AN EXPANSION IN TERMS OF ASSOCIATED LEGENDRE FUNCTIONS

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§ 1. *Introductory.* If the right-hand side of the expansion

$$(1-2\mu h+h^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty} h^n P_n(\mu) \qquad \text{.....................................(1)}$$

is integrated m times from 1 to μ, it becomes

$$(\mu^2-1)^{\frac{1}{2}m}\sum_{n=0}^{\infty} h^n P_n^{-m}(\mu).$$

In Hobson's treatise on *Spherical and Ellipsoidal Harmonics*, page 105, it is stated that the corresponding integration of the left-hand side gives rise to the function

$$\frac{\Gamma(\frac{1}{2}-m)}{2^m\,\Gamma(\frac{1}{2})}(1-2\mu h+h^2)^{m-\frac{1}{2}}\,h^{-m},$$

together with a rational expression which involves only powers of h. If, however, the integration is carried out step by step it is seen that, after the first step, the rational expression involves powers of μ also and that it is of the form

$$\frac{\Gamma(\frac{1}{2}-m)}{2^m\,\Gamma(\frac{1}{2})}h^{-m}\sum_{n=0}^{2m-1}(-1)^n\,{}^{2m-1}C_n\,h^n f_n^m(\mu),$$

where, for $n=0, 1, 2, \ldots, m-1$, $f_n^m(\mu)$ is a polynomial in μ of degree n; and, for

$n=m, m+1, \ldots, 2m-1,$ $\qquad f_n^m(\mu)=f_{2m-1-n}^m(\mu).$

When $\mu=1$, $f_n^m(\mu)$ takes the value 1.

These results will be established in § 3, and it will be shown that the polynomials $f_n^m(\mu)$ can be expressed in terms of Associated Legendre Functions of the Second Kind. In § 2 some formulae, old and new, on which the proof is based, are given.

§ 2. *Associated Legendre Function Formulae.* If m is not a positive integer and if h is sufficiently small, then *

$$(1-2\mu h+h^2)^{m-\frac{1}{2}}=\frac{(\mu^2-1)^{\frac{1}{2}m}\,\Gamma(\frac{1}{2})}{2^{-m}\,\Gamma(\frac{1}{2}-m)}\sum_{n=0}^{\infty} h^n\frac{\Gamma(n-2m+1)}{n!}P_{n-m}^m(\mu), \qquad \text{................(2)}$$

where, if n is zero or a positive interger and m is not integral,

$$P_{n-m}^m(\mu)=\frac{2^{n-m}\,\Gamma(n-m+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(n-2m+1)}(\mu^2-1)^{-\frac{1}{2}m}\mu^n$$
$$\times F(-\tfrac{1}{2}n, \tfrac{1}{2}-\tfrac{1}{2}n;\ \tfrac{1}{2}+m-n;\ \mu^{-2}). \qquad \text{..............................(3)}$$

Now the right-hand side of (3) is equal to

$$\frac{\sin 2m\pi}{\pi\cos m\pi}\frac{\Gamma(\frac{1}{2})\,\Gamma(2m-n)}{2^{m-n}\,\Gamma(m-n+\frac{1}{2})}\frac{(\mu^2-1)^{\frac{1}{2}m}}{\mu^{2m-n}}F\left(\begin{matrix}\frac{2m-n}{2}, & \frac{2m-n+1}{2}; & \frac{1}{\mu^2}\\ \frac{1}{2}+m-n & & \end{matrix}\right).$$

* *Quart. J. of Math.* (Oxford) XIV. (1943), 1, 2.

Therefore, if n is zero or a positive integer and m is not integral,

$$P_{n-m}^{m}(\mu) = \frac{2 \sin m\pi}{\pi} Q_{m-n-1}^{m}(\mu). \quad \text{.....(4)}$$

The formula

$$P_n^m(\mu) = \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)} P_n^{-m}(\mu) + 2\frac{\sin m\pi}{\pi} Q_n^m(\mu) \quad \text{.....(5)}$$

will also be required.

§ 3. *Proof of the Expansion.* On replacing n by $n-m$ in (5), that formula becomes

$$\Gamma(n-2m+1)\, P_{n-m}^{m}(\mu) = \Gamma(n+1)\, P_{n-m}^{-m}(\mu) + 2\frac{\sin m\pi}{\pi}\Gamma(n-2m+1)\, Q_{n-m}^{m}(\mu). \quad \text{......(6)}$$

Hence, if m tends to a positive integral value and $n \geqq 2m$,

$$\Gamma(n-2m+1)\, P_{n-m}^{m}(\mu) \to n!\, P_{n-m}^{-m}(\mu), \quad \text{.....(7)}$$

a well-known formula.

Again, from (6),

$$\Gamma(n-2m+1)\, P_{n-m}^{m}(\mu) = n!\, P_{n-m}^{-m}(\mu) + (-1)^n \frac{2\sin m\pi}{\sin 2m\pi}\frac{1}{\Gamma(2m-n)} Q_{n-m}^{m}(\mu),$$

and therefore, when m tends to a positive integral value and $n = m, m+1, \ldots, 2m-1$,

$$\Gamma(n-2m+1)\, P_{n-m}^{m}(\mu) \to n!\, P_{n-m}^{-m}(\mu) + (-1)^{m+n}\frac{1}{\Gamma(2m-n)} Q_{n-m}^{m}(\mu). \quad \text{.....(8)}$$

Next, from (4),

$$\Gamma(n-2m+1)\, P_{n-m}^{m}(\mu) = (-1)^n \frac{2\sin m\pi}{\sin 2m\pi}\frac{1}{\Gamma(2m-n)} Q_{m-n-1}^{m}(\mu),$$

and therefore, when m tends to a positive integral value and $n = 0$ 1, 2, ..., $m-1$,

$$\Gamma(n-2m+1)\, P_{n-m}^{m}(\mu) \to (-1)^{m+n}\frac{1}{\Gamma(2m-n)} Q_{m-n-1}^{m}(\mu). \quad \text{.....(9)}$$

Thus, from (9), (8) and (7), when m tends to a positive integral value, (2) becomes

$$(1-2\mu h+h^2)^{m-\frac{1}{2}} = \frac{(\mu^2-1)^{\frac{1}{2}m}\,\Gamma(\frac{1}{2})}{2^{-m}\,\Gamma(\frac{1}{2}-m)} \times \left\{ \begin{array}{l} \sum\limits_{n=0}^{m-1}(-1)^{m+n}h^n \dfrac{1}{n!\,\Gamma(2m-n)} Q_{m-n-1}^{m}(\mu) \\ + \sum\limits_{n=m}^{2m-1}(-1)^{m+n}h^n \dfrac{1}{n!\,\Gamma(2m-n)} Q_{n-m}^{m}(\mu) + \sum\limits_{n=m}^{\infty} h^n P_{n-m}^{-m}(\mu) \end{array} \right\}.$$

Thus, finally,

$$\begin{aligned} \frac{\Gamma(\frac{1}{2}-m)}{2^m\,\Gamma(\frac{1}{2})}(1-2\mu h+h^2)^{m-\frac{1}{2}} &= (\mu^2-1)^{\frac{1}{2}m}\sum_{n=0}^{\infty} h^{n+m} P_n^{-m}(\mu) \\ &+ \sum_{n=0}^{m-1}(-1)^{m+n}h^n \frac{(\mu^2-1)^{\frac{1}{2}m}}{n!\,(2m-1-n)!} Q_{m-n-1}^{m}(\mu) \\ &+ \sum_{n=m}^{2m-1}(-1)^{m+n}h^n \frac{(\mu^2-1)^{\frac{1}{2}m}}{(2m-1-n)!\,n!} Q_{n-m}^{m}(\mu). \end{aligned} \quad \text{.....(10)}$$

On replacing n by $2m-1-n$ in the last line it is seen that the coefficient of h^{2m-1-n} is equal to minus the coefficient of h^n in the second last line. But

$$f_n^m(\mu) = \frac{2^m\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-m)}\frac{n!\,(2m-1-n)!}{(2m-1)!} \times \text{coefficient of } (-h)^n$$

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in these two lines. Therefore

$$f_{2m-1-n}^{m}(\mu) = f_n^m(\mu),$$

where $n=0, 1, 2, \ldots, m-1$.

Again, if $n=0, 1, 2, \ldots, m-1$,

$$f_n^m(\mu) = \frac{2^m\,\Gamma(\frac{1}{2}+m)}{\Gamma(\frac{1}{2})\Gamma(2m)}\,\frac{(\mu^2-1)^m}{\mu^{2m-n}}\,\frac{\Gamma(\frac{1}{2})\Gamma(2m-n)}{2^{m-n}\,\Gamma(m-n+\frac{1}{2})}$$
$$\times F\left(\frac{2m-n}{2},\ \frac{2m-n+1}{2};\ m-n+\tfrac{1}{2};\ \frac{1}{\mu^2}\right)$$
$$= \frac{2^n\,\Gamma(\frac{1}{2}+m)\,\Gamma(2m-n)}{\Gamma(2m)\,\Gamma(m-n+\frac{1}{2})}\,\mu^n\,F\left(\begin{matrix}-\frac{1}{2}n,\ \frac{1}{2}-\frac{1}{2}n\\ m-n+\frac{1}{2}\end{matrix};\ \frac{1}{\mu^2}\right).$$

From this it is clear that $f_n^m(\mu)$ is a polynomial in μ of degree n.

Lastly, let $\mu \to 1$ and apply Gauss's Theorem ; then

$$f_n^m(1) = \frac{2^n\,\Gamma(\frac{1}{2}+m)\,\Gamma(2m-n)}{\Gamma(2m)\,\Gamma(m-n+\frac{1}{2})}\,\frac{\Gamma(m-n+\frac{1}{2})\Gamma(m)}{\Gamma(m-\frac{1}{2}n+\frac{1}{2})\,\Gamma(m-\frac{1}{2}n)} = 1.$$

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