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EXACT AND ANALYTIC-NUMERICAL SOLUTIONS OF STRONGLY COUPLED MIXED DIFFUSION PROBLEMS

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Abstract This paper deals with the construction of exact and analytical-numerical solutions with *a priori* error bounds for systems of the type $u_t = Au_{xx}$, $A_1u(0,t)+B_1u_x(0,t) = 0$, $A_2u(1,t)+B_2u_x(1,t) = 0$, $0 < x < 1$, $t > 0$, $u(x,0) = f(x)$, where A_1, A_2, B_1 and B_2 are matrices for which no simultaneous diagonalizable hypothesis is assumed, and A is a positive stable matrix. Given an admissible error ε and a bounded subdomain D, an approximate solution whose error with respect to an exact series solution is less than ε uniformly in D is constructed.

Keywords: coupled diffusion problem; coupled boundary conditions; vector boundary-value differential system; analytic-numerical solution; Moore–Penrose pseudoinverse

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1. Introduction and preliminaries

Coupled partial differential systems with coupled boundary-value conditions are frequent in quantum mechanical scattering problems [**1**,**19**,**27**], chemical physics [**16**,**17**,**22**], thermoelastoplastic modelling [**13**], coupled diffusion problems [**8**,**20**,**29**], and other fields. In this paper we consider systems of the type

$$u_t(x,t) - Au_{xx}(x,t) = 0, \quad 0 < x < 1, \quad t > 0, \tag{1.1}$$

$$A_1u(0,t) + B_1u_x(0,t) = 0, \quad t > 0, \tag{1.2}$$

$$A_2u(1,t) + B_2u_x(1,t) = 0, \quad t > 0, \tag{1.3}$$

$$u(x,0) = f(x), \quad 0 \leqslant x \leqslant 1, \tag{1.4}$$

where the unknown $u = (u_1, u_2, \dots, u_m)^{\mathrm{T}}$ and $f = (f_1, f_2, \dots, f_m)^{\mathrm{T}}$ are m-dimensional vectors, A_i, B_i, $i = 1,2$ are $m \times m$ complex matrices, elements of $\mathbb{C}^{m\times m}$, and A is a positive stable matrix

$$\mathrm{Re}(z) > 0 \text{ for all eigenvalues } z \text{ of } A. \tag{1.5}$$

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We assume that

$$\left.\begin{array}{l}\text{The block matrix } \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} \text{ is invertible and} \\ \text{not all its blocks } A_1,\ A_2,\ B_1,\ B_2 \text{ are singular.}\end{array}\right\} \tag{1.6}$$

Conditions on the function $f(x)$ and on the matrix coefficients will be determined in order to guarantee the existence of a series solution of the problem, as well as the construction of analytic-numerical finite-sum approximations with a prefixed accuracy in a bounded subdomain. Mixed problems of the above type, but with Dirichlet conditions $u(0,t) = 0$, $u(1,t) = 0$ instead of equations (1.2) and (1.3), have been treated in [**15**, **23**].

The organization of the paper is as follows. In § 2, the vector eigenvalue differential problem

$$\left.\begin{array}{c} X''(x) + \lambda^2 X(x) = 0, \quad 0 < x < 1, \quad \lambda \geqslant 0, \\ A^j X(0) + B_1 A^j X'(0) = 0, \\ A_2 A^j X(1) + B_2 A^j X'(1) = 0, \\ 0 \leqslant j \leqslant p-1, \quad p \geqslant 1, \end{array}\right\} \tag{1.7}$$

is studied. Sufficient conditions for the existence of eigenvalues are given. Using a separation-of-variables technique, an exact series solution of problems (1.1)–(1.4) is constructed in § 3. In § 4, a procedure for the construction of a finite-sum approximation with a prefixed accuracy is given, by truncation of the exact infinite-series solution and appropriate approximations of the eigenvalues.

Throughout this paper, the set of all the eigenvalues of a matrix C in $\mathbb{C}^{m\times m}$ is denoted by $\sigma(C)$ and its 2-norm denoted by $\|C\|$ is defined by [**11**, p. 56]

$$\|C\| = \sup_{z\neq o} \frac{\|Cz\|_2}{\|z\|_2},$$

where, for a vector y in $\mathbb{C}^m$, $\|y\|_2$ denotes the usual Euclidean norm of y. By [**11**, p. 556], it follows that

$$\|e^{tC}\| \leqslant e^{t\alpha(C)} \sum_{k=0}^{m-1} \frac{\|\sqrt{m}C\|^k t^k}{k!}, \quad t \geqslant 0, \tag{1.8}$$

where $\alpha(C) = \max\{\mathrm{Re}(w);\ w \in \sigma(C)\}$. The conjugate transpose of C is denoted by C^*. If B is a matrix in $\mathbb{C}^{n\times m}$, we denote by $B^\dagger$ its Moore–Penrose pseudoinverse. An account of examples, properties and applications of this concept may be found in [**6**] and [**26**], and $B^\dagger$ can be efficiently computed with the MATLAB package. The kernel of B, denoted by $\ker B$, coincides with the image of the matrix $I - B^\dagger B$, denoted by $\mathrm{Im}(I - B^\dagger B)$, see [**6**]. We say that a subspace E of $\mathbb{C}^m$ is invariant by the matrix $A \in \mathbb{C}^{m\times m}$ if $A(E) \subset E$. Hence, property $A(\ker G) \subset \ker G$ is equivalent to the condition $GA(I - G^\dagger G) = 0$. We conclude this section with an algebraic result that will play an important role in the following.

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Lemma 1.1. *Let M and N be matrices in $\mathbb{C}^{m\times m}$, then*

$$\ker M \cap \ker N = \operatorname{Im}\{(I - M^\dagger M)\{I - [N(I - M^\dagger M)]^\dagger [N(I - M^\dagger M)]\}\}.$$

Proof. If $v \in \ker M \cap \ker N$, then $Mv = 0$, and, by Theorem 2.3.2 of [**26**, p. 24], $v = \operatorname{Im}(I - M^\dagger M)d$, where d is an arbitrary vector in $\mathbb{C}^m$. Hence

$$v \in \operatorname{Im}(I - M^\dagger M) \supset \operatorname{Im}\{(I - M^\dagger M)\{I - [N(I - M^\dagger M)]^\dagger [N(I - M^\dagger M)]\}\}.$$

Conversely, let $v \in \operatorname{Im}\{(I - M^\dagger M)\{I - [N(I - M^\dagger M)]^\dagger [N(I - M^\dagger M)]\}$. Then, for some $z \in \mathbb{C}^m$, one gets

$$v = (I - M^\dagger M)\{I - [N(I - M^\dagger M)]^\dagger [N(I - M^\dagger M)]\}z.$$

Hence, and using that $M = MM^\dagger M$, it follows that

$$Mv = (M - MM^\dagger M)\{I - [N(I - M^\dagger M)]^\dagger [N(I - M^\dagger M)]\}z = 0$$

and

$$Nv = \{N(I - M^\dagger M) - [N(I - M^\dagger M)][N(I - M^\dagger M)]^\dagger [N(I - M^\dagger M)]\}z = 0.$$

Thus $v \in \ker M \cap \ker N$, and the result is established. □

The set of all the real numbers will be denoted by $\mathbb{R}$, and the set of all non-negative integers will be denoted by $\mathbb{N}$. If A is a matrix in $\mathbb{C}^{m\times m}$, we denote $\beta(A) = \min\{\operatorname{Re}(w);\ w \in \sigma(A)\}$, and if $\beta(A) > 0$ and $t \geqslant 0$, from (1.8) one gets

$$\|e^{-tA}\| \leqslant e^{-t\beta(A)} \sum_{k=0}^{m-1} \frac{\|\sqrt{m}A\|^k t^k}{k!}, \quad t \geqslant 0.$$

2. Vector eigenvalue differential systems

Vector Sturm–Liouville differential systems of the form

$$\begin{gathered} -(P(x)y')' + Q(x)y = \lambda W(x)y, \quad a \leqslant x \leqslant b, \\ A_1^* y(a) + A_2^* P(a)y'(a) = 0, \\ B_1^* y(b) + B_2^* P(b)y'(b) = 0, \end{gathered}$$

where P, Q and W are symmetric $m \times m$ matrix functions of x with P and W positive definite for all $x \in [a, b]$, y is an m-vector function of x, λ is a scalar parameter, and A_1, A_2, B_1 and B_2 are matrices in $\mathbb{C}^{m\times m}$, such that $A_1^* A_2 = A_2^* A_1$, $B_1^* B_2 = B_2^* B_1$, and (A_1, A_2), (B_1, B_2) are full-rank $\mathbb{C}^{m\times 2m}$ matrices which have been treated in [**3,4,12,18**]. In this section, we consider vector eigenvalue differential problems of the type (1.7). Suppose that

$$A_1 = I. \tag{2.1}$$

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Under this hypothesis, the general solution of the vector equation $X'' + \lambda^2 X = 0$ is given by

$$X_\lambda(x) = \begin{cases} \sin(\lambda x)D_\lambda + \cos(\lambda x)E_\lambda, & D_\lambda, E_\lambda \in \mathbb{C}^m, \quad \lambda > 0, \\ D_0 + xE_0, & D_0, E_0 \in \mathbb{C}^m, \quad \lambda = 0. \end{cases} \tag{2.2}$$

Condition $X(0) + B_1X'(0) = 0$ implies $D_\lambda = -\lambda B_1 E_\lambda$, if $\lambda > 0$ and $D_0 = -B_1E_0$. Hence, (2.2) takes the form

$$X_\lambda(x) = \begin{cases} (\cos(\lambda x) - \lambda B_1 \sin(\lambda x))E_\lambda, & E_\lambda \in \mathbb{C}^m, \quad \lambda > 0, \\ (Ix - B_1)E_0, & E_0 \in \mathbb{C}^m, \quad \lambda = 0. \end{cases} \tag{2.3}$$

By imposing the remaining boundary-value conditions $A^jX(0) + B_1A^jX'(0) = 0$ for $1 \leqslant j \leqslant p-1$ and $A_2A^jX(1) + B_2A^jX'(1) = 0$, $0 \leqslant j \leqslant p-1$, one gets the following conditions on the vector E_λ, for $\lambda > 0$:

$$(A^jB_1 - B_1A^j)E_\lambda = 0, \quad 1 \leqslant j \leqslant p-1, \quad \lambda > 0, \tag{2.4}$$

$$[-\lambda(\cos(\lambda)A_2 - \lambda\sin(\lambda)B_2)A^jB_1 + (\sin(\lambda)A_2 + \lambda\cos(\lambda)B_2)A^j]E_\lambda = 0,$$
$$\text{for } 0 \leqslant j \leqslant p-1, \quad \lambda > 0. \tag{2.5}$$

Taking into account (2.4), conditions (2.5) can be written in the form

$$[\sin(\lambda)(A_2 + \lambda^2 B_2B_1) + \lambda\cos(\lambda)(B_2 - A_2B_1)]A^jE_\lambda = 0,$$
$$\text{for } 0 \leqslant j \leqslant p-1, \quad \lambda > 0. \tag{2.6}$$

Since we seek non-zero vectors E_λ, by (2.6) one gets that

$$L(\lambda) = (A_2 + \lambda^2 B_2B_1)\sin(\lambda) + (B_2 - A_2B_1)\lambda\cos(\lambda) \text{ is singular}, \quad \lambda > 0. \tag{2.7}$$

Assume that the block matrix

$$\begin{bmatrix} I & B_1 \\ A_2 & B_2 \end{bmatrix} \tag{2.8}$$

is invertible. By (2.8) and the properties of the Schur complement of a matrix [**5**, p. 93], one gets that $B_2 - A_2B_1$ is invertible, and condition (2.7) implies $\sin(\lambda) \neq 0$. Hence, condition (2.7) is equivalent to

$$A_2 + \lambda^2 B_2B_1 + \lambda\cot(\lambda)(B_2 - A_2B_1) \text{ singular}, \quad \lambda > 0,$$

or

$$(B_2 - A_2B_1)^{-1}A_2 + \lambda^2(B_2 - A_2B_1)^{-1}B_2B_1 + \lambda I\cot(\lambda)I \text{ singular}, \quad \lambda > 0.$$

Hence,

$$\lambda\cot(\lambda) \in \sigma((A_2B_1 - B_2)^{-1}A_2 + \lambda^2(A_2B_1 - B_2)^{-1}B_2B_1), \quad \lambda > 0. \tag{2.9}$$

Let us introduce the matrices

$$\hat{A}_2 = (A_2B_1 - B_2)^{-1}A_2, \qquad \hat{B}_2 = (A_2B_1 - B_2)^{-1}B_2, \tag{2.10}$$

and note that

$$\hat{B}_2 = \hat{A}_2B_1 - I. \tag{2.11}$$

Hence condition (2.9) can be written in the form

$$\lambda \cot(\lambda) \in \sigma(\hat{A}_2 + \lambda^2(\hat{A}_2B_1^2 - B_1)), \quad \lambda > 0. \tag{2.12}$$

Assume that matrices $\hat{A}_2$ and B_1 have real eigenvalues $\alpha \in \sigma(\hat{A}_2)$ and $\beta \in \sigma(B_1)$ and a common eigenvector $v \in \mathbb{C}^m$ associated to them:

$$(B_1 - \beta I)v = (\hat{A}_2 - \alpha I)v = 0, \quad v \in \mathbb{C}^m, \quad v \neq 0, \quad (\alpha, \beta) \in \mathbb{R}^2. \tag{2.13}$$

Then

$$[\hat{A}_2 + \lambda^2(\hat{A}_2B_1^2 - B_1)]v = [\alpha + \lambda^2(\alpha\beta^2 - \beta)]v,$$

and, for $\lambda > 0$, one gets

$$\left.\begin{array}{l}\alpha + \lambda^2(\alpha\beta^2 - \beta) \text{ is a real eigenvalue of } \hat{A}_2 + \lambda^2(\hat{A}_2B_1^2 - B_1) \\ \text{and } v \text{ is an eigenvector associated with } \alpha + \lambda^2(\alpha\beta^2 - \beta),\end{array}\right\} \tag{2.14}$$

and

$$\lambda \cot(\lambda) = \alpha + \lambda^2(\alpha\beta^2 - \beta), \quad \lambda > 0, \tag{2.15}$$

has a sequence of positive roots. Note that by (2.4) and (2.7), eigenfunctions $X_\lambda(x)$ are given by (see (2.3))

$$X_\lambda(x) = \{\cos(\lambda x) - \lambda B_1 \sin(\lambda x)\}E_\lambda, \quad E_\lambda \in \mathbb{C}^m, \quad \lambda > 0, \tag{2.16}$$

where vectors E_λ satisfy

$$H_\lambda E_\lambda = 0, \quad \lambda > 0, \tag{2.17}$$

where H_λ is the matrix in $\mathbb{C}^{(2p-1)m \times m}$ defined by

$$H_\lambda = \begin{bmatrix} B_1A - AB_1 \\ B_1A^2 - A^2B_1 \\ \vdots \\ B_1A^{p-1} - A^{p-1}B_1 \\ \hat{A}_2 + \lambda^2(\hat{A}_2B_1^2 - B_1) - (\alpha + \lambda^2(\alpha\beta^2 - \beta))I \\ [\hat{A}_2 + \lambda^2(\hat{A}_2B_1^2 - B_1) - (\alpha + \lambda^2(\alpha\beta^2 - \beta))I]A \\ \vdots \\ [\hat{A}_2 + \lambda^2(\hat{A}_2B_1^2 - B_1) - (\alpha + \lambda^2(\alpha\beta^2 - \beta))I]A^{p-1} \end{bmatrix}. \tag{2.18}$$

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If for $\lambda = 0$, by imposing to $X_0(x) = (Ix - B_1)E_0$ given by (2.3), the boundary-value conditions $A^j X(0) + B_1 A^j X'(0) = 0$ for $1 \leqslant j \leqslant p-1$ and $A_2 A^j X(1) + B_2 A^j X'(1) = 0$, $0 \leqslant j \leqslant p-1$, it follows that $E_0 \in \mathbb{C}^m$ must verify

$$(B_1 A^j - A^j B_1)E_0 = 0, \quad 1 \leqslant j \leqslant p-1, \tag{2.19}$$

$$A_2 A^j (I - B_1)E_0 + B_2 A^j E_0 = 0, \quad 0 \leqslant j \leqslant p-1. \tag{2.20}$$

Note that condition (2.19) is also verified for $j = 0$. Substituting condition (2.19) into (2.20) one gets

$$\begin{aligned} A_2 A^j E_0 - A_2 B_1 A^j E_0 + B_2 A^j E_0 = 0, \\ (A_2 - A_2 B_1 + B_2) A^j E_0 = 0, \quad 0 \leqslant j \leqslant p-1. \end{aligned} \tag{2.21}$$

By the definition of $\hat{A}_2$ given by (2.10), it follows that $A_2 = (A_2 B_1 - B_2)\hat{A}_2$, and, thus, condition (2.21) can be written in the form

$$(A_2 B_1 - B_2)(\hat{A}_2 - I) A^j E_0 = 0, \quad 0 \leqslant j \leqslant p-1. \tag{2.22}$$

Since $A_2 B_1 - B_2$ is invertible, condition (2.22) is equivalent to

$$(\hat{A}_2 - I) A^j E_0 = 0, \quad 0 \leqslant j \leqslant p-1.$$

Thus, conditions (2.19) and (2.20) are equivalent to the condition

$$H_0 E_0 = 0, \tag{2.23}$$

where

$$H_0 = \begin{bmatrix} B_1 A - A B_1 \\ B_1 A^2 - A^2 B_1 \\ \vdots \\ B_1 A^{p-1} - A^{p-1} B_1 \\ \hat{A}_2 - I \\ (\hat{A}_2 - I)A \\ \vdots \\ (\hat{A}_2 - I)A^{p-1} \end{bmatrix}. \tag{2.24}$$

Note that taking $\lambda = 0$ and $\alpha = 1$ in (2.18), one gets H_0 defined by (2.24). By (2.17) and (2.23), the existence of eigenfunctions associated with $\lambda \geqslant 0$ is granted if the matrix H_λ, defined by (2.18) for $\lambda > 0$ and by (2.24) for $\lambda = 0$, satisfies

$$\operatorname{rank} H_\lambda < m, \quad \lambda \geqslant 0. \tag{2.25}$$

Furthermore, under condition (2.25) and Theorem 2.3.2 in [**26**, p. 24], if equation $H_\lambda E_\lambda = 0$ is compatible, its solution set is given by

$$E_\lambda = (I - H_\lambda^\dagger H_\lambda) S_\lambda, \quad S_\lambda \in \mathbb{C}^m.$$

Assume that apart from condition (2.13), vector v satisfies

$$\{A^j v;\ 1 \leqslant j \leqslant p-1\} \subset \operatorname{Ker}(\hat{A}_2 - \alpha I), \tag{2.26}$$

then v satisfies $H_\lambda v = 0$ for all the positive solutions λ of equation (2.15). Note that condition (2.26) is granted if apart from (2.13), we assume that

$$\operatorname{Ker}(B_1 - \beta I) \cap \operatorname{Ker}(\hat{A}_2 - \alpha I) \text{ is an invariant subspace of } A.$$

Summarizing, the following result has been established.

Theorem 2.1. *Let $A \in \mathbb{C}^{m\times m}$, p an integer, $p \geqslant 1$, and suppose that the matrix*

$$\begin{bmatrix} I & B_1 \\ A_2 & B_2 \end{bmatrix}$$

is invertible in $\mathbb{C}^{2m\times 2m}$. Let $\hat{A}_2$, $\hat{B}_2$ be defined by (2.10) and assume condition (2.13) for some vector $v \in \mathbb{C}^m$. Let H_λ be defined by (2.18) for $\lambda > 0$ and by (2.24) if $\lambda = 0$.

(i) *A positive solution, λ, of equation (2.15) is an eigenvalue of problem (1.7) if* $\operatorname{rank} H_\lambda < m$, *and $\lambda_0 = 0$ is an eigenvalue if* $\operatorname{rank} H_0 < m$.

(ii) *If apart from condition (2.13) the vector v satisfies (2.26), then problem (1.7) admits a countable set $\mathcal{F}(\alpha,\beta) = \{\lambda_n;\ n \in \mathbb{N}\}$ of real eigenvalues with $\lim_{n\to\infty} \lambda_n = +\infty$.*

(iii) *If $\lambda_n \geqslant 0$ is an eigenvalue of problem (1.7), then eigenfunctions associated to λ_n are given by*

$$X_{\lambda_n}(x) = \begin{cases} \{\cos(\lambda_n x) - \lambda_n B_1 \sin(\lambda_n x)\} E_{\lambda_n}, & \lambda_n > 0, \\ (Ix - B_1) E_0, & \lambda_0 = 0, \end{cases}$$

where $E_{\lambda_n} = (I - H_{\lambda_n}^{\dagger} H_{\lambda_n}) S_{\lambda_n}$, where S_{λ_n} is an arbitrary vector in $\mathbb{C}^m$.

Remark 2.2. With respect to the localization of the eigenvalues of problem (1.7), it is easy to show that the sequence $\{\lambda_k\}_{k\geqslant 1}$ of non-negative roots of equation (2.15) verifies the following cases.

Case 1. $\beta(1-a) > 0$. If

$$\begin{array}{rll} \alpha > 1, & k\pi < \lambda_k < \frac{1}{2}(2k+1)\pi, & k \geqslant 1, \\ 0 \leqslant \alpha < 1, & (k-1)\pi < \lambda_k < \frac{1}{2}(2k-1)\pi, & k \geqslant 1, \\ \alpha < 0, & (k-1)\pi < \lambda_k < k\pi, & k \geqslant 1. \end{array}$$

Thus, in all the subcases, one gets

$$(k-1)\pi < \lambda_k < k + \tfrac{1}{2}\pi, \quad k \geqslant 1, \tag{2.27}$$

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Case 2. $\beta(1-\alpha)=0$. If

$$\begin{array}{rlr} \alpha>1, & k\pi<\lambda_k<\frac{1}{2}(2k+1)\pi, & k\geqslant 1,\\ \alpha=1, & \lambda_0=0 \text{ and } k\pi<\lambda_k<\frac{1}{2}(2k+1)\pi, & k\geqslant 1,\\ 0<\alpha<1, & (k-1)\pi<\lambda_k<\frac{1}{2}(2k-1)\pi, & k\geqslant 1,\\ \alpha=0, & \lambda_k=\frac{1}{2}(2k-1)\pi, & k\geqslant 1,\\ \alpha<0, & \frac{1}{2}(2k-1)\pi<\lambda_k<k\pi, & k\geqslant 1. \end{array}$$

So in all the subcases for $k\geqslant 1$ one gets (2.27).

Case 3. $\beta(1-\alpha)<0$. If

$$\begin{array}{rlr} \alpha>1, & (k-1)\pi<\lambda_k<(k+1)\pi, & k\geqslant 1,\\ 0<\alpha<1, & (k-1)\pi<\lambda_k<k\pi, & k\geqslant 1,\\ \alpha\leqslant 0, & \frac{1}{2}(2k-1)\pi<\lambda_k<k\pi, & k\geqslant 1. \end{array}$$

Thus, in all the cases the positive solutions λ_k of (2.15) verify (2.27).

Remark 2.3. The study of the problem with $B_1=I$,

$$\left.\begin{array}{c} X''(x)+\lambda^2X(x)=0,\quad 0<x<1,\\ A_1A^jX(0)+A^jX'(0)=0,\\ A_2A^jX(1)+B_2A^jX'(1)=0,\\ 0\leqslant j\leqslant p-1,\quad p\geqslant 1, \end{array}\right\} \tag{2.28}$$

is analogous to problem (1.7). It is easy to check that the problems

$$\left.\begin{array}{c} X''(x)+\lambda^2X(x)=0,\quad 0<x<1,\\ A_1A^jX(0)+B_1A^jX'(0)=0,\\ A_2A^jX(1)+B_2A^jX'(1)=0,\\ 0\leqslant j\leqslant p-1,\quad p\geqslant 1, \end{array}\right\} \tag{2.29}$$

where $A_2=I$ or $B_2=I$, can be reduced to the previous cases considering the change of variables defined by

$$y=y(x)=1-x,\quad 0\leqslant x\leqslant 1. \tag{2.30}$$

Thus, the approach developed is applicable to any problem of the type

$$\left.\begin{array}{c} X''(x)+\lambda^2X(x)=0,\quad 0<x<1,\\ A_1A^jX(0)+B_1A^jX'(0)=0,\\ A_2A^jX(1)+B_2A^jX'(1)=0,\\ 0\leqslant j\leqslant p-1,\quad p\geqslant 1, \end{array}\right\} \tag{2.31}$$

where some of the block entries are the identity matrix. Finally, it is important to point out that the hypothesis $A_i = I$ for $i = 1$ or $i = 2$, or $B_i = I$ for $i = 1$ or $i = 2$, does not involve a lack of generality. In fact, if in problem (2.31) one verifies that some A_i (respectively, B_i) is invertible, premultiplying the corresponding boundary condition of (2.31) by A_i^{-1} (respectively, B_i^{-1}), one achieves a previously considered problem.

3. Construction of an exact series solution

Let us seek solutions of the boundary-value problems (1.1)–(1.3) under hypotheses (2.1) and (2.8). A separation-of-variables technique suggests

$$v_\lambda(x,t) = T_\lambda(t)X_\lambda(x), \quad T_\lambda(t) \in \mathbb{C}^{m\times m}, \quad X_\lambda(x) \in \mathbb{C}^m, \quad \lambda \geqslant 0, \tag{3.1}$$

where

$$T'_\lambda(t) + \lambda^2 A T_\lambda(t) = 0, \quad t \geqslant 0, \quad \lambda \geqslant 0, \tag{3.2}$$

$$\left.\begin{aligned} X''_\lambda(x) + \lambda^2 X_\lambda(x) &= 0, \quad 0 < x < 1, \quad \lambda \geqslant 0, \\ X_\lambda(0) + B_1 X'_\lambda(0) &= 0, \\ A_2 X_\lambda(1) + B_2 X'_\lambda(1) &= 0. \end{aligned}\right\} \tag{3.3}$$

The solution of (3.2) satisfying $T_\lambda(0) = I$ is $T_\lambda(t) = \exp(-\lambda^2 At)$, but, although $v_\lambda(x,t)$ defined by (3.1) satisfies (1.1)

$$\begin{aligned} \frac{\partial}{\partial t}(v_\lambda(x,t)) - A\frac{\partial^2}{\partial x^2}(v_\lambda(x,t)) &= T'_\lambda(t)X_\lambda(x) - AT_\lambda(t)X''_\lambda(x) \\ &= -\lambda^2 AT_\lambda(t)X_\lambda(x) + AT_\lambda(t)\lambda^2 X_\lambda(x) = 0, \end{aligned}$$

condition (1.2) is not granted because

$$\begin{aligned} v_\lambda(0,t) + B_1\frac{\partial}{\partial x}(v_\lambda(0,t)) &= T_\lambda(t)X_\lambda(0) + B_1 T_\lambda(t)X'_\lambda(0) \\ &= \exp(-\lambda^2 At)X_\lambda(0) + B_1\exp(-\lambda^2 At)X'_\lambda(0), \end{aligned} \tag{3.4}$$

and the last equation does not vanish because matrix B_1 does not commute with A. However, if X_λ satisfies (1.7) instead of (3.3), where p is the degree of the minimal polynomial of A, then $T_\lambda(t) = \exp(-\lambda^2 At)$ can be expressed as a matrix polynomial of A [**9**, p. 557],

$$T_\lambda(t) = \exp(-\lambda^2 At) = b_0(t,\lambda)I + b_1(t,\lambda)A + \cdots + b_{p-1}(t,\lambda)A^{p-1}, \tag{3.5}$$

where $b_j(t,\lambda)$, $0 \leqslant j \leqslant p-1$ are scalars. Under the boundary-value conditions of (1.7) it follows that

$$v_\lambda(0,t) + B_1\frac{\partial}{\partial x}(v_\lambda(0,t)) = \sum_{j=0}^{p-1} b_j(t,\lambda)\{A^j X_\lambda(0) + B_1 A^j X'_\lambda(0)\} = 0, \qquad t \geqslant 0,$$

and

$$A_2 v_\lambda(1,t) + B_2 \frac{\partial}{\partial x}(v_\lambda(1,t)) = \sum_{j=0}^{p-1} b_j(t,\lambda)\{A_2 A^j X_\lambda(1) + B_1 A^j X'_\lambda(1)\} = 0, \quad t \geqslant 0.$$

Assume the notation and hypotheses of Theorem 2.1 and let $\{\lambda_n\}_{n=1}^{\infty}$ be the sequence of positive eigenvalues of problem (1.7). The candidate series solution of problems (1.1)–(1.4) is given by

$$U(x,t) = \begin{cases} X_0(x)E_0 + \sum_{n\geqslant 1} \exp(-\lambda_n^2 At) X_{\lambda_n}(x), & 0 \in \mathcal{F}(\alpha,\beta), \\ \sum_{n\geqslant 1} \exp(-\lambda_n^2 At) X_{\lambda_n}(x), & 0 \notin \mathcal{F}(\alpha,\beta), \end{cases} \tag{3.6}$$

where X_{λ_n} is defined by Theorem 2.1, for appropriate vectors E_{λ_n} to be determined. Consider the case where $0 \notin \mathcal{F}(\alpha,\beta)$. Associated to problem (1.7) we introduce the scalar Sturm–Liouville problem

$$\left.\begin{aligned} X''(x) + \lambda^2 X(x) = 0, \quad 0 < x < 1, \\ X(0) + \beta X'(0) = 0, \\ \alpha X(1) + (\alpha\beta - 1)X'(1) = 0. \end{aligned}\right\} \tag{3.7}$$

For the sake of well-posedness, assume that function $f(x)$ appearing in (1.4) satisfies the property

$$\left.\begin{array}{l} f(x) \text{ is twice continuously differentiable in } [0,1] \\ \text{and } f(0) + \beta f'(0) = 0,\ \alpha f(1) + (\alpha\beta - 1) f'(1) = 0. \end{array}\right\} \tag{3.8}$$

By the convergence theorem in series of Sturm–Liouville functions (see [**14**, ch. 11], [**10**, p. 90] and [**7**]), each component $f_i(x)$ of f, for $1 \leqslant i \leqslant m$ admits a series representation, absolute and uniformly convergent in [0, 1], of the form

$$f_i(x) = \sum_{n\geqslant 1} \{\sin(\lambda_n x) + \lambda_n \beta \cos(\lambda_n x)\} e_{\lambda_n}(i), \quad 0 \leqslant x \leqslant 1,$$

where

$$e_{\lambda_n}(i) = \frac{\int_0^1 f_i(x)\{\sin(\lambda_n x) + \beta\lambda_n \cos(\lambda_n x)\}\,\mathrm{d}x}{\int_0^1 \{\sin(\lambda_n x) + \beta\lambda_n \cos(\lambda_n x)\}^2\,\mathrm{d}x}, \quad n \geqslant 1, \quad 1 \leqslant i \leqslant m. \tag{3.9}$$

Note that if we define vectors $E_{\lambda_n} \in \mathbb{C}^m$ by

$$E_{\lambda_n} = \begin{bmatrix} e_{\lambda_n}(1) \\ e_{\lambda_n}(2) \\ \vdots \\ e_{\lambda_n}(m) \end{bmatrix}, \tag{3.10}$$

then $U(x,t)$, defined by

$$U(x,t)=\sum_{n\geqslant 1}\exp(-\lambda_n^2At)\{\sin(\lambda_n x)-\beta\lambda_n\cos(\lambda_n x)\}E_{\lambda_n}, \tag{3.11}$$

satisfies $U(x,0)=f(x)$, $0\leqslant x\leqslant 1$.

For the case where $0\in\mathcal{F}(\alpha,\beta)$ and $\lambda_0=0$ is an eigenvalue, we consider the scalar Sturm–Liouville problem (3.7) with $\alpha=1$:

$$\left.\begin{aligned} X''(x)+\lambda^2X(x)=0,\quad 0<x<1,\\ X(0)+\beta X'(0)=0,\\ \alpha X(1)+(\beta-1)X'(1)=0. \end{aligned}\right\} \tag{3.12}$$

If function $f(x)$ appearing in (1.4) satisfies condition (3.8) with $\alpha=1$, and if apart from $e_{\lambda_n}(i)$, defined by (3.10), one considers

$$e_0(i)=\frac{\displaystyle\int_0^1 f_i(x)(x-\beta)\,\mathrm{d}x}{\displaystyle\int_0^1(x-\beta)^2\,\mathrm{d}x},\quad 1\leqslant i\leqslant m,\quad E_0=\begin{bmatrix} e_0(1)\\ e_0(2)\\ \vdots\\ e_0(m)\end{bmatrix}, \tag{3.13}$$

then $U(x,t)$, defined by

$$U(x,t)=(x-\beta)E_0+\sum_{n\geqslant 1}\exp(-\lambda_n^2At)\{\sin(\lambda_n x)-\beta\lambda_n\cos(\lambda_n x)\}E_{\lambda_n}, \tag{3.14}$$

satisfies the initial condition (1.4). Note that in order to satisfy conditions (1.1)–(1.3), vectors E_{λ_n} must verify the conditions of Theorem 2.1. By definition of vector E_{λ_n}, these conditions are satisfied if

$$H_0f(x)=0,\quad H_{\lambda_n}f(x)=0,\quad (B_1-\beta I)f(x)=0,\quad 0\leqslant x\leqslant 1. \tag{3.15}$$

Note that by definition of H_{λ_n}, condition (3.15) holds if

$$(\hat{A}_2-\alpha I)A^jf(x)=0=(B_1-\beta I)A^jf(x),\quad 0\leqslant x\leqslant 1,\quad 0\leqslant j\leqslant p-1. \tag{3.16}$$

Conversely, if the conditions in (3.15) hold true, then

$$(B_1A^j-A^jB_1)f(x)=(B_1-\beta I),\quad 0\leqslant x\leqslant 1,\quad 0\leqslant j\leqslant p-1,$$

and

$$\begin{aligned}\{\hat{A}_2+\lambda^2(\hat{A}_2B_1^2-B_1)-[\alpha+\lambda^2(\alpha\beta^2-\beta)]I\}A^jf(x)\\ =\{\hat{A}_2+\lambda^2(\hat{A}_2\beta_1^2-\beta I)-[\alpha+\lambda^2(\alpha\beta^2-\beta)]I\}A^jf(x)\\ =(1+\lambda^2\beta^2)(\hat{A}_2-\alpha I)A^jf(x)=0,\quad 0\leqslant x\leqslant 1,\quad 0\leqslant j\leqslant p-1.\end{aligned}$$

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Thus, conditions (3.15) and (3.16) are equivalent, and it is clear that (3.16) is equivalent to the condition

$$\left.\begin{array}{l}(\hat{A}_2 - \alpha I)f(x) = 0 = (B_1 - \beta I)f(x),\ 0 \leqslant x \leqslant 1, \text{ and} \\ \ker(\hat{A}_2 - \alpha I) \cap \ker(B_1 - \beta I) \text{ is an invariant subspace of } A.\end{array}\right\} \tag{3.17}$$

By Lemma 1.1, taking $M = \hat{A}_2 - \alpha I$, $N = B_1 - \beta I$, condition (3.17) can be written in the compact form

$$f(x) \in \operatorname{Im} H(\alpha,\beta), \quad 0 \leqslant x \leqslant 1, \quad \text{and} \quad [I - H(\alpha,\beta)(H(\alpha,\beta))^{\dagger}]AH(\alpha,\beta) = 0, \tag{3.18}$$

where

$$\left.\begin{array}{c}H(\alpha,\beta) = (I - M_\alpha^{\dagger} M_\alpha)\{I - [N_\beta(I - M_\alpha^{\dagger} M_\alpha)]^{\dagger}[N_\beta(I - M_\alpha^{\dagger} M_\alpha)]\}, \\ M_\alpha = \hat{A}_2 - \alpha I, \quad N_\beta = B_1 - \beta I.\end{array}\right\} \tag{3.19}$$

Note that condition (3.18) means that $f(x)$ lies in $\operatorname{Im} H(\alpha,\beta)$ and that $\operatorname{Im} H(\alpha,\beta)$ is an invariant subspace of the matrix A. With respect to the convergence of the series (3.11) or (3.14)—as well as their partial differentiability with respect to the variable t once, and x twice, for $0 < x < 1$, $t > 0$—note that if $t_0 > 0$ and $D(t_0) = \{(x,t);\ 0 \leqslant x \leqslant 1, t \geqslant t_0 > 0\}$ by inequality (1.8) and condition (1.5), the series appearing by twice termwise partial differentiation with respect to x and once with respect to t, in (3.11), takes the form

$$\sum_{n\geqslant 1} \lambda_n^2 \exp(-\lambda_n^2 At) X_{\lambda_n}(x), \qquad \sum_{n\geqslant 1} (-\lambda_n^2) A \exp(-\lambda_n^2 At) X_{\lambda_n}(x),$$

and is uniformly convergent in $D(t_0)$. By the differentiation theorem of functional series [**2**, p. 403], the series (3.11) or (3.14) define rigorous solutions of problems (1.1)–(1.4), and the following result has been established.

Theorem 3.1. *Let A be a positive stable matrix in $\mathbb{C}^{m\times m}$, assume that*

$$\begin{bmatrix} I & B_1 \\ A_2 & B_2 \end{bmatrix}$$

is invertible and that there exist real numbers α and β satisfying (2.13). If $\hat{A}_2$ and $\hat{B}_2$ are defined by (2.10), $H(\alpha,\beta)$ by (3.19), and $f(x)$ is twice continuously differentiable in $[0,1]$, satisfying (3.8) and (3.18), then problems (1.1)–(1.4) admit a well-posed solution given by (3.11) or (3.14), where vectors E_{λ_n} are defined by (3.10) for $n \geqslant 1$ and by (3.13) for $n = 0$.

Remark 3.2. Condition (3.8) together with (3.18) are equivalent to

$$\left.\begin{array}{r}f(0) + B_1 f'(0) = 0, \\ A_2 f(1) + B_2 f'(1) = 0,\end{array}\right\} \tag{3.8$'$}$$

and (3.18). In fact, premultiplying the second condition of (3.8′) by $(A_2B_1 - B_2)^{-1}$ and taking into account (2.11), one gets (3.8).

Now we are interested in the construction of an exact series solution of problems (1.1)–(1.4) for more general functions $f(x)$ than those considered in Theorem 3.1. Assume that

$$\left.\begin{aligned} \Lambda = \{\alpha(1), \cdots, \alpha(k)\} \text{ are the distinct real eigenvalues of } \hat{A}_2, \\ \Omega = \{\beta(1), \cdots, \beta(k)\} \text{ are the distinct real eigenvalues of } B_1, \end{aligned}\right\} \tag{3.20}$$

and let $H(\alpha(i), \beta(j))$ be the matrix defined by (3.19) for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant s$. Recall that by Lemma 1.1, condition $\ker(\hat{A}_2 - \alpha(i)I) \cap \ker(B_1 - \beta(j)I) \neq 0$ is equivalent to the condition $H(\alpha(i), \beta(j)) \neq 0$. Consider the subset of $\Lambda \times \Omega$ defined by

$$\mathcal{S} = \{(\alpha(i_l), \beta(j_l)) \in \Lambda \times \Omega;\ H(\alpha(i_l), \beta(j_l)) \neq 0,\ 1 \leqslant l \leqslant q\}, \tag{3.21}$$

and the block matrix in $\mathbb{C}^{m \times mq}$ defined by

$$\mathcal{H} = [H(\alpha(i_1), \beta(j_1)), H(\alpha(i_2), \beta(j_2)), \cdots, H(\alpha(i_q), \beta(j_q))]. \tag{3.22}$$

Assume that $f(x)$ is twice continuously differentiable in $[0, 1]$ such that

$$(I - HH^{\dagger})f(x) = 0, \quad 0 \leqslant x \leqslant 1, \tag{3.23}$$

$$\left.\begin{aligned} H^{\dagger}(f(0) + \beta(j_l)f'(0)) = 0, \\ [0 \cdots 0\, H(\alpha(i_l), \beta(j_l))\, 0 \cdots 0]H^{\dagger}[\alpha(i_l)f(1) + (\alpha(i_l)\beta(j_l) - 1)f'(1)] = 0, \end{aligned}\right\} \quad 1 \leqslant l \leqslant q. \tag{3.24}$$

Since, by Lemma 1.1, one gets

$$\operatorname{Im} H(\alpha(i_l), \beta(j_l)) = \ker(\hat{A}_2 - \alpha(i)I) \cap \ker(B_1 - \beta(j)I), \tag{3.25}$$

then $\operatorname{Im} H$ is the direct sum of the subspaces $S_l = \operatorname{Im} H(\alpha(i_l), \beta(j_l))$, $1 \leqslant l \leqslant q$, and the projection $g_l(x)$ of the $f(x)$ on the subspace S_l is given by

$$g_l(x) = [0 \cdots 0\, H(\alpha(i_l), \beta(j_l))\, 0 \cdots 0]H^{\dagger}f(x), \quad 1 \leqslant l \leqslant q, \quad 0 \leqslant x \leqslant 1, \tag{3.26}$$

because

$$g_l(x) \in \operatorname{Im} H(\alpha(i_l), \beta(j_l)) = S_l, \tag{3.27}$$

and, by (3.23), one gets

$$\sum_{l=1}^{q} g_l(x) = HH^{\dagger}f(x), \quad 0 \leqslant x \leqslant 1. \tag{3.28}$$

By the hypothesis on $f(x)$, it follows that $g_l(x)$ is twice continuously differentiable in $[0, 1]$, and, by (3.24), one gets

$$\left.\begin{aligned} g_l(0) + \beta(j_l)g_l'(0) = 0, \\ \alpha(i_l)g_l(1) + (\alpha(i_l)\beta(j_l) - 1)g_l'(1) = 0, \quad 1 \leqslant l \leqslant q, \end{aligned}\right\} \tag{3.29}$$

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If the subspace $\operatorname{Im} H(\alpha(i_l), \beta(j_l))$ is invariant by the matrix A, or

$$I - [H(\alpha(i_l), \beta(j_l))[H(\alpha(i_l), \beta(j_l))]^{\dagger}]AH(\alpha(i_l), \beta(j_l)) = 0, \quad 1 \leqslant l \leqslant q, \tag{3.30}$$

by (3.26), (3.27) and (3.29) together with Theorem 3.1, one gets a series $U(x,t,l)$ defined by

$$U(x,t,l) = \begin{cases} \sum_{n \geqslant 1} \mathrm{e}^{(-\lambda_n^2(l)At)} \{\sin \lambda_n(l)x - \beta(j_l)\lambda_n(l) \cos \lambda_n(l)x\} E_{\lambda_n(l)} + (x - \beta(j_l))E_0(l), \\ \qquad\qquad 0 \in \mathcal{F}(\alpha(i_l), \beta(j_l)), \\ \sum_{n \geqslant 1} \mathrm{e}^{(-\lambda_n^2(l)At)} \{\sin \lambda_n(l)x - \beta(j_l)\lambda_n(l) \cos \lambda_n(l)x\} E_{\lambda_n(l)}, \\ \qquad\qquad 0 \notin \mathcal{F}(\alpha(i_l), \beta(j_l)), \end{cases} \tag{3.31}$$

where $\mathcal{F}(\alpha(i_l), \beta(j_l))$, $\lambda_n(l)$ and $E_{\lambda_n(l)}$ are given by Theorem 3.1, is a solution of problems (1.1)–(1.3) together with the initial condition

$$U(x, 0, l) = g_l(x), \quad 0 \leqslant x \leqslant 1. \tag{3.32}$$

By (3.28) and (3.32), one gets that

$$u(x,t) = \sum_{l=1}^{q} U(x,t,l), \tag{3.33}$$

is a solution of problems (1.1)–(1.4). Summarizing, the following result has been established.

Theorem 3.3. *Let A be a matrix in $\mathbb{C}^{m \times m}$ satisfying (1.5), and assume hypothesis (1.6), where $A_1 = I$. Let $\mathcal{S}$ and $\mathcal{H}$ be defined by (3.21) and (3.22), respectively. Let $\hat{A}_2$ be defined by (2.10), and $f(x)$ is a twice continuously differentiable function in $[0,1]$ satisfying (3.23) and (3.24). Under hypothesis (3.30), $u(x,t)$—defined by (3.33), where $U(x,t,l)$ is defined by (3.31), $1 \leqslant l \leqslant q$—is a solution of problems (1.1)–(1.4), with $A_1 = I$.*

Remark 3.4. Taking into account Remark 2.3, a solution of problems (1.1)–(1.4) can be constructed in an analogous way under the hypotheses (1.6) and (1.5) and certain conditions on $f(x)$.

The following example illustrates that the hypotheses of Theorem 3.3 are easy to check.

Example 3.5. Consider problems (1.1)–(1.4), where $A_1 = I$ in $\mathbb{C}^{4 \times 4}$,

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & -3 \\ -0 & 0 & 0 & -1 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 1 & -1 & 0 & -1 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Here, the block matrix

$$\begin{bmatrix} I & B_1 \\ A_2 & B_2 \end{bmatrix}$$

is invertible, with

$$\hat{A}_2 = (A_2B_1 - B_2)^{-1}A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\sigma(A) = \{1,2\}, \quad \sigma(B_1) = \{2,-1,1\} \quad \text{and} \quad \sigma(\hat{A}_2) = \{0,2\}.$$

With the above notation we have

$$M_0 = \hat{A}_2, \qquad M_2 = \hat{A}_2 - 2I = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & -2 & 0 & -1 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix},$$

$$N_2 = B_1 - 2I = \begin{bmatrix} -1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 3 \\ 0 & 0 & 0 & -3 \end{bmatrix}, \qquad N_{-1} = B_1 + I = \begin{bmatrix} 2 & 0 & 0 & -2 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$N_1 = B_1 - I = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & -2 \end{bmatrix}, \qquad M_0^\dagger = \frac{1}{6}\begin{bmatrix} 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -4 & -1 & 0 \end{bmatrix},$$

$$M_2^\dagger = \frac{1}{8}\begin{bmatrix} -2 & 0 & -2 & 0 \\ -1 & -4 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \qquad N_2(I - M_0^\dagger M_0) = \begin{bmatrix} -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 \\ -1 & 0 & -1 & -1 \end{bmatrix},$$

$$N_2(I - M_2^\dagger M_2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad N_{-1}(I - M_0^\dagger M_0) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$N_{-1}(I - M_2^\dagger M_2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad N_1(I - M_0^\dagger M_0) = \frac{2}{3}\begin{bmatrix} -1 & 0 & -1 & -1 \\ 0 & \frac{3}{2} & 0 & 0 \\ -1 & 0 & -1 & -1 \\ -1 & 0 & -1 & -1 \end{bmatrix},$$

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$$N_1(I - M_2^\dagger M_2) = O.$$

Hence,

$$[N_2(I - M_0^\dagger M_0)]^\dagger = \frac{1}{9}\begin{bmatrix} -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 \\ -1 & 0 & -1 & -1 \end{bmatrix}, \qquad [N_2(I - M_2^\dagger M_2)]^\dagger = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$[N_{-1}(I - M_0^\dagger M_0)]^\dagger = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad [N_{-1}(I - M_2^\dagger M_2)]^\dagger = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$[N_1(I - M_0^\dagger M_0)]^\dagger = \frac{1}{6}\begin{bmatrix} -1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & -1 \\ -1 & 0 & -1 & -1 \end{bmatrix}, \qquad [N_1(I - M_2^\dagger M_2)]^\dagger = O.$$

Matrices $H(\alpha, \beta)$ defined by (3.19) take the values

$$H(0,2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad H(0,-1) = \frac{1}{3}\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix},$$

$$H(2,1) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad H(2,2) = H(2,-1) = H(0,1) = O.$$

Matrix H defined by (3.22) is

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\begin{bmatrix} H(0,2) & 0 & 0 \end{bmatrix} H^\dagger = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & H(2,1) \end{bmatrix} H^\dagger = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix}0 & H(0,-1) & 0\end{bmatrix} H^{\dagger} = \frac{1}{2}\begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 1\\ 1 & 0 & 0 & 1\end{bmatrix}, \qquad I - HH^{\dagger} = \frac{1}{2}\begin{bmatrix}1 & 0 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0\end{bmatrix}.$$

If we impose on $f = (f_1, f_2, f_3, f_4)^{\mathrm{T}}$ the condition (3.23), it follows that

$$(I - HH^{\dagger})f(x) = 0, \quad \text{then } f_1(x) = f_4(x), \quad 0 \leqslant x \leqslant 1.$$

Projections $g_l(x)$ defined by (3.26) are

$$g_1(x) = \begin{bmatrix}H(0,2) & 0 & 0\end{bmatrix} H^{\dagger} f(x) = \begin{pmatrix}0 & f_2(x) & 0 & 0\end{pmatrix}^{\mathrm{T}},$$
$$g_2(x) = \begin{bmatrix}0 & H(0,-1) & 0\end{bmatrix} H^{\dagger} f(x) = \begin{pmatrix}f_1(x) & 0 & f_1(x) & f_1(x)\end{pmatrix}^{\mathrm{T}},$$
$$g_3(x) = \begin{bmatrix}0 & 0 & H(2,1)\end{bmatrix} H^{\dagger} f(x) = \begin{pmatrix}0 & 0 & f_3(x) f_1(x) & 0\end{pmatrix}^{\mathrm{T}}.$$

Since

$$[H(0,2)]^{\dagger} = \begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}, \qquad [H(0,-1)]^{\dagger} = \frac{1}{3}\begin{bmatrix}1 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\\ 1 & 0 & 1 & 1\\ 1 & 0 & 1 & 1\end{bmatrix},$$

$$[H(2,1)]^{\dagger} = \begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},$$

and

$$[I - H(0,2)[H(0,2)]^{\dagger}]AH(0,2) = 0,$$
$$[I - H(0,-1)[H(0,-1)]^{\dagger}]AH(0,-1) = 0,$$
$$[I - H(2,1)[H(2,1)]^{\dagger}]AH(2,1) = 0,$$

Thus, condition (3.30) holds true and the subspaces $\operatorname{Im} H(0,2)$, $\operatorname{Im} H(0,-1)$, and also $\operatorname{Im} H(2,1)$, are invariant by the stable matrix A. Well-posedness conditions (3.24) take the form

$$\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}(f(0) + 2f'(0)) = 0 \quad \text{or} \quad f_2(0) + 2f_2'(0) = 0,$$

$$\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} f'(1) = 0 \quad \text{or} \quad f_2'(1) = 0,$$

$$\frac{1}{2}\begin{bmatrix}1&0&0&1\\0&0&0&0\\1&0&0&1\\1&0&0&1\end{bmatrix}(f(0)-f'(0))=0 \quad\text{or}\quad f_1(0)-f_1'(0)=0,$$

$$\frac{1}{2}\begin{bmatrix}1&0&0&1\\0&0&0&0\\1&0&0&1\\1&0&0&1\end{bmatrix}f'(1)=0 \quad\text{or}\quad f_1'(1)=0,$$

$$\frac{1}{2}\begin{bmatrix}0&0&0&0\\0&0&0&0\\-1&0&2&-1\\0&0&0&0\end{bmatrix}(f(0)+f'(0))=0 \quad\text{or}\quad f_3(0)-f_1(0)+f_3'(0)-f_1'(0)=0,$$

$$\frac{1}{2}\begin{bmatrix}0&0&0&0\\0&0&0&0\\-1&0&2&-1\\0&0&0&0\end{bmatrix}(2f(1)+f'(1))=0 \quad\text{or}\quad 2f_3(1)-2f_1(1)+f_3'(1)-f_1'(1)=0.$$

Summarizing the corresponding problems (1.1)–(1.4) is well-posed and satisfies hypotheses of Theorem 3.3 if $f(x)$ is twice continuously differentiable in $[0,1]$ and verifies the conditions

$$f_1(x)=f_4(x),\quad 0\leqslant x\leqslant 1,$$
$$f_1(0)-f_1'(0)=0,\qquad f_2(0)+2f_2'(0)=0,\qquad f_3(0)+f_3'(0)=2f_1(0),$$
$$f_1'(1)=0,\qquad f_2'(1)=0,\qquad 2f_3(1)+f_3'(1)=2f_1(1).$$

4. Analytic-numerical solutions with prefixed accuracy

The series solution of problems (1.1)–(1.4) provided by Theorem 3.3 presents some computational difficulties. Firstly, the infiniteness of the series. Secondly, eigenvalues are not exactly computable because equation (2.15) is not solvable in a closed form. It is important to point out here that eigenvalues of the coupled problems (1.1)–(1.4) and eigenfunctions are built up in terms of scalar Sturm–Liouville problems of the type (3.7) or (3.29). In spite of well-known efficient numerical algorithms for the computations of eigenvalues [**18**, **24**, **25**], it is interesting to study the admissible tolerance in the approximate eigenvalues according with a prefixed accuracy. Finally, as the computation of matrix exponentials appearing in the exact solution of problems (1.1)–(1.4) is not an easy task (see [**21**]), we also approximate matrix exponentials by appropriate matrix polynomials of certain degree. In this section we address the following question. Given an admissible error $\varepsilon>0$ and a bounded subdomain $D(t_0,t_1)=\{(x,t);\ 0\leqslant x\leqslant 1,\ 0\leqslant t_0\leqslant t\leqslant t_1\}$, how do we construct an approximation that avoids the above-quoted difficulties and whose error with respect to the exact solution is less than ε uniformly in $D(t_0,t_1)$. By

Theorem 3.3 it is sufficient to develop the approach when the exact series solution is the given by Theorem 3.1.

To fix ideas we seek to approximate the series $U(x,t)$ defined by (3.11), where vector E_{λ_n} is given by (3.9)–(3.10). By applying Parseval's inequality (see [**3**, p. 223] and [**7**]) to the scalar Sturm–Liouville problem (3.7), one gets

$$|e_{\lambda_n}(i)|^2 \leqslant \int_0^1 |f_i(x)|^2 \,\mathrm{d}x, \quad n \geqslant 1, \quad 1 \leqslant i \leqslant m, \tag{4.1}$$

$$\|E_{\lambda_n}\|^2 \leqslant \sum_{j=1}^{m} \int_0^1 |f_j(x)|^2 \,\mathrm{d}x = \int_0^1 \|f(x)\|_2^2 \,\mathrm{d}x = F^2, \quad n \geqslant 1. \tag{4.2}$$

By Theorem 3.1 we have $\beta \in \sigma(B_1)$, and, by (3.17), one gets $B_1(E_{\lambda_n}) = \beta E_{\lambda_n}$ and

$$X_{\lambda_n}(x) = \{\sin(\lambda_n x) - \lambda_n \beta \cos(\lambda_n x)\} E_{\lambda_n}, \quad n \geqslant 1, \quad 0 \leqslant x \leqslant 1. \tag{4.3}$$

By (3.32)–(4.3), it follows that

$$\|X_{\lambda_n}(x)\| \leqslant F(1 + \lambda_n \|B_1\|), \quad 0 \leqslant x \leqslant 1, \quad n \geqslant 1. \tag{4.4}$$

By (1.8) for $t_1 \geqslant t \geqslant t_0$, one gets

$$\|\mathrm{e}^{-\lambda_n^2 A t}\| \leqslant \mathrm{e}^{-\beta(A) t_0 \lambda_n^2} \sum_{j=0}^{m-1} \frac{(\|A\| t_1 \sqrt{m})^j}{j!} \lambda_n^{2j}. \tag{4.5}$$

Let φ_k and ϕ_k be the scalar functions defined for $s > 0$ by

$$\varphi_k(s) = (k+2)\ln(s) - s^2 \beta(A) t_0, \quad \phi_k(s) = \mathrm{e}^{-s^2 \beta(A) t_0} s^k, \quad 0 \leqslant k \leqslant 2m-1. \tag{4.6}$$

Since

$$\varphi_k'(s) = \frac{(k+2)}{s} - 2s\beta(A) t_0,$$

it follows that

$$\varphi_k'(s) < 0, \quad \text{if } s > s_k = \left(\frac{k+2}{\beta(A) t_0}\right)^{1/2}, \quad 0 \leqslant k \leqslant 2m-1.$$

Take $s_k' \geqslant s_k$ such that

$$(k+2)\ln(s) - s^2 \beta(A) t_0 < 0, \quad s \geqslant s_k' \geqslant s_k, \quad 0 \leqslant k \leqslant 2m-1, \tag{4.7}$$

then, by (4.7), it follows that

$$\phi_k(s) = \mathrm{e}^{-s^2 \beta(A) t_0} s^k < (1 + \|B_1\|)^{-1} s^{-2}, \quad s \geqslant s_k', \quad 0 \leqslant k \leqslant 2m-1. \tag{4.8}$$

Since $\lim_{n\to\infty} \lambda_n = +\infty$ and $\lambda_n < \lambda_{n+1}$, let n_0 be the first positive integer so that

$$\lambda_{n_0} > s^* = \max\{s_k';\ 0 \leqslant k \leqslant 2m-1\}. \tag{4.9}$$

By (4.3)–(4.9), it follows that

$$\|e^{-\lambda_n^2 At} X_{\lambda_n}(x)\| \leqslant F \sum_{j=0}^{m-1} \frac{(\phi_{2j}(\lambda_n) + \|B_1\|\phi_{2j+1}(\lambda_n))}{j!} (\|A\| t_1 \sqrt{m})^j$$
$$\leqslant \lambda_n^{-2} \left(\sum_{j=0}^{m-1} \frac{(\|A\| t_1 \sqrt{m})^j}{j!} \right), \quad 0 \leqslant x \leqslant 1, \quad 0 < t_0 \leqslant t \leqslant t_1, \quad n \geqslant n_0.$$

Since $\lambda_n > (n-1)\pi$, $n \geqslant 1$ (see Remark 2.2), if we denote by L the constant

$$L = F\left(\sum_{j=0}^{m-1} \frac{(\|A\| t_1 \sqrt{m})^j}{j!} \right), \tag{4.10}$$

then

$$\left\| \sum_{n>n_0} e^{-\lambda_n^2 At} X_{\lambda_n}(x) \right\|_2 \leqslant L \sum_{n>n_0} \lambda_n^{-2} \leqslant \frac{L}{\pi^2} \sum_{n>n_0} n^{-2}. \tag{4.11}$$

Since $\sum_{n\geqslant 1} n^{-2} = \frac{1}{6}\pi^2$, taking $n_1 > n_0$ so that

$$\sum_{n=1}^{n_1} n^{-2} > \pi^2 \left(\frac{1}{6} - \frac{\varepsilon}{3L} \right), \tag{4.12}$$

by (4.11) and (4.12), one gets

$$\left\| \sum_{n>n_1} e^{-\lambda_n^2 At} X_{\lambda_n}(x) \right\|_2 \leqslant \tfrac{1}{3}\varepsilon, \quad 0 \leqslant x \leqslant 1, \quad 0 < t_0 \leqslant t. \tag{4.13}$$

Thus, the finite sum

$$V(x,t,n_1) = \sum_{n=1}^{n_1} e^{-\lambda_n^2 At} X_{\lambda_n}(x), \tag{4.14}$$

satisfies

$$\|U(x,t) - V(x,t,n_1)\|_2 < \tfrac{1}{3}\varepsilon, \quad 0 \leqslant x \leqslant 1, \quad 0 < t_0 \leqslant t. \tag{4.15}$$

The approximation $V(x,t,n_1)$ involves computation of the exact eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{n_1}$, which is not easy in practice. Now we study the admissible tolerance when one considers approximate eigenvalues $\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_{n_1}$, building up the approximation of $V(x,t,n_1)$ defined by

$$\left.\begin{aligned} \tilde{V}(x,t,n_1) &= \sum_{n=1}^{n_1} e^{-\tilde{\lambda}_n^2 At} X_{\tilde{\lambda}_n}(x), \\ X_{\tilde{\lambda}_n}(x) &= \{\sin(\tilde{\lambda}_n x) - \tilde{\lambda}_n \beta \cos(\tilde{\lambda}_n x)\} E_{\tilde{\lambda}_n}, \quad E_{\tilde{\lambda}_n} = \begin{bmatrix} e_{\tilde{\lambda}_n}(1) \\ \vdots \\ e_{\tilde{\lambda}_n}(m) \end{bmatrix}, \end{aligned}\right\} \tag{4.16}$$

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where, for $1 \leqslant i \leqslant m$, $e_{\tilde{\lambda}_n}(i)$ is defined replacing λ_n by $\tilde{\lambda}_n$ in (3.9). Note that we can write

$$\begin{aligned}
&\mathrm{e}^{-\tilde{\lambda}_n^2 At} X_{\tilde{\lambda}_n}(x), -\mathrm{e}^{-\lambda_n^2 At} X_{\lambda_n}(x) \\
&\quad = \mathrm{e}^{-\tilde{\lambda}_n^2 At}\{\sin(\tilde{\lambda}_n x) - \tilde{\lambda}_n \beta \cos(\tilde{\lambda}_n x)\} E_{\tilde{\lambda}_n} - \mathrm{e}^{-\lambda_n^2 At}\{\sin(\lambda_n x) - \lambda_n \beta \cos(\lambda_n x)\} E_{\lambda_n} \\
&\quad = (\mathrm{e}^{-\tilde{\lambda}_n^2 At} - \mathrm{e}^{-\lambda_n^2 At})\{\sin(\tilde{\lambda}_n x) - \tilde{\lambda}_n \beta \cos(\tilde{\lambda}_n x)\} E_{\tilde{\lambda}_n} \\
&\qquad + \mathrm{e}^{-\tilde{\lambda}_n^2 At}\{\sin(\tilde{\lambda}_n x) - \tilde{\lambda}_n \beta \cos(\tilde{\lambda}_n x) - \sin(\lambda_n x) + \lambda_n \beta \cos(\lambda_n x)\} E_{\tilde{\lambda}_n} \\
&\qquad + \mathrm{e}^{-\lambda_n^2 At}\{\sin(\lambda_n x) - \lambda_n \beta \cos(\lambda_n x)\}(E_{\tilde{\lambda}_n} - E_{\lambda_n}).
\end{aligned}$$

Let $I(\rho)$ be defined by

$$I(\rho) = \int_0^1 \{\sin(\rho x) - \beta\rho\cos(\rho x)\}^2 \,\mathrm{d}x, \quad \rho > 0, \tag{4.17}$$

and let γ, Λ and Λ_1 be positive constants chosen so that

$$\left.\begin{aligned}
\inf\{I(\rho);\ \rho = \lambda_n,\ \rho = \tilde{\lambda}_n,\ 1 \leqslant n \leqslant n_1\} \geqslant \gamma^{-1}; \\
0 < \Lambda_1 < \min\{\lambda_1, \tilde{\lambda}_1\}, \quad \max\{\lambda_n, \tilde{\lambda}_n;\ 1 \leqslant n \leqslant n_1\} \leqslant \Lambda.
\end{aligned}\right\} \tag{4.18}$$

It is easy to show that

$$\left.\begin{aligned}
|\sin(\tilde{\lambda}_n x) - \tilde{\lambda}_n \beta \cos(\tilde{\lambda}_n x) - \sin(\lambda_n x) + \lambda_n \beta \cos(\lambda_n x)| \leqslant (1 + \beta| + |\beta|\lambda_n)|\lambda_n - \tilde{\lambda}_n|, \\
|\sin(\tilde{\lambda}_n x) - \tilde{\lambda}_n \beta \cos(\tilde{\lambda}_n x)| \leqslant 1 + |\tilde{\lambda}_n|, \quad 0 \leqslant x \leqslant 1.
\end{aligned}\right\} \tag{4.19}$$

By (3.9), for $1 \leqslant i \leqslant m$, one gets

$$\begin{aligned}
e_{\lambda_n}(i) - e_{\tilde{\lambda}_n}(i) &= \frac{(I(\lambda_n) - I(\tilde{\lambda}_n)) \int_0^1 f_i(x)\{\sin(\tilde{\lambda}_n x) - \tilde{\lambda}_n \beta \cos(\tilde{\lambda}_n x)\}\,\mathrm{d}x}{I(\lambda_n) I(\tilde{\lambda}_n)} \\
&\quad + \frac{\int_0^1 f_i(x)\{\sin(\lambda_n x) - \lambda_n \beta \cos(\lambda_n x) - \sin(\tilde{\lambda}_n x) + \tilde{\lambda}_n \beta \cos(\tilde{\lambda}_n x)\}\,\mathrm{d}x}{I(\lambda_n)};
\end{aligned} \tag{4.20}$$

by the Cauchy–Schwarz inequality for integrals it follows that

$$\int_0^1 |f_i(x)\{\sin(\tilde{\lambda}_n x) - \tilde{\lambda}_n \beta \cos(\tilde{\lambda}_n x)\}|\,\mathrm{d}x \leqslant \left(\int_0^1 |f_i(x)|^2\,\mathrm{d}x\right)^{1/2} (I(\tilde{\lambda}_n))^{1/2}; \tag{4.21}$$

and by (4.19)

$$\begin{aligned}
&\int_0^1 |f_i(x)\{\sin(\lambda_n x) - \lambda_n \beta \cos(\lambda_n x) - \sin(\tilde{\lambda}_n x) + \tilde{\lambda}_n \beta \cos(\tilde{\lambda}_n x)\}|\,\mathrm{d}x \\
&\qquad \leqslant (1 + \|B_1\|(1 + \lambda_n))\left(\int_0^1 |f_i(x)|^2\,\mathrm{d}x\right)^{1/2} |\lambda_n - \tilde{\lambda}_n|, \quad 1 \leqslant i \leqslant m.
\end{aligned} \tag{4.22}$$

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By (4.20)–(4.22) it follows that

$$|e_{\lambda_n}(i) - e_{\tilde{\lambda}_n}(i)| \leqslant \left\{ \frac{|I(\lambda_n) - I(\tilde{\lambda}_n)|}{(I(\tilde{\lambda}_n))^{1/2}} + |\lambda_n - \tilde{\lambda}_n|(1 + \|B_1\|(1+\lambda_n)) \right\} \frac{(\int_0^1 |f_i(x)|^2 \, \mathrm{d}x)^{1/2}}{I(\lambda_n)}.$$

Note that

$$I(\lambda_n) - I(\tilde{\lambda}_n) = \int_0^1 (\sin(\lambda_n x) - \lambda_n \beta \cos(\lambda_n x) + \sin(\tilde{\lambda}_n x) - \tilde{\lambda}_n \beta \cos(\tilde{\lambda}_n x)) \times (\sin(\lambda_n x) - \lambda_n \beta \cos(\lambda_n x) - \sin(\tilde{\lambda}_n x) + \tilde{\lambda}_n \beta \cos(\tilde{\lambda}_n x)) \, \mathrm{d}x,$$

and by (4.18) and (4.19) one gets

$$|I(\lambda_n) - I(\tilde{\lambda}_n)| \leqslant |\lambda_n - \tilde{\lambda}_n|(1 + \|B_1\|(1 + \lambda_n + \tilde{\lambda}_n))^2,$$

$$|e_{\lambda_n}(i) - e_{\tilde{\lambda}_n}(i)| \leqslant (1 + (I(\lambda_n))^{-1/2})(I(\lambda_n))^{-1}(1 + \|B_1\|(1 + \lambda_n + \tilde{\lambda}_n))^2 \times \left(\int_0^1 |f_i(x)|^2 \, \mathrm{d}x \right)^{1/2} |\lambda_n - \tilde{\lambda}_n|,$$

$$|e_{\lambda_n}(i) - e_{\tilde{\lambda}_n}(i)| \leqslant 4(\gamma^{1/2} + 1)\gamma(1 + \|B_1\|)^2(1 + \Lambda) \times \left(\int_0^1 |f_i(x)|^2 \, \mathrm{d}x \right)^{1/2} |\lambda_n - \tilde{\lambda}_n|, \quad 1 \leqslant n \leqslant n_1,$$

$$\|E_{\lambda_n} - E_{\tilde{\lambda}_n}\|_2 \leqslant 4(\gamma^{1/2} + 1)\gamma(1 + \|B_1\|)^2(1 + \Lambda) \times \left(\int_0^1 \|f_i(x)\|_2^2 \, \mathrm{d}x \right)^{1/2} |\lambda_n - \tilde{\lambda}_n|, \quad 1 \leqslant n \leqslant n_1. \tag{4.23}$$

By definition of $E_{\tilde{\lambda}_n}$ we have

$$\|E_{\tilde{\lambda}_n}\|_2 \leqslant 2(1 + \tilde{\lambda}_n \|B_1\|) \left(\int_0^1 \|f_i(x)\|_2^2 \, \mathrm{d}x \right)^{1/2}, \quad 1 \leqslant n \leqslant n_1. \tag{4.24}$$

By (1.8) and (4.18) one gets

$$\left. \begin{aligned} \|\mathrm{e}^{-\lambda_n^2 A t}\| &\leqslant \mathrm{e}^{-t_0 \beta(A) \Lambda_1^2} \sum_{j=0}^{m-1} \frac{(\Lambda^2 t_1 \|A\| \sqrt{m})^j}{j!}, \\ \left\|\mathrm{e}^{-\tilde{\lambda}_n^2 A t}\right\| &\leqslant \mathrm{e}^{-t_0 \beta(A) \Lambda_1^2} \sum_{j=0}^{m-1} \frac{(\Lambda^2 t_1 \|A\| \sqrt{m})^j}{j!}, \quad 1 \leqslant n \leqslant n_1, \quad t_0 \leqslant t \leqslant t_1. \end{aligned} \right\} \tag{4.25}$$

Let us write

$$\mathrm{e}^{-t\lambda_n^2 A} - \mathrm{e}^{-t\tilde{\lambda}_n^2 A} = \mathrm{e}^{-t\tilde{\lambda}_n^2 A}(\mathrm{e}^{-t(\lambda_n^2 - \tilde{\lambda}_n^2)A} - I).$$

By (1.8), (4.18) and the mean value theorem, under the hypothesis $|\lambda_n - \tilde{\lambda}_n| < 1$ one gets

$$\|\mathrm{e}^{-t\lambda_n^2 A} - \mathrm{e}^{-t\tilde{\lambda}_n^2 A}\| \leqslant \|\mathrm{e}^{-t\tilde{\lambda}_n^2 A}\|(\mathrm{e}^{-t(\lambda_n^2-\tilde{\lambda}_n^2)\|A\|} - 1), \quad 1 \leqslant n \leqslant n_1,$$

$$\|\mathrm{e}^{-t\lambda_n^2 A} - \mathrm{e}^{-t\tilde{\lambda}_n^2 A}\| \leqslant \left(\mathrm{e}^{-t_0\beta(A)\Lambda_1^2}\left[\sum_{j=0}^{m-1}\frac{(\Lambda^2 t_1\|A\|\sqrt{m})^j}{j!}\right]4\Lambda\|A\|t_1\mathrm{e}^{2\|A\|\Lambda t_1}\right)|\lambda_n - \tilde{\lambda}_n|. \tag{4.26}$$

By (4.16), (4.23), (4.24), (4.25) and (4.26), assuming that $|\lambda_n - \tilde{\lambda}_n| < 1$, $1 \leqslant n \leqslant n_1$, $t_0 \leqslant t \leqslant t_1$, it follows that

$$\|\mathrm{e}^{-t\tilde{\lambda}_n^2 A}X_{\tilde{\lambda}_n}(x) - \mathrm{e}^{-t\lambda_n^2 A}X_{\lambda_n}(x)\|_2 \leqslant K|\lambda_n - \tilde{\lambda}_n|, \quad 1 \leqslant n \leqslant n_1, \quad t_0 \leqslant t \leqslant t_1, \tag{4.27}$$

where

$$\left.\begin{aligned} K &= 4\left[\sum_{j=0}^{m-1}\frac{(\Lambda^2 t_1\|A\|\sqrt{m})^j}{j!}\right](1+\Lambda)^2\mathrm{e}^{-t_0\beta(A)\Lambda_1^2}K_1,\\ K_1 &= (\gamma^{1/2}+1)\gamma(1+\|B_1\|)^2 + \|A\|t_1\mathrm{e}^{2\|A\|\Lambda t_1} + \left(\int_0^1 \|f_i(x)\|_2^2\,\mathrm{d}x\right)^{1/2}. \end{aligned}\right\} \tag{4.28}$$

Given $\varepsilon > 0$ and n_1, consider approximations $\tilde{\lambda}_n$ of λ_n for $1 \leqslant n \leqslant n_1$, so that

$$|\lambda_n - \tilde{\lambda}_n| < \min\left(1, \frac{\varepsilon}{3n_1 K}\right), \quad 1 \leqslant n \leqslant n_1,$$

then by (4.14), (4.16), (4.27) and (4.28) it follows that

$$\|V(x,t,n_1) - \tilde{V}(x,t,n_1)\|_2 < \tfrac{1}{3}\varepsilon, \quad t_0 \leqslant t \leqslant t_1, \quad 0 \leqslant x \leqslant 1. \tag{4.29}$$

By Theorem 11.2.4 of [**11**, p. 550], one gets

$$\left\|\mathrm{e}^{-t\tilde{\lambda}_n^2 A} - \sum_{k=0}^{q}\frac{(-\tilde{\lambda}_n^2 tA)^k}{k!}\right\| \leqslant \frac{m}{(q+1)!}(\tilde{\lambda}_n^2 t_1\|A\|)^{q+1}\mathrm{e}^{t_1\tilde{\lambda}_n^2\|A\|}, \quad t_0 \leqslant t \leqslant t_1; \tag{4.30}$$

and by (4.18) and (4.19),

$$\|X_{\tilde{\lambda}_n}(x)\|_2 \leqslant 2(1+\Lambda\|B_1\|)^2\left(\int_0^1 \|f(x)\|_2^2\,\mathrm{d}x\right)^{1/2}, \quad 1 \leqslant n \leqslant n_1, \quad 0 \leqslant x \leqslant 1.$$

Since

$$\lim_{q\to\infty}\frac{(\Lambda^2 t_1\|A\|)^{q+1}}{(q+1)!} = 0,$$

take the first positive integer q_0 such that

$$\frac{(\Lambda^2 t_1\|A\|)^{q_0+1}}{(q_0+1)!} < \frac{\varepsilon}{6n_1\mathrm{e}^{\|A\|\Lambda^2 t_1}(1+\Lambda\|B_1\|)^2(\int_0^1 \|f(x)\|_2^2\,\mathrm{d}x)^{1/2}}, \tag{4.31}$$

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then, if we define

$$\tilde{u}(x,t,n_1,q_0) = \sum_{n=1}^{n_1}\sum_{k=0}^{q_0} \frac{(-\tilde{\lambda}_n^2 tA)^k}{k!} X_{\tilde{\lambda}_n}(x), \tag{4.32}$$

by (4.16), (4.31) and (4.32) one gets

$$\|\tilde{V}(x,t,n_1) - \tilde{u}(x,t,n_1,q_0)\|_2 < \tfrac{1}{3}\varepsilon, \quad t_0 \leqslant t \leqslant t_1, \quad 0 \leqslant x \leqslant 1, \tag{4.33}$$

and by (4.15), (4.29) and (4.33) one concludes that

$$\|U(x,t) - \tilde{u}(x,t,n_1,q_0)\|_2 < \varepsilon, \qquad t_0 \leqslant t \leqslant t_1, \quad 0 \leqslant x \leqslant 1. \tag{4.34}$$

Summarizing, the following result has been established.

Theorem 4.1. *With the hypotheses and the notation of Theorem 3.1, let* $\varepsilon > 0$, $t_0 > 0$ *and* $D(t_0,t_1) = \{(x,t);\ 0 \leqslant x \leqslant 1,\ t_0 \leqslant t \leqslant t_1\}$. *Let* γ, Λ *and* Λ_1 *be defined by (4.18). Let* n_1 *be chosen by (4.12) and* q_0 *by (4.31). Let* $\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_{n_1}$ *be approximations of eigenvalues* $\lambda_1, \lambda_2, \dots, \lambda_{n_1}$ *satisfying*

$$|\lambda_n - \tilde{\lambda}_n| < \min\left(1, \frac{\varepsilon}{3n_1K}\right), \quad 1 \leqslant n \leqslant n_1,$$

where K *is given by (4.28). Then* $\tilde{u}(x,t,n_1,q_0)$, *defined by (4.32), is an approximation of the exact solution* $U(x,t)$ *of problems (1.1)–(1.4), given by Theorem 3.1, satisfying (4.34).*

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