FACIAL STRUCTURES FOR THE POSITIVE LINEAR MAPS BETWEEN MATRIX ALGEBRAS

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ABSTRACT. Let $\mathcal{P}$ denote the convex set of all positive linear maps from the matrix algebra $M_n(\mathbb{C})$ into itself. We construct a join homomorphism from the complete lattice $\mathcal{F}(\mathcal{P})$ of all faces of $\mathcal{P}$ into the complete lattice $\mathcal{J}(\mathcal{V})$ of all join homomorphisms between the lattice $\mathcal{V}$ of all subspaces of $\mathbb{C}^n$. We also characterize all maximal faces of $\mathcal{P}$.

1. **Introduction.** Let M_n be the C^*-algebra of all $n \times n$ matrices over the complex field, and $\mathcal{P}$ the convex set of all positive linear maps on M_n, that is, those linear maps that send the set of positive semi-definite matrices into itself. The structure of $\mathcal{P}$ turns out to be very complicated, and several authors have tried to decompose $\mathcal{P}$ into another simpler convex subsets. We say that a positive linear map ϕ between C^*-algebras is *decomposable* if it is the sum of a completely positive linear map and a completely copositive linear map. Although every positive linear map between M_2 is decomposable [8, 9], this is not the case in general [1, 2, 5]. Such an example of an indecomposable positive linear map was obtained by adjusting diagonal entries and attaching minus signs at offdiagonal entries. In order to find another class of such maps, the author [4] has considered positive linear maps which fix diagonal entries, and showed that these maps are decomposable in M_3, but there are indecomposable positive linear maps among them whenever $n \geq 4$ [3].

In the course of studying positive linear maps which fix diagonals, it turns out that the boundary structure of $\mathcal{P}$ plays an essential role. Because the boundary of $\mathcal{P}$ consists of nontrivial faces, this leads us naturally to study general facial structures of $\mathcal{P}$. We denote by $\mathcal{F}(\mathcal{P})$ the complete lattice of all faces of the convex set $\mathcal{P}$. We also denote by $\mathcal{V}$ the complete lattice of all subspaces of the n-dimensional vector space $\mathbb{C}^n$, and $\mathcal{J}(\mathcal{V})$ the complete lattice of all join homomorphisms from $\mathcal{V}$ into itself.

In this note, we show that there is a well-defined join homomorphism Φ from $\mathcal{F}(\mathcal{P})$ into $\mathcal{J}(\mathcal{V})$, and characterize all maximal faces of the convex set $\mathcal{F}(\mathcal{P})$. After we collect some elementary facts on faces of a general convex set in a Euclidean space in Section 2, we apply these to the convex set $\mathcal{P}$ in Section 3 to construct the above mentioned join homomorphism. In Section 4, we show that every maximal face of $\mathcal{P}$ is of the form

$$F[p, \eta] = \{\phi \in \mathcal{P} : \phi(p)\eta = 0\},$$

where p is a one dimensional projection and η is a nonzero vector. We also examine the face arising from the Choi map [1, 2] to see that the join homomorphism Φ is not injective. For the case of the convex set of all completely positive linear maps from a C^*-algebra into a matrix algebra, we note that there is an another approach [7] to investigate the facial structure, as was pointed out by the referee. Throughout this note, we write $a \geq 0$ if a is a positive semi-definite matrix.

2. **Faces of a convex set.** Let C be a nonempty convex subset of a Euclidean space $\mathbb{R}^\nu$. Recall that a point x of C is said to be a *relative interior point* (we will just call an *interior point*, in short) of C if for each $y \in C$ there is $t > 1$ such that $(1 - t)y + tx \in C$. We will denote by int C the set of all interior points of C, which is nothing but the relative interior of C with respect to the affine manifold generated by C (see [6, Theorem 6.4]). Note that int C is nonempty whenever C is nonempty. A point $y \in C$ is also said to be a *boundary point* if it is not an interior point, and the set of all boundary points of C will be denoted by ∂C. If C is a closed convex subset of $\mathbb{R}^\nu$, then ∂C coincides with the set $C \setminus \text{int } C$. The following simple proposition will be useful to find boundary points of a convex set. Note that this proposition has been already used implicitly in [3, 4].

PROPOSITION 2.1. *Let x_0 be a fixed interior point of a convex set C. Then a point $y \in C$ is a boundary point if and only if the following condition holds:*

(2.1) $\sup\{t : (1 - t)x_0 + ty \in C\} = 1.$

PROOF.   It is clear that if y satisfies the condition (2.1) then y is a boundary point. Assume that y does not satisfy the condition. Then there is $t > 1$ such that $(1 - t)x_0 + ty \in C$. For a given $z \in C$ there is $s < 0$ such that $(1 - s)x_0 + sz \in C$, because x_0 is an interior point. Put $r = \frac{s(t-1)}{t-s}$. Then we see that $r < 0$ and

$$(1 - r)y + rz = \frac{1-s}{t-s}[(1-t)x_0 + ty] + \frac{t-1}{t-s}[(1-s)x_0 + sz] \in C.$$

This shows that y is an interior point of C.   ∎

Recall that a convex subset F of a convex set C is said to be a *face* of C if the following property holds:

(2.2) $x, y \in C, (1 - t)x + ty \in F$ for some $t \in (0, 1) \Longrightarrow x, y \in F.$

A *proper face* of F is a face of C which is neither C itself nor empty. Note that every proper face of C is contained in the boundary of C. It is clear that a face of a face is a face. It is also clear that the intersection of faces is again a face, and so there is a unique smallest face containing a given subset. For a family $\{F_i : i \in I\}$ of faces, we denote by $\vee_{i \in I} F_i$ the smallest face containing every F_i. In this way, the set $\mathcal{F}(C)$ of all faces of a convex set C is a complete lattice with respect to the partial order induced by the set inclusions. It is well known [6, Corollary 18.1.2] that if two faces of C have a common interior point then they coincide. The following Lemma is actually contained in the Proof of [6, Theorem 18.2].

LEMMA 2.2.   *Let C_1 be a convex subset of a convex set C, and F the smallest face of C containing C_1. Then we have* int $C_1 \subseteq$ int F.

LEMMA 2.3.   *Let C_i be a convex subset of a convex set C for $i = 0, 1$, and C_2 the convex subset of C generated by C_0 and C_1. Then we have $x_t = (1 - t)x_0 + tx_1$ is an interior point of C_2 for each $t \in (0, 1)$ and $x_i \in$ int C_i with $i = 0, 1$.*

PROOF.   Take an arbitrary point $y_s = (1-s)y_0 + sy_1 \in C_2$ with $s \in [0, 1]$ and $y_i \in C_i$ for $i = 0, 1$. We denote by $[x, y]$ the line segment between points x and y. Since x_i is an interior point of C_i, we can take $z_i \in C_i$ such that $x_i \in$ int$[y_i, z_i]$ for $i = 0, 1$. We denote by C_3 the convex subset generated by the line segments $[y_0, z_0]$ and $[y_1, z_1]$. It is clear then that x_t is a relative interior point of C_3. (Note that the hyperplane generated by C_3 is at most 3-dimensional.) Therefore, there is $w \in C_3$ such that $x_t \in$ int$[y_s, w]$. Since $w \in C_3 \subseteq C_2$, the conclusion follows.   ∎

PROPOSITION 2.4.   *Let x_i be an interior point of a face F_i of a convex set, for $i = 0, 1$. Then the point $x_t = (1 - t)x_0 + tx_1$ is an interior point of $F_0 \vee F_1$ for each $t \in (0, 1)$.*

PROOF.   Let C_2 be the convex subset generated by F_0 and F_1. Then by Lemma 2.3, we see that x_t is an interior point of C_2. Because $F_1 \vee F_2$ is the smallest face containing C_2, the Proof is complete by Lemma 2.2.   ∎

A proper face F of a convex set C is said to be *maximal* if every proper face containing F coincides with F. Note that every proper face F of C is contained in a maximal face because C is of finite dimension. With above Lemmas, it is easy to characterize maximal faces.

PROPOSITION 2.5.   *Let F be a proper face of a convex set C. Then the following are equivalent:*
   (i)   *F is a maximal face of C.*
   (ii) *If F is a face of F_1 and F_1 is a face of C then $F = F_1$ or $F_1 = F$.*
   (iii) *If $x \in$ int F, $y \in C \setminus F$ then $(1 - t)x + ty \in$ int C for each $t \in (0, 1)$.*
   (iv) *If $x \in$ int F, $y \in \partial C \setminus F$ then $(1 - t)x + ty \in$ int C for each $t \in (0, 1)$.*
   (v)   *There is $x \in$ int F with the property: If $y \in \partial C \setminus F$, then $(1 - t)x + ty \in$ int C for each $t \in (0, 1)$.*
   (vi) *There is $x \in$ int F with the property: If $y \in \partial C \setminus F$, then $(1 - t)x + ty \in$ int C for some $t \in (0, 1)$.*
   (vii) *There is $x \in F$ with the property: If $y \in \partial C \setminus F$, then $(1 - t)x + ty \in$ int C for some $t \in (0, 1)$.*

PROOF.   The directions (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) are clear. For the direction (vii) $\Rightarrow$ (i), assume that x is a point of F with the property in (vii) and F_1 is a proper face of C with $F \subseteq F_1$. If there is $y \in F_1 \setminus F$, then $y \in \partial C \setminus F$ and so $(1 - t)x + ty \in$ int C for some $t \in (0, 1)$. On the other hand, the convexity of F_1 implies that $(1 - t)x + ty \in F_1 \subseteq \partial C$, which is a contradiction. For the remaining implication (ii) $\Rightarrow$ (iii), let $x \in$ int F and $y \in C \setminus F$. We denote by F_1 the smallest face containing the

convex set generated by F and y. Then, we have $(1 - t)x + ty \in \text{int}\, F_1$ for each $t \in (0, 1)$, by Lemmas 2.2 and 2.3. But, we have $F_1 = C$ by the condition (ii), because F is a proper face of F_1.   ∎

3. **Join homomorphisms between lattices.** Let L be a complete lattice. We denote by $\mathcal{J}(L)$ the set of all join homomorphisms from L into L. We define the partial order in $\mathcal{J}(L)$ by

$$h \leq k \iff h(p) \leq k(p) \text{ for each } p \in L.$$

We note that an arbitrary subset of $\mathcal{J}(L)$ has the least upper bound. Indeed, if $\{h_i : i \in I\}$ is a family of join homomorphisms between L, then it is easy to see that their least upper bound in $\mathcal{J}(L)$ is given by

$$\vee_{i \in I} h_i : p \longmapsto \vee_{i \in I} h_i(p), \qquad p \in L.$$

Therefore, $\mathcal{J}(L)$ is a complete lattice. If L has 0 and 1, then $\mathcal{J}(L)$ also has 0 and 1 with the constant homomorphisms.

We denote by $\mathcal{P}$ the convex set of all positive linear maps from $M_n(\mathbb{C})$ into itself, and $\mathcal{V}$ the complete lattice of all self-adjoint projections in $\mathbb{C}^n$. For $\phi \in \mathcal{P}$, we define a map $\tilde{\phi} : \mathcal{V} \to \mathcal{V}$ by

(3.1) $$\tilde{\phi}(p) = R[\phi(p)], \qquad p \in \mathcal{V},$$

where $R[x]$ denotes the range projection of $x \in M_n(\mathbb{C})$.

LEMMA 3.1.   *The function $\tilde{\phi}$ in (3.1) is a join homomorphism from $\mathcal{V}$ into itself.*

PROOF.   For $p, q \in \mathcal{V}$, we denote by λ and μ the smallest and largest positive eigenvalues of $p + q$, respectively. Then we have

$$\lambda(p \vee q) \leq p + q \leq \mu(p \vee q).$$

Therefore, we have $\lambda \phi(p \vee q) \leq \phi(p) + \phi(q)$, which implies that

$$\tilde{\phi}(p \vee q) \leq R[\phi(p) + \phi(q)] = R[\phi(p)] \vee R[\phi(q)] = \tilde{\phi}(p) \vee \tilde{\phi}(q).$$

From the relation $p \leq p \vee q$, we infer that $\phi(p) \leq \phi(p \vee q)$, and $\tilde{\phi}(p) \leq \tilde{\phi}(p \vee q)$. Similarly, we also have $\tilde{\phi}(q) \leq \tilde{\phi}(p \vee q)$, and so we conclude that

$$\tilde{\phi}(p) \vee \tilde{\phi}(q) \leq \tilde{\phi}(p \vee q).$$ ∎

LEMMA 3.2.   *Let C be a nonempty convex subset of $\mathcal{P}$, and F a face of C. If $\phi \in F$ and $\psi \in \text{int}\, F$, then we have $\tilde{\phi} \leq \tilde{\psi}$.*

PROOF.   Note that there is a $\rho \in F$ such that $\psi = (1 - t)\phi + t\rho$ with $t \in (0, 1)$. Let $p \in \mathcal{V}$. If ξ is a null vector of $\psi(p)$, then it is also a null vector of $\phi(p)$, because $\phi(p)$ and $\rho(p)$ are positive semi-definite. This says that the null space of $\phi(p)$ includes the null space of $\psi(p)$. Therefore, we have $\tilde{\phi}(p) \leq \tilde{\psi}(p)$.   ∎

Let C be an arbitrary convex subset of $\mathcal{P}$. If ϕ and ψ are interior points of a common face in C, then we see that $\tilde{\phi} = \tilde{\psi}$ by Lemma 3.2. For a given face of C, we define

$$(3.2) \qquad\qquad \Phi_C(F) = \tilde{\phi}$$

with an interior point ϕ of F. We also define $\Phi_C(\emptyset) = 0_{\mathcal{J}(\mathcal{V})}$. In this way, we get a function Φ_C from the lattice $\mathcal{F}(C)$ into the lattice $\mathcal{J}(\mathcal{V})$. The above Lemma says that Φ_C is order-preserving. Actually, it is a join homomorphism.

THEOREM 3.3.   *For any convex subset C of $\mathcal{P}$, the map $\Phi_C : \mathcal{F}(C) \to \mathcal{J}(\mathcal{V})$ is a join homomorphism.*

PROOF.   Let F_i be a face of C for $i = 0, 1$. If one of F_i is empty, then it is clear that

$$(3.3) \qquad\qquad \Phi_C(F_0 \vee F_1) = \Phi_C(F_0) \vee \Phi_C(F_1).$$

Assume that F_i is nonempty with an interior point ϕ_i, for $i = 0, 1$. By Proposition 2.4, we see that $\psi = \frac{1}{2}(\phi_0 + \phi_1)$ is an interior point of $F_0 \vee F_1$. For each $p \in \mathcal{V}$, we have

$$R[\psi(p)] = R[\phi_0(p) + \phi_1(p)] = R[\phi_0(p)] \vee R[\phi_1(p)],$$

and so, it follows that $\tilde{\psi} = \tilde{\phi_0} \vee \tilde{\phi_1}$. Therefore, we have the relation (3.3).   ∎

4. Maximal faces.   We note that the trace map

$$\mathrm{tr} : X \mapsto \mathrm{tr}(X)I$$

is a typical example of an interior point of $\mathcal{P}$. We also note that $\tilde{\mathrm{tr}} = 1_{\mathcal{J}(\mathcal{V})}$. It follows from Lemma 3.2 that $\tilde{\phi} = 1_{\mathcal{J}(\mathcal{V})}$ whenever ϕ is an interior point of $\mathcal{P}$. For a nonzero vector ξ, we denote by $[\xi] \in \mathcal{V}$ the one-dimensional projection onto the subspace spanned by ξ.

PROPOSITION 4.1.   *For a positive linear map $\phi \in \mathcal{P}$, the following are equivalent:*
  (i) ϕ is an interior point of $\mathcal{P}$.
 (ii) $\tilde{\phi} = 1_{\mathcal{J}(\mathcal{V})}$.
(iii) $\phi([\xi])$ is nonsingular for each one-dimensional projection $[\xi] \in \mathcal{V}$.

PROOF.   It suffices to prove the direction (iii) $\Rightarrow$ (i). We denote by $\mathcal{V}_1$ the subset of $\mathcal{V}$ consisting of all one-dimensional projections. By the assumption (iii), we see that the function

$$[\xi] \mapsto \|\phi([\xi])^{-1}\| : \mathcal{V}_1 \to \mathbb{R}$$

is a continuous function on $\mathcal{V}_1$, which is compact under the usual topology. Therefore, we can choose $s > 1$ such that $\|\phi([\xi])^{-1}\| < s$ for each $[\xi] \in \mathcal{V}_1$. From this, we see that

$$s\phi([\xi]) \geq I = \mathrm{tr}(I) \geq \mathrm{tr}([\xi]), \qquad [\xi] \in \mathcal{V}_1,$$

which implies that $s\phi \geq \mathrm{tr}$. Therefore, it follows that

$$\left(1 - \frac{s}{s-1}\right)\mathrm{tr} + \frac{s}{s-1}\phi \geq \left(1 - \frac{s}{s-1}\right)\mathrm{tr} + \frac{1}{s-1}\,\mathrm{tr} = 0.$$

Because $\frac{s}{s-1} > 1$, we conclude that $\phi \in \text{int } \mathcal{P}$ by Proposition 2.1. ∎

It is well known [6, Corollary 6.5.1] that if L is a hyperplane which contains an interior point of a convex set C, then

(4.1) $$\text{int}(L \cap C) = L \cap \text{int } C.$$

For each positive definite (invertible) matrix K, the map

$$\text{tr}_K : X \longmapsto K^{1/2} \text{tr}(X) K^{1/2}$$

is an interior point of $\mathcal{P}$ by Proposition 4.1. We denote by $\mathcal{P}_K$ the convex subset of $\mathcal{P}$ consisting of $\phi \in \mathcal{P}$ which sends the identity I to K. By the relation (4.1), we see that Proposition 4.1 holds when $\mathcal{P}$ is replaced by $\mathcal{P}_K$ for a positive definite matrix K. Now, we characterize maximal faces of $\mathcal{P}$. We begin with the following simple Lemma:

LEMMA 4.2. *For any nonzero vectors $\xi, \eta \in \mathbb{C}^n$, there is a unital positive linear map $\phi \in \mathcal{P}$ with the following properties:*

(i) $\ker \phi([\xi])$ is one dimensional subspace spanned by η.

(ii) $\phi(q)$ is nonsingular whenever $q \in \mathcal{V}$ is a projection of rank one which is different from $[\xi]$.

PROOF. For $x \in M_n(\mathbb{C})$, we define $\sigma(x) \in M_n(\mathbb{C})$ by

$$[\sigma(x)]_{ij} = \begin{cases} 0, & i \neq j, \\ \frac{1}{n} \text{tr } x, & i = j = 1, 2, \ldots, n-1, \\ \frac{1}{n-1}(x_{22} + \cdots + x_{nn}), & i = j = n. \end{cases}$$

Then the map σ satisfies the conditions with $[\xi] = e_{11} \in \mathcal{V}$ and $\eta = e_n \in \mathbb{C}^n$, where $\{e_{ij}\}$ and $\{e_i\}$ are the usual matrix units and the orthonormal basis. We may assume that η is a unit vector. Now, let u and v be unitaries with $e_{11} = u^*[\xi]u$ and $v\eta = e_n$, respectively. Then the map $x \longmapsto v^*\sigma(u^*xu)v$ satisfies the required conditions. ∎

For nonzero vectors $\xi, \eta \in \mathbb{C}^n$, we define

(4.2) $$F[\xi, \eta] = \{\phi \in \mathcal{P} : \phi([\xi])\eta = 0\}.$$

Then it is clear that $F[\xi, \eta]$ is a convex subset of $\mathcal{P}$, which is nonempty by Lemma 4.2. We write $\phi_t = (1 - t)\phi_0 + t\phi_1$ for each $t \in [0, 1]$. Assuming that $\phi_0, \phi_1 \in \mathcal{P}$ and $\phi_t \in F[\xi, \eta]$ for some $t \in (0, 1)$, we have $\phi_0([\xi])\eta = \phi_1([\xi])\eta = 0$ because $\phi_t([\xi])$ is positive semi-definite. Therefore, we see that $F[\xi, \eta]$ is a face of $\mathcal{P}$.

For nonzero vectors $\xi, \eta \in \mathbb{C}^n$, we also define the join homomorphism $J[\xi, \eta]$ from $\mathcal{V}$ into $\mathcal{V}$ by

(4.3) $$J[\xi, \eta](p) = \begin{cases} 1 \in \mathcal{V}, & p \in \mathcal{V}, p \neq [\xi], \\ [\eta]^\perp \in \mathcal{V}, & p = [\xi] \in \mathcal{V}. \end{cases}$$

Then it is clear that ϕ satisfies the two conditions in Lemma 4.2 if and only if

(4.4) $$\tilde{\phi} = J[\xi, \eta].$$

LEMMA 4.3. *Let ϕ be an interior point of the face $F[\xi, \eta]$, then $\tilde{\phi} = J[\xi, \eta]$.*

PROOF. By Lemma 4.2, we can take $\psi \in F[\xi, \eta]$ such that $\tilde{\psi} = J[\xi, \eta]$. Then we see that $\tilde{\phi} \geq \tilde{\psi}$ by Lemma 3.2, and so $\tilde{\phi} = 1_{\mathcal{J}(\mathcal{V})}$ or $\tilde{\phi} = J[\xi, \eta]$. The conclusion follows from Proposition 4.1, since ϕ is a boundary point of $\mathcal{P}$. ∎

THEOREM 4.4. *Let $\mathcal{P}$ be the convex set of all positive linear maps on the matrix algebra $M_n(\mathbb{C})$. Then we have the following:*

(i) *For each pair (ξ, η) of nonzero vectors in $\mathbb{C}^n$, the set $F[\xi, \eta]$ is a maximal face of $\mathcal{P}$.*

(ii) *If F is a maximal face of $\mathcal{P}$ then there is a pair of nonzero vectors (ξ, η) in $\mathbb{C}^n$ such that $F = F[\xi, \eta]$.*

(iii) *If F_0 and F_1 are two maximal faces of $\mathcal{P}$ then they are affine isomorphic each other.*

PROOF. Take an interior point ϕ_0 of $F[\xi, \eta]$, and $\phi_1 \in \mathcal{P} \setminus F[\xi, \eta]$. For the proof of (i), it suffices to show that $\phi_t = (1 - t)\phi_0 + t\phi_1$ is an interior point of $\mathcal{P}$ for $t \in (0, 1)$, by Proposition 2.5. Assuming that ϕ_t is a boundary point of $\mathcal{P}$, there is $[\zeta] \in \mathcal{V}$ such that $\phi_t([\zeta])$ is singular with a nonzero null vector ω. Therefore, it follows that

$$\phi_0([\zeta])\omega = \phi_1([\zeta])\omega = 0.$$

By Lemma 4.3, we have $[\zeta] = [\xi]$ and $[\omega] = [\eta]$, and so it follows that $\phi_1 \in F[\xi, \eta]$, which is a contradiction.

If F is a maximal face of $\mathcal{P}$ with an interior point ϕ, then there is $[\xi] \in \mathcal{V}$ such that $\phi([\xi])$ is singular with a nonzero null vector η by Proposition 4.1. Therefore, $\phi \in F[\xi, \eta]$, and so it follows that $F \subseteq F[\xi, \eta]$. The maximality implies $F = F[\xi, \eta]$. For the last assertion, let $F_i = F[\xi_i, \eta_i]$ for $i = 0, 1$. Take unitaries u and v such that $u^*[\xi_1]u = [\xi_0]$ and $v\eta_1 = \eta_0$, and define the affine isomorphism $\phi \mapsto \alpha_\phi$ between $\mathcal{P}$ by

$$\alpha_\phi : x \mapsto v^*\phi(u^*xu)v, \qquad x \in M_n(\mathbb{C})$$

as in the proof of Lemma 4.2. Then we see that $\phi \in F_0$ if and only if $\alpha_\phi \in F_1$.   ∎

In the preceding discussions and Theorem 4.4, the convex set $\mathcal{P}$ may be replaced by the convex set $\mathcal{P}_K$ for a positive definite matrix K. In this case, the face $F[\xi, \eta]$ should be replaced by

$$(4.5) \qquad\qquad F_K[\xi, \eta] =: F[\xi, \eta] \cap \mathcal{P}_K.$$

5. **Examples.** The author [3, 4] has considered the positive linear maps which fix diagonal entries, and showed that such maps are of the forms

$$(5.1) \qquad\qquad \phi_{A,B} : X \mapsto A \circ X + B \circ X^{\mathrm{t}} + I \circ X,$$

for self-adjoint matrices A and B with zero diagonals, where $A \circ X$ is the Hadamard product of A and X, and X^{t} denotes the transpose of X. The set Δ of all such maps is a face of the convex set $\mathcal{P}_I$, and diagonal map $X \mapsto X \circ I$ is an interior point of Δ. Considering the join homomorphism in $\mathcal{J}(\mathcal{V})$ induced by the diagonal map, it is easy to see that

$$(5.2) \qquad\qquad \Delta = \bigcap_{i=1}^{n} \left[\bigcap \{ F_I[e_i, e_j] : j = 1, 2, \ldots, n, \quad j \neq i \} \right],$$

where $\{e_i\}$ is the usual orthonormal basis of $\mathbb{C}^n$.

Choi [1, 2] found an example of an indecomposable positive linear map in $M_3(\mathbb{C})$ which generates an extreme ray. This map is given by

$$(5.3) \qquad \gamma: [a_{ij}] \mapsto \frac{1}{2}\begin{bmatrix} a_{11}+a_{33} & -a_{12} & -a_{13} \\ -a_{21} & a_{22}+a_{11} & -a_{23} \\ -a_{31} & -a_{32} & a_{33}+a_{22} \end{bmatrix}$$

Assume that ξ is a nonzero vector in $\mathbb{C}^3$ and the image of $[\xi]$ under the above map is singular. Then, by a direct calculation, we see that ξ is one of the following vectors

$$\xi_1 = (1,0,0), \qquad \xi_2 = (0,1,0), \qquad \xi_3 = (0,0,1), \qquad \xi_\alpha = (e^{ia}, e^{ib}, e^{ic}),$$

and the corresponding null spaces are generated by

$$\eta_1 = (0,0,1), \qquad \eta_2 = (1,0,0), \qquad \eta_3 = (0,1,0), \qquad \eta_\alpha = (e^{ia}, e^{ib}, e^{ic}),$$

respectively, where $\alpha = (a, b, c) \in \mathbb{R}^3$. Although the Choi map γ is an extreme point of $\mathcal{P}_I$, it should be noted that the face

$$(5.4) \qquad \Gamma = \Big(\bigcap_{i=1}^{3} F_I[\xi_i, \eta_i]\Big) \bigcap \Big(\bigcap_{\alpha \in \mathbb{R}^3} F_I[\xi_\alpha, \eta_\alpha]\Big)$$

contains another map different from the Choi map. In fact, the two faces Δ and Γ have a nonempty intersection. For example, the map

$$X \mapsto \frac{1}{2}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \circ X, \qquad X \in M_3$$

lies in $\Delta \cap \Gamma$.

We note that $\tilde{\gamma}: \mathcal{V} \to \mathcal{V}$ sends each $[\xi_i]$ to $[\eta_i]^\perp$ for each $i = 1, 2, 3$, $\alpha \in \mathbb{R}^3$, and sends another $p \in \mathcal{V}$ to 1. By definition of Γ, it is clear that $\tilde{\phi} \leq \tilde{\gamma}$ for each $\phi \in \Gamma$. Therefore, we have $\tilde{\phi} = \tilde{\gamma}$ for each interior point $\phi \in \text{int}\,\Gamma$. This means that the join homomorphism $\Phi_{\mathcal{P}_I}$ is not injective, since γ lies on the boundary of Γ.

ADDED IN PROOF (FEBRUARY 13, 1996). The author has shown that every maximal face of the convex cone of all s-positive linear maps from M_m into M_n corresponds to an $m \times n$ matrix whose rank is less than or equal to s, in the papers [*On the convex set of completely positive linear maps in matrix algebras*, Math. Proc. Cambridge Philos. Soc., to appear] and [*Boundaries of the cone of positive linear maps and its subcones in matrix algebras*, preprint].

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