Let $NP = R$ and have projections X and Y on OX and OY respectively. Then if NP makes an angle θ with OX

$$X = R\cos\theta \qquad Y = R\sin\theta$$

$$\therefore \quad R = X\cos\theta + Y\sin\theta.$$

But $$\tan\theta = \text{gradient of } NP = \frac{b}{a}$$

$$\therefore \quad \cos\theta = \frac{a}{d} \quad \text{and} \quad \sin\theta = \frac{b}{d} \text{ where } d = \pm\sqrt{a^2+b^2}.$$

Again, if N is (α, β), $X = x' - \alpha$ and $Y = y' - \beta$,

$$\therefore \quad R = \frac{a(x'-\alpha) + b(y'-\beta)}{d}$$

$$= \frac{ax' + by' + c}{d}$$

since $$a\alpha + b\beta + c = 0.$$

Also $$x' - \alpha = R\cos\theta = \frac{a(ax'+by'+c)}{d^2}$$

Hence $$\alpha = x' - \frac{a(ax'+by'+c)}{a^2+b^2}$$

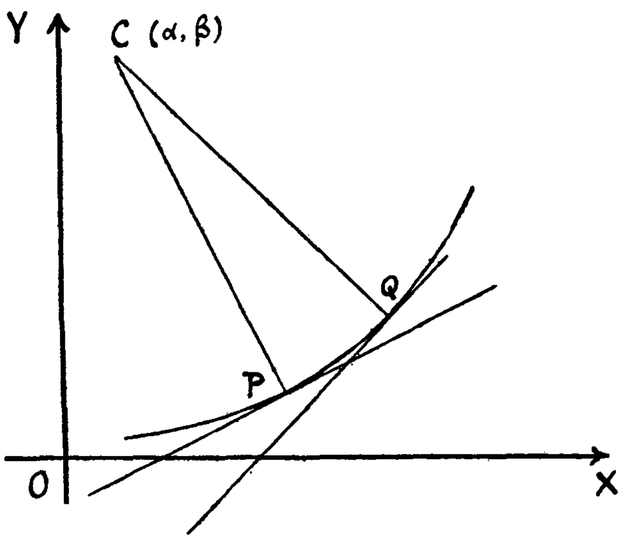
and $$\beta = y' - \frac{b(ax'+by'+c)}{a^2+b^2}$$

R. J. T. Bell.

Note on the Determination of Centres of Curvature. —For determining the Cartesian co-ordinates of the centre of curvature of a plane curve two methods are principally used in the text-books. One of these (see for example, Edwards, "Differential Calculus," p 266, § 339) having previously established the formula for the radius of curvature, derives the coordinates of the centre of curvature by using the circular functions of the angle "ψ" which the tangent to the curve at the point considered makes with OX. Since, however, for the same tangent, and therefore the same "ψ," the curve may be either convex or concave towards OX (and accordingly the centre of curvature have quite different positions), this method introduces an essential ambiguity, and is not to be recommended.

The second method (Edwards, p. 279, § 353) is developed by finding the equation of the osculating circle to the curve, and so determining its centre and radius free from any possible ambiguity. But this requires some previous considerable discussion of Contact of Plane Curves, and so is unsuitable for elementary classes.

The following method, which determines centre and radius of curvature together, *seems* to me at once free from ambiguity and at the same time fairly elementary; it assumes, in fact, no more than that the limiting position of the intersection of the normals at two points P, Q on a curve tends, as Q approaches P indefinitely closely, to a definite fixed position on the normal at P. The method is not, so far as I am aware, indicated in any of the text-books.



I. Let PC be the normal at any point $P(x, y)$ of the curve, $C(\alpha, \beta)$ the corresponding centre of curvature, and y' the gradient of the tangent at P. Then

$$y' = -\frac{x-\alpha}{y-\beta} \quad \text{.....................................} \quad \text{(i)}$$

and, differentiating this equation,

$$y'' = -\frac{(y-\beta)-(x-\alpha)\,y'}{(y-\beta)^2}$$

$$= -\frac{1-\dfrac{x-\alpha}{y-\beta}\,y'}{(y-\beta)}$$

$$= -\frac{1+y'^2}{y-\beta}, \text{ from (i).}$$

$\therefore$ Solving for β,

$$\beta = y + \frac{1+y'^2}{y''}, \quad \text{..................................} \quad \text{(ii)}$$

and substituting in (i) we find

$$\alpha = x - \frac{y'(1+y'^2)}{y''} \quad \text{..............................} \quad \text{(iii)}$$

which are the usual expressions for the coordinates of the centre of curvature.

Then for the radius of curvature,

$$\rho \equiv PC = \sqrt{(x-\alpha)^2+(y-\beta)^2}$$
$$= \{1+y'^2\}^{\frac{3}{2}} / y'', \quad \text{..........................} \quad \text{(iv)}$$

as usual, and the equation to the osculating circle can at once be written down.

II. At first sight it might perhaps appear that this process determined the coordinates of an arbitrary fixed point, C, on the normal at P, which we had, without authority, *called* the centre of curvature. That this is not so is at once evident on appeal to the limit-idea involved in the differentiation of equation (i).

For that differentiation may be written

$$\underset{\delta x \to 0}{\mathcal{L}} \frac{(y+\delta y)' - (y)'}{\delta x} = \underset{\delta x \to 0}{\mathcal{L}} - \frac{\dfrac{x+\delta x - \alpha}{y + \delta y - \beta} - \dfrac{x-\alpha}{y-\beta}}{\delta x},$$

in which form it is readily recognisable as the expression of the simultaneous solution of the equations of the normals at $P(x, y)$ and $Q\ (x+\delta x,\ y+\delta y)$, δx tending towards zero.

The only assumption involved is, therefore, as stated, that of the definition of the centre of curvature as the limiting position of the intersection of these normals.

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