Su\| \leqslant \alpha \|u\| + \beta, \tag{3.1}$$

for some non-negative constants α, β. Further assume that $f: I \times \mathbb{R}^3 \to \mathbb{R}$ is continuous and satisfies the following properties:

(a) *there is a constant $M \geqslant 0$ such that $x_1 f(t, x_0, x_1, 0) > 0$, for all (t, x_0, x_1) in the set $I \times \{(p, q) \in \mathbb{R}^2 : |p| \leqslant \alpha |q| + \beta$ and $|q| > M\}$*

and

(b) $|f(t, x_0, x_1, x_2)| \leqslant C(t, x_0, x_1)x_2^2 + D(t, x_0, x_1)$ *for all* $(t, x_0, x_1, x_2) \in I \times \mathbb{R}^3$, *where* $C(t, x_0, x_1)$ *and* $D(t, x_0, x_1)$ *are non-negative functions bounded on bounded subsets of* $I \times \mathbb{R}^2$.

Then the problem (1.1) has at least one solution.

First, we shall establish a *priori* estimates, independent of λ, for the solutions of the family of problems

$$(3.2_\lambda) \qquad u^{(2)} - u = \lambda\Big[f\Big(t,\, Su,\, u,\, u^{(1)}\Big) - u\Big],\ t \in [0,\, 1],$$

for $\lambda \in [0,\, 1]$. Below Lemma 3.2, the bounds for solutions u_λ are obtained, and the bounds for $u_\lambda^{(1)}$ are established in Lemma 3.3. The technique used follows closely that in [5].

Lemma 3.2. *Let f and S be as in Theorem 3.1. Then for any solution u_λ of (3.2_λ), $\|u\|_\lambda \leqslant M$.*

Proof: Since for $\lambda \in (0, 1)$, $f_\lambda \colon I \times \mathbb{R}^3 \to \mathbb{R}$ defined as $f_\lambda(t, x_0, x_1, x_2) = \lambda f(t, x_0, x_1, x_2) + (1-\lambda)x_1$ also satisfies the condition (a) of Theorem 3.1 with the same M, it is enough to show that for any solution u of (1.1), we have $\|u\| \leqslant M$. Let u be a solution to (1.1) and let $t_0 \in [0, 1]$ be a point where $|u|$ attains its maximum. Then

$$(3.3) \qquad |[Su](t_0)| \leqslant \|Su\| \leqslant \alpha\,\|u\| + \beta = \alpha\,|u(t_0)| + \beta.$$

Assume that $u(t_0) > M$. If (i) holds then $t_0 \in (0, 1)$. Thus, for (i) and (ii), $u^{(1)}(t_0) = 0$. Then

$$0 \geqslant u^{(2)}(t_0)u(t_0) = f(t_0,\, [Su](t_0),\, u(t_0),\, 0)u(t_0)$$

contradicting (a). The case $u(t_0) < M$ follow similarly. ∎

Lemma 3.3. *Let f and S be as in Theorem 3.1. Then there exists a constant $M_1 > 0$ such that for any solution u_λ of (3.2_λ), $\left\|u_\lambda^{(1)}\right\| \leqslant M_1$.*

Proof: Let $P = \sup\{C(t, x_0, x_1) \colon t \in I,\, |x_0| \leqslant \alpha M + \beta,\, |x_1| \leqslant M\}$ and $Q = \sup\{D(t, x_0, x_1) \colon t \in I,\, |x_0| \leqslant \alpha M + \beta,\, |x_1| \leqslant M\} + M$.

From Lemma 3.2, we have $\|u_\lambda\| \leqslant M$. Moreover,

$$|[Su_\lambda](t)| \leqslant \alpha\,\|u_\lambda\| + \beta \leqslant \alpha M + \beta.$$

Thus

$$(3.4) \qquad \begin{aligned} u_\lambda''(t) \leqslant |u_\lambda''(t)| &\leqslant \Big|f_\lambda\Big(t,\, [Su_\lambda](t),\, u_\lambda(t)u_\lambda^{(1)}(t)\Big)\Big| \\ &\leqslant \Big|\lambda f\Big(t,\, [Su_\lambda](t),\, u_\lambda(t),\, u_\lambda^{(1)}(t)\Big)\Big| + (1-\lambda)\,|u_\lambda(t)| \\ &\leqslant P\Big(u_\lambda^{(1)}(t)\Big)^2 + Q, \end{aligned}$$

where P and Q are independent of λ. From (3.4)

$$\frac{\left|2Pu_\lambda^{(1)}u_\lambda^{(2)}\right|}{P[u_\lambda^{(1)}]^2+Q} \leqslant \left|2Pu_\lambda^{(1)}\right|.$$

Now u_λ' vanishes at least once in $[0, 1]$. By a standard argument (see (5), for example), we obtain

$$|u_\lambda'(t)| \leqslant \left(\frac{Q}{P}\left(e^{4PM}-1\right)\right)^2.$$

$\square$

Proof of Theorem 3.1: Consider the family of problems (3.2_λ) for $\lambda \in [0, 1]$. From Lemmas 3.2 and 3.3, there exist constants M and M_1 independent of λ such that $\|u_\lambda\| \leqslant M$ and $\left\|u_\lambda^{(1)}\right\| \leqslant M_1$ for all $\lambda \in [0, 1]$. In addition, the continuity of f_λ on

$$I \times [-(\alpha M+\beta), (\alpha M+\beta)] \times [-M, M] \times [-M_1, M_1]$$

implies that there is $M_2 \geqslant 0$ such that $\left\|u^{(2)}\right\| \leqslant M_2$, for all $\lambda \in [0, 1]$.

Now let E denote the Banach space $C_B^2[0, 1]$. Define an operator $L\colon C_B^2[0, 1] \to C[0, 1]$ by $Lu = u^{(2)} - u$. It is easy to show that L has a bounded inverse L^{-1}.

Consider a family of maps $F_\lambda\colon C_B^1[0, 1] \to C[0, 1]$ defined by

$$F_\lambda u(t) = f_\lambda\Big(t, [Su](t), u(t), u^{(1)}(t)\Big)$$

and the completely continuous embedding $j\colon C_B^2[0, 1] \to C_B^1[0, 1]$.

Note that F_λ is well defined since $C_B^1[0, 1] \subseteq C_B[0, 1] \subseteq Y$. As in [5], set $H(u, \lambda) = L^{-1}F_\lambda ju$. This is a compact homotopy and is also fixed point free on the boundary of V, where $V = \{u \in E\colon \left\|u^{(2)}\right\| < 1 + \max(M, M_1, M_2)\}$. Since $H_0 = H(\cdot, 0)$ is the zero map, it is essential and so is H_1. In particular H_1 has a fixed point which is the required solution to the problem (1.1) and this completes the proof. $\square$

Corollary 3.4. *Let $\mathbb{B}$ and S be as in Theorem 3.1 and let us assume that $f\colon I \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function satisfying*

(a′) there ia a constant $M \geqslant 0$ such that $x_1 f(t, x_0, x_1, 0) \geqslant 0$ for all (t, x_0, x_1) in the set

$$I \times \{(p, q) \in \mathbb{R}^2 : |p| \leqslant \alpha|q| + \beta \text{ and } |q| > M\}$$

and (b) of Theorem 3.1. Then the problem (1.1) has at least one solution.

Proof: For $k \in \mathbb{N}$, define $f_k(t, x_0, x_1, x_2)$ as $f(t, x_0, x_1, x_2) + x_1/k$. Each f_k satisfies (a) of Theorem 3.1 and hence for every k, there is at least one solution

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$u_k \in C^2[0, 1]$ to the problem

$$u^{(2)} = f_k\left(t, Su, u, u^{(1)}\right), t \in [0, 1],$$
$$u \in \mathbb{B}.$$

As in the proof of Theorem 3.1, we now obtain a bound for u_k in $\|\cdot\|_2$-norm, namely $\|u_k\|_2 \leqslant N$, where N is independent of k. Continuity of f and S and the Ascoli-Arzela theorem imply the existence of a subsequence u_{k_j} which converges to a solution of the problem (1.1). This completes the proof of the theorem. ∎

REMARK 3.1. As in [5], Theorem 3.1 can be extended to cover the following boundary conditions:

(iii) $$u(0) = u(1),$$ $$u^{(1)}(0) = u^{(1)}(1),$$ $$-\mu u(0) + \xi u^{(1)}(0) = 0, \mu\xi > 0.$$

(iv) $au(1) + bu^{(1)}(1) = 0, a, b > 0.)$

(v) $u(0) = 0, au(1) + bu^{(1)}(1) = 0, a \geqslant 0, b > 0.$

(vi) $u(1) = 0, -\mu u(0) + \xi u^{(1)}(0) = 0, \mu \geqslant 0, \xi > 0.$

(vii) $u^{(1)}(0) = 0, au(1) + bu^{(1)}(1) = 0, a > 0, b \geqslant 0.$

(viii) $u^{(1)}(1) = 0, -\mu u(0) + \xi u^{(1)}(0) = 0, \mu > 0, \xi \geqslant 0.$

(ix) $u(0) = -u(1), u^{(1)}(0) = -u^{(1)}(1).$

The following Theorem can be easily proved by making minor modifications in the proof of Theorem 3.1.

THEOREM 3.5. *Let T and $\mathbb{B}$ be as in Theorem 3.1 and assume that $S: Y \to C[0, 1]$ is a continuous operator with*

$$\|Su\| \leqslant \alpha \|u\|^\delta + \beta, \tag{3.5}$$

where $\delta \geqslant 0$ is a constant. Further assume that $f: I \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function satisfying

(c) there is a constant $M \geqslant 0$ such that $x_1 f(t, x_0, x_1, 0) > 0$ for all (t, x_0, x_1) in the set

$$I \times \{(p, q) \in \mathbb{R}^2 : |p| \leqslant \alpha |q|^\delta + \beta \text{ and } |q| > M\}$$

and (b) of Theorem 3.1. Then the problem (1.1) has at least one solution.

Below we consider an integral type of monotonicity condition on the nonlinear function f and obtain existence of solution to problem (1.1).

Theorem 3.6. *Let Y and S be as in Theorem 3.1 and assume that $f\colon I\times\mathbb{R}^3 \to \mathbb{R}$ is a continuous function satisfying*

(d) there is a constant $M \geqslant 0$ such that

$$\int_0^t v^{(1)}(s)\Big[f\Big(s,\, Sv(s),\, v(s),\, v^{(1)}(s)\Big)\Big]ds > 0,\ 0 < t \leqslant 1,$$

for $|v(t)| > M$, $v \in C^1[0, 1]$ with $v(0) = 0$, and condition (b) of Theorem 3.1. Then the problem (1.1) with $\mathbb{B}$ as in (i) has at least one solution.

Proof: It is enough to show that for any solution u of (1.1) with $\mathbb{B}$ as in (i), the estimate $|u(t)| \leqslant M$ for $t \in [0, 1]$ holds. By (i), $|u|$ attains its maximum at some $t_0 \in (0, 1)$. Then $u^{(1)}(t_0) = 0$. Suppose that $|u(t_0)| > M$, now the condition (d) implies

$$\int_0^{t_0} u^{(1)}(s)u^{(2)}(s)ds > 0.$$

So we obtain

$$-[u^{(1)}(0)]^2 > 0$$

which is a contradiction, and hence $|u(t_0)| \leqslant M$. The rest of the proof follows from Theorem 3.1. ∎

Remark 3.2. As in Corollary 3.4, we may replace the condition (d) in the previous Theorem by

(d') There is a constant $M > 0$ such that

$$\int_0^t v^{(1)}(s)\Big[f\Big(s,\, Sv(s),\, v(s),\, v^{(1)}(s)\Big)\Big]ds \geqslant 0,\ 0 < t \leqslant 1$$

for $|v(t)| > M$, $v \in C^1[0, 1]$ with $v(0) = 0$.

4. Integro-differential equations

In this section, we consider equations of the type

$$u^{(2)}(t) = f\Big(t,\, \int_0^t k(s,\, t)u(s)ds,\, u(t),\, u^{(1)}(t)\Big),\ t \in [0, 1] \qquad (4.1)$$
$$u \in \mathbb{B}$$

and

$$u^{(2)}(t) = f\Big(t,\, \int_0^1 G(s,\, t,\, u(s))ds,\, u(t),\, u^{(1)}(t)\Big),\ t \in [0, 1] \qquad (4.2)$$
$$u \in \mathbb{B},$$

where B stands for the boundary conditions (i) or (ii).

Theorem 4.1. *Let $k\colon I \times I \to \mathbb{R}$ be continuous. Let $G\colon I \times I \times \mathbb{R} \to \mathbb{R}$ be continuous such that for some $m, q \geqslant 0$,*

$$|G(s, t, p)| \leqslant m\,|p| + q \tag{4.3}$$

for all $p \in \mathbb{R}$ and $s, t \in I$. Then,

(A) *the equation (4.1) has a solution if f satisfies (a) and (b) of Theorem 3.1 with $\alpha \geqslant \sup_{s,t\in I} |k(s, t)|$ and $\beta \geqslant 0$,*

(B) *the equation (4.2) has a solution if f satisfies (a) and (b) of Theorem 3.1 with $\alpha \geqslant m$ and $\beta \geqslant q$.*

Proof: Consider the problem (4.1). Let Y be $C[0, 1]$ and let $S\colon Y \to C[0, 1]$ be an integral operator defined by (1.2). Then S satisfies (3.1) with α and β as in (A). The proof of (A) is completed by the application of Theorem 3.1.

Consider the boundary value problem (4.2). With $Y = C[0, 1]$, let $S\colon Y \to C[0, 1]$ be an integral operator, which is defined by (1.3). Then from (4.3), S is continuous and satisfies (3.1) with α and β as in (B). An application of Theorem 3.1 now completes the rest of the proof. ∎

As an application of Theorem 4.1, we have the following existence results for third and fourth order boundary value problems:

$$y^{(3)} = f\Big(t, y, t^{(1)}, y^{(2)}\Big), t \in [0, 1], y \in \mathbb{B}, \tag{4.4}$$

where $\mathbb{B}$ stands for one of the boundary conditions

(x) $\quad y(c) = 0, \qquad y^{(1)}(0) = 0, \qquad y^{(1)}(1) = 0$

(xi) $\quad y(c) = 0, \qquad y^{(2)}(0) = 0, \qquad y^{(2)}(1) = 0$

with $c \in [0, 1]$, and

$$y^{(4)} = f\Big(t, y, y^{(2)}, y^{(3)}\Big), \quad t \in [0, 1], \quad y \in \mathbb{B}, \tag{4.5}$$

where $\mathbb{B}$ stands for one of the boundary conditions

(xii) $\quad y(0) = y(1) = y^{(1)}(0) = y^{(1)}(1) = 0$

(xiii) $\quad y(0) = y(1) = y^{(2)}(0) = y^{(2)}(1) = 0.$

Corollary 4.2. *Let f satisfy (a) and (b) of Theorem 3.1 with $\alpha = 0$ and $\beta = 0$. Then there exists at least one solution for each of the boundary value problems (4.4) and (4.5).*

Proof: Consider the problem (4.4) with $\mathbb{B}$ as in (x). We first observe that it is sufficient to show the existence of a solution $u \in C^2[0, 1]$ to (4.1) with $k(s, t) = 1$.

Indeed, if u is a solution of (4.1), with $\mathbb{B}$ as in (i), then $y(t) = \int_0^t u(s)ds$ is a solution of the problem (4.4) with $\mathbb{B}$ as in (x). Similarly the problem (4.4) with $\mathbb{B}$ as in (xi) is reduced to the problem (4.4) with $\mathbb{B}$ as in (ii). Hence by putting $k(s,t) \equiv 1$, the proof is completed by an application of (A) of Theorem 3.1.

For the problem (4.5), define $\widetilde{G}: I \times I \to \mathbb{R}$ as

$$\widetilde{G}(s,t) = \begin{cases} s(1-t), & s \leqslant t \\ t(1-s), & s > t \end{cases} \tag{4.6}$$

and set $\widetilde{G}(s,t,r) = G(s,t)r$, for all $r \in R$. Now we have $m = 1/4$ and $q = 0$ in (4.3). Consider the boundary condition (xii). The transformation $u(t) = y^{(2)}(t)$ reduces the problem (4.5) to that of (4.2). Indeed, if $u \in C^2[0,1]$ is a solution to the problem (4.2) with $\mathbb{B}$ as in (i), then $y \in C^4[0,1]$ defined as $y(t) = \int_0^1 G(s,t,u(s))ds$ is a solution to the problem (4.5) with $\mathbb{B}$ as in (xii). Similarly the problem (4.4) with $\mathbb{B}$ as in (xiii) is reduced to the problem (4.2) with $\mathbb{B}$ as in (ii). Hence an application of (B) of Theorem 3.1 now completes the rest of the proof. $\square$

5. Functional differential equations

In this section, we consider the following functional differential equation:

$$\begin{aligned} &u^{(2)}(t) = f\Big(t, u(g(t)), u(t), u^{(1)}(t)\Big), && t \in [0,1], \\ &u(t) = \varphi_1(t), && t \in [a,0], \\ &u(t) = \varphi_2(t), && t \in [1,b] \end{aligned} \tag{5.1}$$

where $g, \varphi_1, \varphi_2, a$ and b are as in (III) of Section 1. We say that $u \in C[a,b]$ is a solution of (5.1) if $u|_{[0,1]} \in C^2[0,1]$ and u satisfies equation (5.1).

Theorem 5.1. *Let $q = \max\Big(\sup_{t\in[a,0]} |\varphi_1(t)|, \sup_{t\in[1,b]} |\varphi_2(t)|\Big)$. Further, let $f: I \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function satisfying (a) and (b) of Theorem 3.1 with $\alpha \geqslant 1$ and $\beta \geqslant q$. Then the problem (5.1) has a solution.*

Proof: Let $\mathbb{B}$ stand for the boundary condition (i). For $u \in C_B[0,1]$, we define $Su: I \to \mathbb{R}$ as in (1.4). Su is continuous since g, φ_1 and φ_2 are continuous and $\varphi_1(0) = \varphi_2(1) = 0$. Thus we have an operator $S: C_B[0,1] \to C[0,1]$. We now claim that S is a continuous affine map, and hence satisfies (3.1).

Let $\theta \in C_B[0,1]$ be the zero function. Then $Su - S\theta$ is given by

$$[Su - S\theta](t) = \begin{cases} u(g(t)), & g(t) \in [0,1] \\ 0, & g(t) \in [a,0] \cup [1,b]. \end{cases}$$

Defining $T\colon C_B[0, 1] \to C[0, 1]$ by $Tu = Su - S\theta$, it is easy to verify that T is linear. Now

$$\|Tu\| = \sup_{t\in[0,1]} |Tu(t)| = \sup_{g(t)\in[0,1]} |u(g(t))| \leqslant \sup_{s\in[0,1]} |u(s)| = \|u\|.$$

Thus, T is continuous and so is S. Further, S satisfies (3.1) with $\alpha \geqslant 1$ and $\beta \geqslant q$. Now an application of Theorem 3.1 yields the existence of $w \in C^2[0, 1]$ satisfying

$$w^{(2)}(t) = f\Big((t, [Sw])(t), w(t), w^{(1)}(t)\Big), \ t \in [0, 1]$$
$$w(0) = w(1) = 0.$$

Setting $u \in C[a, b]$ as

$$u(s) = \begin{cases} \varphi_1(s), & s \in [a, 0], \\ w(s), & s \in [0, 1], \\ \varphi_2(s), & s \in [1, b], \end{cases}$$

u satisfies (5.1) and this completes the proof. ∎

6. Examples and comparisions

First we compare our results for the problem (4.4) with a result of O'Regan [**10**].

(I) Consider the following differential equation:

$$(6.1) \qquad \begin{aligned} &y^{(3)} = 1 + y^2 + \left(y^{(1)}\right)^3 + \left(y^{(2)}\right)^2, \\ &y \in \mathbb{B}, \end{aligned}$$

where $\mathbb{B}$ stands for the boundary conditions (viii), (ix) or

$$(\text{xiv}) \qquad \begin{aligned} &dy(c) + ey^{(1)}(c) = 0, \ d \neq 0, \\ &-\mu y^{(1)}(0) + \xi y^{(2)}(0) = 0, \ \mu, \xi > 0, \\ &ay^{(1)}(1) + by^{(2)}(1) = 0, \ a, b > 0. \end{aligned}$$

Theorem 6.2. ([**10**]). *Let $f\colon I \times \mathbb{R}^3 \to \mathbb{R}$ be continuous. Suppose that*

(e) *there is a constant $M \geqslant 0$ such that $x_1 f(t, x_0, x_1, 0) > 0$ for $|x_1| > M$ and $(t, x_0) \in I \times \mathbb{R}$ and*

(f) $|f(t, x_0, x_1, x_2)| \leqslant C(t, x_0, x_1)x_2^2 + D(t, x_0, x_1)(t, x_0, x_1, x_2) \in I \times \mathbb{R}^3$,

where $C(t, x_0, x_1)$ and $D(t, x_0, x_1)$ are non-negative functions bounded on bounded subsets of $I \times \mathbb{R}$.

Then the problem (4.4), where $\mathbb{B}$ stands for either of the boundary conditions (x), (xi) or (xiv), has at least one solution.

Clearly (e) of the above theorem implies (a) of Theorem 3.1. Since the function f defined as $f(t, x_0, x_1, x_2) = 1 + x_0^2 + x_1^3 + x_2^2$ does not satisfy (e). Theorem 6.2 does not guarantee that the problem (6.1) has a solution. But below we shall now show that the problem (6.1) has at least one solution using Corollary 4.2 and Remark 3.1.

Consider $x_1 f(t, x_0, x_1, 0) = x_1 + x_1 x_0^2 + x_1^4$. Let $\alpha, \beta \geqslant 0$ be given. Suppose $|x_0| \leqslant \alpha |x_1| + \beta$. We have

$$\left|x_1 + x_1 x_0^2\right| \leqslant |x_1| + |x_1| (\alpha |x_1| + \beta)^2 = \phi(|x_1|),$$

where ϕ is a polynomial of degree at most 3.

Let $M = \max\{3\alpha^2, (6\alpha\beta)^{1/2}, [3(\beta^2+1)]^{1/2}\}$. Now for $|x_1| > M$,

$$x_1^4 > \phi(|x_1|) \geqslant \left|x_1 + x_1 x_0^2\right|. \tag{6.1}$$

If $x_1 > 0$, then trivially $x_1 + x_1 x_0^2 + x_1^4 > 0$. For $x_1 < 0$, $x_1 + x_1 x_0^2 \leqslant 0$ and by (6.1) for $|x_1| > M$, $x_1 + x_1 x_0^2 + x_1^4 > 0$.

To verify the condition (b) in Theorem 3.1, take $C \equiv 1$ and $D(t, x_0, x_1) = 1 + x_0^2 + |x_1|^3$. Therefore, Corollary 4.2 guarantees that the problem 6.1 has at least one solution for each of the boundary condition (x) and (xii).

For the boundary condition (xiv) we proceed as follows. Define the operator $S \colon Y = C[0, 1] \to C[0, 1]$ as

$$[Su(t)] = eu(c)/d + \int_c^t u(s)ds.$$

S is a continuous linear operator which satisfies (3.1) of Theorem 3.1 with $\alpha \geqslant 1 + |e/d|$ and $\beta \geqslant 0$. We now apply the extension of Theorem 3.1 as indicated in Remark 3.1, and obtain existence of a solution u to the problem (1.1) with $\mathbb{B}$ as in (iv). Defining $y(t) = [Su](t)$, it is easy to see that y is a solution to the problem (6.1) with $\mathbb{B}$ given by (xiv).

(II) Consider the following non-linear integro-differential equation:

$$u^{(2)}(t) = 1 + \left(\int_0^t [u(s)]^k ds\right)^p + [u(t)]^m + \left(u^{(1)}(t)\right)^2, \qquad u \in \mathbb{B}, \tag{6.2}$$

where $\mathbb{B}$ stands for the boundary conditions (i) or (ii) and k, p, m are positive in tegers such $kp < m$ and m is odd.

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The operator $S\colon C[0, 1] \to C[0, 1]$ defined as

$$[Su](t) = \int_0^t [u(s)]^k ds$$

satisfies the condition (3.5) fo Theorem 3.5 for $\alpha = 1$, $\beta = 0$ and $\delta = k$. As in the previous example, we can show that the function defined as $f(t, x_0, x_1, x_2) = 1 + x_0^p + x_1^m + x_2^2$ satisfies (c) of Theorem 3.5 and (b) of Theorem 3.1. Hence by Theorem 3.5, the problem (6.2) has at least one solution.

(III) Consider the boundary value problem:

$$y^{(3)} = \phi[y^{(1)}(t)] + \frac{y^{(2)}(t)[y(t)]^2}{1 + |y^{(2)}(t)|} + y^{(2)}(t)|y^{(2)}|, \qquad (6.3)$$
$$y \in \mathbb{B}$$

where $\mathbb{B}$ stands for the boundary condition (viii) of Section 4 with $c = 0$ and $\phi\colon \mathbb{R} \to \mathbb{R}$ is defined as

$$\phi(r) = \begin{cases} -\sin|r|, & \text{if } r \in (-\infty, \pi), \\ r - \pi, & \text{if } r \in (\pi, \infty). \end{cases}$$

As in Corollary 4.2, it is enough to consider the existence of solution to the following second order equation:

$$u^{(2)} = \phi[u(t)] + \frac{u^{(1)}(t)[Su(t)]^2}{1 + |u^{(1)}(t)|} + u^{(1)}(t)|u^{(1)}(t)|, u \in \mathbb{B}, \qquad (6.4)$$

where $\mathbb{B}$ stands for the boundary condition (i) and Su is defined as $[Su](t) = \int_0^t u(s)ds$.

Set $f(t, x_0, x_1, x_2) = \phi(x_1) + (x_2 x_0^2)/(1 + |x_2|) + x_2|x_2|$. For $v \in C^1[0, 1]$ with $v(0) = 0$, we have,

$$\begin{aligned} &\int_0^t v^{(1)}(s)\Big[f\Big(s, Sv(s), v(s), v^{(1)}(s)\Big)\Big]ds \\ &\quad = \int_0^t \Big\{\phi[v(s)]v'(s) + \frac{[v'(s)]^2[Sv(s)]^2}{1 + |v'(s)|} + [v'(s)]^2|v'(s)|\Big\}ds \\ &\quad \geqslant \int_{v(0)}^{v(t)} \phi(s)ds = \int_0^{v(t)} \phi(s)ds. \end{aligned}$$

Let $v(t) \leqslant 0$. Then

$$\int_0^{v(t)} \phi(s)ds = -\int_{v(t)}^0 -[\sin(-s)]ds = 1 - \cos[v(t)] \geqslant 0. \qquad (6.5)$$

For $v(t) \geqslant \pi$,

$$\int_0^{v(t)} \phi(s)ds = \int_0^{\pi} \phi(s)ds + \int_{\pi}^{v(t)} \phi(s)ds$$
$$= [v(t)]^2/2 - \pi v(t) + \pi^2/2 - 2.$$

Thus, for $v(t) \geqslant \pi + 2$

$$\int_0^{v(t)} \phi(s)ds \geqslant 0. \tag{6.6}$$

From (6.5) and (6.6) (d$'$) of Remark (3.2) is satisfied for $M = \pi + 2$. Now by taking $C(t, x_0, x_1) \equiv \pi + |x_1| + x_0^2$ and $D(t, x_0, x_1) \equiv 1$, the condition (b) of Theorem 3.1 is also satisfied. Therefore by Remark 3.2, the problem (6.4), and hence the problem (6.3), has at least one solution.

REMARK 6.3. For the problem (6.3), neither (a) of Theorem 3.1 of this paper nor the integral type of monotonicity condition in Theorem 3.2 of [**10**] is applicable. Therefore, neither Theorem 3.1 of this paper nor Theorem 3.2 of [**10**] covers this example.

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