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Rotation intervals for a class of maps of the real line into itself

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Abstract. We study a class of maps of the real line into itself which are degree one liftings of maps of the circle and have discontinuities only of a special type. This class contains liftings of continuous degree one maps of the circle, lifting of increasing mod 1 maps and some maps arising from Newton's method of solving equations. We generalize some results known for the continuous case.

0. *Introduction*

We study liftings of maps of a circle into itself which are not necessarily continuous. If a map of a circle is discontinuous, then its lifting to the map of a real line into itself is not determined uniquely up to shifts by integers, as in the case of continuous maps. Therefore some notions used here, as rotation numbers, will actually depend on the lifting, not only on the map of the circle itself.

If a map of a circle is discontinuous, then it is only a matter of introducing a few more discontinuities to consider it as a map of an interval into itself; conversely, a map of an interval into itself can be considered as a map of a circle into itself.

Although throughout most of this paper the maps of the real line are investigated, this is only a means of understanding the dynamics of underlying maps of a circle or of an interval.

1. *Notation, definitions, statement of results*

The points of the real line $\mathbb{R}$ will be denoted usually by capital letters X, Y, Z, T; the points of the circle $S^1=\{z\in\mathbb{C}: |z|=1\}$ by small letters x, y, z; the integers (elements of $\mathbb{Z}$) by small letters i, j, k, l, m, n, p, q, r, s, t, u, v and Greek ν. If we write p/q then we always mean that p, $q\in\mathbb{Z}$ and $q>0$; if we write $n>0$ or $n\geq 0$, we mean that additionally $n\in\mathbb{Z}$. The largest common divisor of p and q will be denoted by (p, q). If φ is a map and (X, Y) an interval then instead of $\varphi((X, Y))$ we shall write $\varphi(X, Y)$.

We denote by $e:\mathbb{R}\to S^1$ the natural projection $e(X)=\exp(2\pi iX)$ (here exceptionally $i=\sqrt{-1}$).

A map $F:\mathbb{R}\to\mathbb{R}$ is called *a lifting* of a map $f: S^1\to S^1$ if $e\circ F=f\circ e$ and there is $k\in\mathbb{Z}$ such that $F(X+1)=F(X)+k$ for all $X\in\mathbb{R}$. This k is called *the degree* of F.

Note that since we do not say anything about continuity here, every f has liftings of all degrees.

A map $F:\mathbb{R}\to\mathbb{R}$ will be called *an old map* (old stands for 'degree one lifting' with the order of letters changed for mnemonic reasons) if $F(X+1)=F(X)+1$ for all $X\in\mathbb{R}$. Clearly, F is an old map if and only if there exists $f: S^1\to S^1$ such that F is a lifting of f of degree one. It is easy to see that if F is an old map then $F(X+k)=F(X)+k$ for all $X\in\mathbb{R}$ and $k\in\mathbb{Z}$ and that iterates of an old map are old maps.

We shall say that a point $X\in\mathbb{R}$ is *periodic* mod 1 of *period* q and *rotation number* p/q for an old map F if $F^q(X)-X=p$ and $F^i(X)-X\notin\mathbb{Z}$ for $i=1,2,\dots,q-1$. Clearly, if F is a lifting of f then X is periodic mod 1 for F if and only if $e(X)$ is periodic for f and their periods are equal.

A map $F:\mathbb{R}\to\mathbb{R}$ will be called *heavy* if for every $X\in\mathbb{R}$ the finite limits

$$F(X+)=\lim_{Y\searrow X}F(Y) \quad\text{and}\quad F(X-)=\lim_{Y\nearrow X}F(Y)$$

exist, and $F(X-)\geq F(X)\geq F(X+)$ (a heavy map can fall down but cannot jump up).

Notice that a heavy map is bounded on bounded sets. Notice also that an iterate of a heavy map need not be heavy.

For an old map F we set

$$a(F)=\inf_{X\in\mathbb{R}}\liminf_{n\to\infty}\frac{1}{n}(F^n(X)-X),$$

$$b(F)=\sup_{X\in\mathbb{R}}\limsup_{n\to\infty}\frac{1}{n}(F^n(X)-X).$$

For a heavy map F we define maps F_l, F_r by

$$F_l(X)=\inf\{F(Y):\ Y\geq X\},$$
$$F_r(X)=\sup\{F(Y):\ Y\leq X\}$$

(cf. [**1**]).

We prove the following theorems.

THEOREM A. *Let $F:\mathbb{R}\to\mathbb{R}$ be an old heavy map. Then*

(a) *if F has a periodic mod 1 point of rotation number p/q then $a(F)\leq p/q\leq b(F)$;*

(b) *if $a(F)<p/q<b(F)$ then F has a periodic mod 1 point of period q and rotation number p/q.*

THEOREM B. *Let $t\mapsto F_t$ be a map from an interval into the space of old heavy maps such that the maps $t\mapsto(F_t)_l$ and $t\mapsto(F_t)_r$ are continuous, ($(F_t)_l$ and $(F_t)_r$ are regarded as elements of the space of maps of $\mathbb{R}$ into itself with the topology of uniform convergence). Then the maps $t\mapsto a(F_t)$ and $t\mapsto b(F_t)$ are also continuous.*

THEOREM C. *Let $F:\mathbb{R}\to\mathbb{R}$ be an old heavy map. If $a(F)$ ($b(F)$) is irrational then for all $\theta>0$ we have $a(F+\theta)>a(F)$ (respectively $b(F+\theta)>b(F)$), where the map $F+\theta$ is defined by $(F+\theta)(X)=F(X)+\theta$.*

THEOREM D. *Let* $F:\mathbb{R}\to\mathbb{R}$ *be an old heavy map, and let* $a(F)\le\alpha\le\beta\le b(F)$. *Then there exists* $T\in\mathbb{R}$ *such that*

$$\liminf_{n\to\infty}\frac{1}{n}(F^n(T)-T)=\alpha,$$

$$\limsup_{n\to\infty}\frac{1}{n}(F^n(T)-T)=\beta.$$

All the above theorems show that the situation for old heavy maps is similar to the one for continuous maps of degree one of a circle. The interval $[a(F), b(F)]$ may be called the *rotation interval* of F. Then theorem A gives a result analogous (although weaker) to the theorem of Šarkovskiĭ type for continuous maps of degree one [**6**]. It must also be related to the results of Hofbauer [3] for monotone mod 1 maps. Theorem B is a generalization of the result of Newhouse, Palis and Takens [**8**]. Theorem C is a generalization of the result of Ito [**5**]. Theorem D shows that in our case the rotation set is closed (as in the continuous case, see [**4**]), even in a strong sense (see [**7**, corollary 1.5]). It also generalizes the result of Bamon, Malta, Pacifico and Takens [**2**].

2. *Heavy maps*

We prove several lemmas on heavy maps.

LEMMA 2.1. *Let F be a heavy map. Then for every* $X\in\mathbb{R}$

(a) $\lim_{\varepsilon\to 0}(\sup\{F(Y): |X-Y|\le\varepsilon\})=F(X-)$;

(b) $\lim_{\varepsilon\to 0}(\inf\{F(Y): |X-Y|\le\varepsilon\})=F(X+)$.

Proof. Fix an arbitrary $\delta>0$. Then there exists $\varepsilon>0$ such that if $Y\in[X-\varepsilon, X)$ then

$$F(X-)-\delta\le F(Y)\le F(X-)+\delta,$$

and if $Y\in(X, X+\varepsilon]$ then

$$F(X+)-\delta\le F(Y)\le F(X+)+\delta.$$

Since $F(X-)\ge F(X)\ge F(X+)$, $|X-Y|\le\varepsilon$ implies $F(X+)-\delta\le F(Y)\le F(-)+\delta$. Therefore

$$\lim_{\varepsilon\to 0}(\sup\{F(Y): |X-Y|\le\varepsilon\})\le F(X-)$$

and

$$\lim_{\varepsilon\to 0}(\inf\{F(Y): |X-Y|\le\varepsilon\})\ge F(X+)$$

The reverse inequalities are obvious. □

LEMMA 2.2. *Let F be a heavy map. Then the maps* F_l *and* F_r *are continuous and non-decreasing.*

Proof. We prove the statement for F_r; the proof for F_l is analogous.

If $X_1\le X_2$ then $\{Y: Y\le X_1\}\subset\{Y: Y\le X_2\}$ and therefore $F_r(X_1)\le F_r(X_2)$. Hence, F_r is non-decreasing.

If $X\in\mathbb{R}$ and $\varepsilon>0$ then

$$F_r(X+\varepsilon)=\max(F_r(X-\varepsilon), \sup\{F(Y): |X-Y|\le\varepsilon\}).$$

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By lemma 2.1 we obtain

$$\lim_{\varepsilon\to 0} F_r(X+\varepsilon)=\max\,(\lim_{\varepsilon\to 0} F_r(X-\varepsilon), F(X-)).$$

For every $Y\in\mathbb{R}$ we have $F(Y)\le F_r(Y)$ and therefore $F(X-)\le \lim_{\varepsilon\to 0} F_r(X-\varepsilon)$. Hence,

$$\lim_{\varepsilon\to 0} F_r(X+\varepsilon)=\lim_{\varepsilon\to 0} F_r(X-\varepsilon).$$

Thus, F_r is continuous. □

For a map $G:\mathbb{R}\to\mathbb{R}$ we denote by Const (G) the union of all open intervals on which G is constant.

LEMMA 2.3. *Let F be a heavy map. Then*
(a) *if* $X\in\mathbb{R}\setminus\text{Const}\,(F_l)$ *then* $F_l(X)=F(X+)$;
(b) *if* $X\in\mathbb{R}\setminus\text{Const}\,(F_r)$ *then* $F_r(X)=F(X-)$.

Proof. We prove (b); the proof of (a) is analogous.

Suppose that $F_r(X)\ne F(X-)$. Since for every $Y\in\mathbb{R}$ we have $F(Y)\le F_r(Y)$, we obtain $F(X-)<F_r(X)$. Therefore, by lemma 2.1(a), if ε is sufficiently small and $|X-Y|\le\varepsilon$, then $F(Y)<F_r(X)$. Hence, if $|X-Y|<\varepsilon$ then $F_r(Y)=F_r(X)$, and consequently $X\in\text{Const}\,(F_r)$. □

LEMMA 2.4. *Let F be a heavy map. Let* $X_i<Y_i$, $F(X_i+)\le X_{i+1}$, $F(Y_i-)\ge Y_{i+1}$ *for* $i=0,1,2,\ldots$. *Then there exist increasing maps* $\psi_i^j:(X_j,Y_j)\to(X_i,Y_i)$ *for all i, j with* $0\le i\le j$, *such that*
(i) $\psi_i^j\circ\psi_j^k=\psi_i^k$ *if* $0\le i\le j\le k$;
(ii) $F^{j-i}\circ\psi_i^j=\text{id}_{(X_j,Y_j)}$ *if* $0\le i\le j$;
(iii) $\psi_i^j(Z)=\psi_i^j(Z+)$ *if* $0\le i\le j$, $Z\in(X_j,Y_j)$;
(iv) *if* $0\le i\le l<j$ *and* $F(X_l+)<X_{l+1}$, *then*

$$\inf\psi_0^i(X_i,Y_i)<\inf\psi_0^j(X_j,Y_j);$$

(v) *if* $0\le i\le l<j$ *and* $F(Y_l-)>Y_{l+1}$, *then*

$$\sup\psi_0^i(X_i,Y_i)>\sup\psi_0^j(X_j,Y_j).$$

Proof. Fix $i\ge 0$. We shall prove first that there exists an increasing map $\varphi_i:(X_{i+1},Y_{i+1})\to(X_i,Y_i)$ such that
(ii′) $F\circ\varphi_i=\text{id}_{(X_{i+1},Y_{i+1})}$;
(iii′) $\varphi_i(Z)=\varphi_i(Z+)$ if $Z\in(X_{i+1},Y_{i+1})$;
(iv′) if $F(X_i+)<X_{i+1}$ then $\inf\varphi_i(X_{i+1},Y_{i+1})>X_i$;
(v′) if $F(Y_i-)>Y_{i+1}$ then $\sup\varphi_i(X_{i+1},Y_{i+1})<Y_i$.

It is easy to see that a map G, defined by

$$G(Z)=\begin{cases}F(X_i+) & \text{if } Z\le X_i,\\ F(Z) & \text{if } X_i<Z<Y_i,\\ F(Y_i-) & \text{if } Z\ge Y_i,\end{cases}$$

is a heavy map. By lemma 2.2, G_r is non-decreasing and continuous, and hence $G_r(\mathbb{R})\supset[F(X+),F(Y-)]$. Since $(X_{i+1},Y_{i+1})\subset(F(X+),F(Y-))$, we can define

φ_i by

$$\varphi_i(Z) = \sup\{T: G_r(T) = Z\}$$

and we have $\varphi_i(Z) \in (X_i, Y_i)$ for all $Z \in (X_{i+1}, Y_{i+1})$.

By definition, if $Z \in (X_{i+1}, Y_{i+1})$, then $\varphi_i(Z) \in (X_i, Y_i) \setminus \mathrm{Const}(G_r)$. By lemma 2.3, we have then

$$G_r(\varphi_i(Z)) = G(\varphi_i(Z)-).$$

If $G(\varphi_i(Z)-) > G(\varphi_i(Z))$ then also $G(\varphi_i(Z)-) > G(\varphi_i(Z)+)$, and consequently if ε is sufficiently small then $G_r(\varphi_i(Z)+\varepsilon) = G_r(\varphi_i(Z))$. This contradicts the definition of $\varphi_i(Z)$. Hence, $G(\varphi_i(Z)-) = G(\varphi_i(Z))$. Thus, we obtain

$$Z = G_r(\varphi_i(Z)) = G(\varphi_i(Z)-) = G(\varphi_i(Z)) = F(\varphi_i(Z)).$$

This proves (ii′).

Since G_r is non-decreasing, φ_i is also non-decreasing. By the definition, it is one-to-one. Hence, it is increasing.

Since φ_i is increasing, $\varphi_i(Z+) \geq \varphi_i(Z)$ for $Z \in (X_{i+1}, Y_{i+1})$. By the continuity of G_r and the definition of φ_i, we obtain equality. This proves (iii′).

Assume that $F(X_i+) < X_{i+1}$. Then $G(X_i) < X_{i+1}$, and since G_r is continuous, we obtain (iv′). The proof of (v′) is analogous.

Now, if $0 \leq i = j$ then we set $\psi_i^j = \mathrm{id}_{(X_i, Y_i)}$; if $0 \leq i < j$ then we set $\psi_i^j = \varphi_i \circ \varphi_{i+1} \circ \cdots \circ \varphi_{j-1}$. The map ψ_i^j is increasing because all φ_i are increasing (the identity map is also increasing).

The property (i) is satisfied by the definition of the maps ψ_i^j. The property (ii) follows from (ii′); the property (iii) from (iii′) and the fact that all φ_k are increasing.

We prove (iv). Write for $0 \leq r \leq s$,

$$\alpha_r^s = \inf \psi_r^s(X_s, Y_s).$$

If $0 \leq m \leq r \leq s \leq t$ and $\alpha_r^s < \alpha_r^t$ ($\alpha_r^s \leq \alpha_r^t$) then, since ψ_m^r is increasing and by (i), we have $\alpha_m^s < \alpha_m^t$ (respectively $\alpha_m^s \leq \alpha_m^t$). In particular, since for $0 \leq r \leq s \leq t$ always $\alpha_s^s \leq \alpha_s^t$ (clearly, $\alpha_s^s = X_s$), we obtain then $\alpha_r^s \leq \alpha_r^t$. Thus, if $0 \leq i \leq l < j$ and $F(X_l+) < X_{l+1}$ then by (iv′), $\alpha_l^l < \alpha_l^{l+1}$. Hence,

$$\alpha_0^i \leq \alpha_0^l < \alpha_0^{l+1} \leq \alpha_0^j,$$

which proves (iv). The proof of (v) is analogous □

3. *Old maps*

We shall use the following three very simple lemmas (see e.g. [**6**], [**7**], [**1**]).

LEMMA 3.1. *Let $F, G: \mathbb{R} \to \mathbb{R}$ be maps such that $F \leq G$. If either F or G is non-decreasing then $F^n \leq G^n$ for all $n \geq 0$.*

LEMMA 3.2. *Let G be an old continuous non-decreasing map. Then for every $X \in \mathbb{R}$ the limit*

$$\lim_{n\to\infty} \frac{1}{n}(G^n(X) - X)$$

exists and is independent of X.

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The above limit is called the *rotation number* of G and is denoted by $\rho(G)$.

LEMMA 3.3. *Let G be an old continuous non-decreasing map, let $p, q \in \mathbb{Z}$, $q > 0$. Then $\rho(G) > p/q$ implies $G^q(X) - X > p$ for every $X \in \mathbb{R}$, and $\rho(G) < p/q$ implies $G^q(X) - X < p$ for every $X \in \mathbb{R}$.*

We shall call a non-empty closed set $B \subset \mathbb{R}$ *minimal* for an old continuous map G if

(i) $G(B) \subset B$;

(ii) if $X \in B$ and $k \in \mathbb{Z}$ then $X + k \in B$;

(iii) every non-empty closed set satisfying (i) and (ii) and contained in B is equal to B.

If G is an old continuous non-decreasing map then it is a lifting of a continuous map $g: S^1 \to S^1$ of degree one. It is easy to see that B is a minimal set for G if and only if $e(B)$ is a mat set for g (see [7]) and $B = e^{-1}(e(B))$.

LEMMA 3.4. *Let G be an old continuous non-decreasing map. Then there exists a set B, minimal for G and disjoint from* Const (G). *If* Const (G) *is non-empty then such B is nowhere dense.*

Proof. Let G be a lifting of g. By theorem A of [7] (note that if G is non-decreasing then the proof of this theorem is very easy) there exists a mat set $A \subset S^1$. We consider two cases.

Case 1. *$\rho(G)$ is irrational.* We take $B = e^{-1}(A)$. Clearly, B is a minimal set for G. Suppose that $B \cap I \neq \varnothing$ for some open interval $I \subset \text{Const}(G)$. Take $x \in A \cap e(I)$. By the minimality of A, there exists $n > 0$ such that $g^n(x) \in e(I)$. Since g is constant on $e(I)$, we have $g^n(g^n(x)) = g^n(x)$ i.e. $g^n(x)$ is a periodic point for g. This contradicts the assumption that $\rho(g)$ is irrational. Hence, $B \cap \text{Const}(g) = \varnothing$.

By proposition 2.6 (a) of [7], A is either equal to S^1 or is nowhere dense. Hence, if Const (G) is non-empty, then A is nowhere dense. Consequently, B is also nowhere dense.

Case 2. *$\rho(G)$ is rational.* By proposition 3.1 of [7], A is a periodic orbit of g. If $A \cap e(\text{Const}(G)) = \varnothing$, we can take as B the set $e^{-1}(A)$ and it satisfies the required conditions. Assume that $A \cap e(\text{Const}(G)) \neq \varnothing$. Let $\rho(G) = p/q$, $(p, q) = 1$. Since Const (G) is non-empty, $G^q - p$ is not the identity map. Therefore there exist Y, $Z \in \mathbb{R}$ such that $Y < Z$, $G^q(Y) = Y + p$, $G^q(Z) = Z + p$ and for all $X \in (Y, Z)$, $G^q(X) \neq X + p$. Clearly, both orbits of $y = e(Y)$ and $z \in e(Z)$ are mat sets. If at least one of them is disjoint from $e(\text{Const}(G))$, we can take its inverse image under e as B and it satisfies all required conditions. Suppose that they both intersect $e(\text{Const}(G))$. Then there exists an open interval $I \subset \mathbb{R}$ such that $g^k(y) \in e(I)$ for some $k \geq 0$ and g is constant on $e(I)$. Since $g^q(y) = y$, we may assume that $0 \leq k < q$. The set $g^{-k}(e(I))$ is an open neighbourhood of Y, and g^q is constant on it. Therefore, if ε is sufficiently small, then

$$(Y - \varepsilon) + p < Y + p = G^q(Y) = G^q(Y - \varepsilon)$$

and

$$(Y + \varepsilon) + p > Y + p = G^q(Y) = G^q(Y + \varepsilon).$$

Analogously, we obtain for ε sufficiently small

$$(Z-\varepsilon)+p<G^q(Z-\varepsilon)$$

and

$$(Z+\varepsilon)+p>G^q(Z+\varepsilon).$$

Hence, if additionally $\varepsilon<\frac{1}{2}(Z-Y)$, we obtain $Y+\varepsilon<Z-\varepsilon$ and

$$G^q(Y+\varepsilon)-(Y+\varepsilon)-p<0,$$
$$G^q(Z-\varepsilon)-(Z+\varepsilon)-p>0.$$

Therefore there exists $x\in(Y+\varepsilon, Z-\varepsilon)$ such that $G^q(X)-X-p=0$, which contradicts the definition of Y and Z. □

4. *Old heavy maps*

Now we prove two lemmas on old heavy maps.

LEMMA 4.1. *Let F be an old heavy map. Then F_l and F_r are old maps.*

Proof. We have

$$F_r(X+1)=\sup\{F(Y): Y\le X+1\}=\sup\{F(Z+1): Z\le X\}$$
$$=\sup\{F(Z)+1: Z\le X\}=F_r(X)+1,$$

and hence F_r is an old map. The proof for F_l is analogous. □

Notice that although we have not used in the proof the assumption that F is heavy, we need this assumption to define F_l and F_r.

LEMMA 4.2. *Let F be an old heavy map, let $p, q\in\mathbb{Z}$, $q>0$. Let $X_i<Y_i$ for $i=0,1,\ldots,q$, $X_q=X_0+p$, $Y_q=Y_0+p$, and $F(X_i+)\le X_{i+1}$, $F(Y_i-)\ge Y_{i+1}$ for $i=0,1,\ldots,q-1$. Then either $F(X_i+)=X_{i+1}$ for all $i\in\{0,1,\ldots,q-1\}$ or $F(Y_i-)=Y_{i+1}$ for all $i\in\{0,1,\ldots,q-1\}$, or there exists $T\in(X_0, Y_0)$ such that $F^i(T)\in(X_i, Y_i)$ for all $i\in\{0,1,\ldots,q-1\}$ and $F^q(T)=T+p$.*

Proof. We define inductively $X_{n+q}=X_n+p$, $Y_{n+q}=Y_n+p$ for $n=1,2,3,\ldots$. Then the hypotheses of lemma 2.4 are satisfied, and hence there exist increasing maps ψ_i^j satisfying the conditions (i)-(v) of lemma 2.4.

Assume that $F(X_l+)<X_{l+1}$ and $F(Y_m-)>Y_{m+1}$ for some $l, m\in\{0,1,\ldots,q-1\}$. Set $\varphi=\psi_0^q$. Then, by (iv), $\inf\varphi(X_q, Y_q)>X_0$, and by (v), $\sup\varphi(X_q, Y_q)<Y_0$.

Therefore the set $\{Z\in(X_0, Y_0): \varphi(Z+p)\ge Z\}$ is non-empty and the point $T=\sup\{Z\in(X_0, Y_0): \varphi(Z+p)\ge Z\}$ belongs to (X_0, Y_0). We claim that $\varphi(T+p)=T$. If $\varphi(T+p)<T$ then for each $Z\in(\varphi(T+p), T)$, we have $\varphi(Z+p)<\varphi(T+p)<Z$, a contradiction. If $\varphi(T+p)>T$, then for each $Z\in(T, \varphi(T+p))$ we have $\varphi(Z+p)>\varphi(T+p)>Z$, also a contradiction. Hence, indeed, $\varphi(T+p)=T$.

From this and from (ii) it follows that

$$F^q(T)=F^q(\varphi(T+p))=T+p.$$

Since for $i=0,1,\ldots,q-1$ we have $F^i\circ\psi_0^i=\mathrm{id}_{(X_i,Y_i)}$, we obtain

$$F^i(T)=F^i(\varphi(T+p))=(F^i\circ\psi_0^i\circ\psi_i^q)(T+p)=\psi_i^q(T+p).$$

But $T+p\in(X_q, Y_q)$ and hence $F^i(T)\in\psi_i^q(X_q, Y_q)\subset(X_i, Y_i)$. □

5. *Periodic mod* 1 *points*

In this section we prove theorem A and derive a corollary to it. The essential part of theorem A can be stated as the following proposition.

PROPOSITION 5.1. *Let F be an old heavy map and let* $\rho(F_l) < p/q < \rho(F_r)$. *Then F has a periodic mod* 1 *point of period q and rotation number* p/q.

Proof. Take $k = p/(p, q)$ and $n = q/(p, q)$. Then $k/n = p/q$ and $(k, n) = 1$.

Since F_l and F_r are old continuous non-decreasing maps (lemmas 2.2 and 4.1), we obtain by lemma 3.3,

$$F_l^n(X) - X < k \quad \text{and} \quad F_r^n(X) - X > k \qquad \text{for all } X \in \mathbb{R}.$$

Since $\rho(F_l) < \rho(F_r)$, we have $F_l \neq F_r$, and therefore F is not non-decreasing. Hence, Const (F_l) and Const (F_r) are non-empty. Thus, by lemma 3.4, there exist nowhere dense sets B_l and B_r, minimal for F_l and F_r respectively, such that $B_l \cap \text{Const}(F_l) = \varnothing$ and $B_r \cap \text{Const}(F_r) = \varnothing$.

We choose the points $Z_l, Z_r \in \mathbb{R}$ in the following way. If $B_l \cap B_r \neq \varnothing$ then we take $Z_l = Z_r \in B_l \cap B_r$. If $B_l \cap B_r = \varnothing$ then, since B_l and B_r are nowhere dense, closed and unbounded from both sides, we can take $Z_l \in B_l$ and $Z_r \in B_r$ such that $Z_r < Z_l$ and $(Z_r, Z_l) \cap (B_l \cup B_r) = \varnothing$.

We have $F_r^n(Z_r) - k > Z_r$. If $Z_r = Z_l$, then $F_r^n(Z_r) - k > Z_l$. If $Z_r < Z_l$ then $(Z_r, Z_l) \cap B_r = \varnothing$ and since $F_r^n(Z_r) - k \in B_r$ (because $Z_r \in B_r$), we obtain $F_r^n(Z_r) - k \geq Z_l$. But in this case $B_r \cap B_l = \varnothing$, and since $Z_l \in B_l$, we have $F_r^n(Z_r) - k \neq Z_l$. Hence, in both cases $F_r^n(Z_r) - k > Z_l$. In an analogous way we obtain $F_l^n(Z_l) - k < Z_r$. Thus,

$$F_l^n(Z_l) - k < Z_r \leq Z_l < F_r^n(Z_r) - k. \tag{5.1}$$

Let m be a non-negative integer. We shall show that

$$\left.\begin{aligned} F_l^m(F_l^n(Z_l) - k) < F_l^m(Z_l), \\ F_r^m(F_r^n(Z_r) - k) > F_r^m(Z_r), \end{aligned}\right\} \tag{5.2}$$

and

$$\left.\begin{aligned} F_l^m(F_l^n(Z_l) - k) < F_r^m(Z_r), \\ F_r^m(F_r^n(Z_r) - k) > F_l^m(Z_l). \end{aligned}\right\} \tag{5.3}$$

Since $F_l^n(Z_l) - k < Z_l$ and F_l is non-decreasing, we have

$$F_l^m(F_l^n(Z_l) - k) \leq F_l^m(Z_l).$$

If equality holds, then

$$F_l^m(Z_l) = F_l^m(F_l^n(Z_l) - k) = F_l^n(F_l^m(Z_l)) - k,$$

and $F_l^m(Z_l)$ is a periodic mod 1 point of F_l with the rotation number k/n. This contradicts the assumption $\rho(F_l) < k/n$. Hence, $F_l^m(F_l^n(Z_l) - k) < F_l^m(Z_l)$. The second inequality of (5.2) follows analogously.

Since $F_l^n(Z_l) - k < Z_r$, $F_l \leq F_r$ and F_l, F_r are non-decreasing, we obtain by Lemma 3.1.,

$$F_l^m(F_l^n(Z_l) - k) \leq F_r^m(Z_r).$$

Suppose that the equality holds. Since its left-hand side belongs to B_l and its right-hand side to B_r, we have $B_l \cap B_r \neq \varnothing$, and consequently $Z_l = Z_r$. By (5.2), we obtain then

$$F_r^m(Z_l) = F_r^m(Z_r) = F_l^m(F_l^n(Z_l) - k) < F_l^m(Z_l),$$

which contradicts lemma 3.1. Hence, $F_l^m(F_l^n(Z_l) - k) < F_r^m(Z_r)$. The second inequality of (5.3) follows analogously.

We set:

$$X_i = \begin{cases} F_l^i(F_l^n(Z_l) - k) & \text{if } 0 \le i < n, \\ F_l^{i-sn}(Z_l - sk) & \text{if } sn \le i < (s+1)n, 1 \le s < (p, q), \\ F_l^n(Z_l) - k - p & \text{if } i = q, \end{cases}$$

$$Y_i = \begin{cases} F_r^i(Z_r) & \text{if } 0 \le i < n, \\ F_r^{i-sn}(F_r^n(Z_r) - k + sk) & \text{if } sn \le i < (s+1)n, 1 \le s < (p, q), \\ Z_r + p & \text{if } i = q. \end{cases}$$

We check that the assumptions of lemma 4.2 are satisfied.

Clearly, $X_q = X_0 + p$ and $Y_q = Y_0 + p$. By (5.3), $X_i < Y_i$ for all i. We have $X_i \in B_l$ and $Y_i \in B_r$ for all i. Since $B_l \cap \text{Const}(F_l) = \varnothing$ and $B_r \cap \text{Const}(F_r) = \varnothing$, and by lemma 2.3, we have $F_l(X_i) = F(X_i+)$ and $F_r(Y_i) = F(Y_i-)$ for $i = 0, 1, \ldots, q-1$. Hence, if n does not divide $i+1$, then $F(X_i+) = X_{i+1}$ and $F(Y_i-) = Y_{i+1}$. By (5.1) and (5.2) we obtain:

if $i = n-1$ and $(p, q) > 1$ then

$$F(X_i+) = F_l^n(F_l^n(Z_l) - k) < F_l^n(Z_l) < Z_l + k = X_{i+1}$$

and

$$F(Y_i-) = F_r^n(Z_r) = Y_{i+1},$$

if $i = jn-1$ and $1 < j < (p, q)$ then

$$F(X_i+) = F_l^n(Z_l + (j-1)k) < Z_l + jk = X_{i+1}$$

and

$$F(Y_i-) = F_r^n(F_r^n(Z_r) + (j-2)k) > F_r^n(Z_r) + (j-1)k = Y_{i+1},$$

if $i = q-1$ and $(p, q) > 1$ then

$$F(X_i+) = F_l^n(Z_l + p - k) = X_{i+1}$$

and

$$F(Y_i-) = F_r^n(F_r^n(Z_r) + p - 2k) > F_r^n(Z_r) + p - k > Z_r - p = Y_i,$$

if $i = q-1$ and $(p, q) = 1$ then

$$F(X_i+) = F_l^n(F_l^n(Z_l) - k) < F_l^n(Z_l) = X_{i+1}$$

and

$$F(Y_i-) = F_r^n(Z_r) > Z_r + k = Y_i.$$

Therefore the assumptions of lemma 4.2 are satisfied and the two first possibilities of its statement do not hold. Therefore the third one holds, namely there exists $T \in (X_0, Y_0)$ such that $F^i(T) \in (X_i, Y_i)$ for $i = 0, 1, \ldots, q$ and $F^q(T) = T + p$. Hence,

T is a periodic mod 1 point of F with rotation number p/q and its period divides q. Denote this period by m. We have $j/m = k/n$ for some $j \in \mathbb{Z}$. Since $(k, n) = 1$, n divides m. Suppose that $m < q$. Then $m = sn$ for some s with $1 \le s < (p, q)$. We have then $X_m = Z_l + sk$ and $Y_m = F_r^n(Z_r) - k + sk$. Since

$$j = \frac{k}{n} \cdot m = \frac{ksn}{n} = sk,$$

we have $F^m(T) = T + sk$. Hence, from $F^m(T) \in (X_m, Y_m)$ it follows that $T \in (Z_l, F_r^n(Z_r) - k)$. But $T \in (X_0, Y_0) = (F_l^n(Z_l) - k, Z_r)$. This contradicts (5.1). Hence, $m = q$. □

THEOREM A. *Let $F: \mathbb{R} \to \mathbb{R}$ be an old heavy map. Then*

(a) *if F has a periodic mod 1 point of rotation number p/q then $a(F) \le p/q \le b(F)$;*

(b) *if $a(F) < p/q < b(F)$ then F has a periodic mod 1 point of period q and rotation number p/q.*

Proof. If F has a periodic mod 1 point X of rotation number p/q then clearly $\lim_{n\to\infty} (1/n)(F^n(X) - X) = p/q$, and hence $a(F) \le p/q \le b(F)$. This proves (a).

Since $F \le F_r$ and F_r is non-decreasing, we have by Lemma 3.1, $F^n \le F_r^n$ for all $n > 0$. Hence, $b(F) \le \rho(F_r)$. Analogously, $\rho(F_l) \le a(F)$. Therefore if $a(F) < p/q < b(F)$ then $p(F_l) < p/q < \rho(F_r)$ and by proposition 5.1, F has a periodic mod 1 point of period q and rotation number p/q. □

COROLLARY 5.2. *Let F be an old heavy map. Then $a(F) = \rho(F_l)$ and $b(F) = \rho(F_r)$.*

Proof. If $\rho(F_l) = \rho(F_r)$ then from

$$\rho(F_l) \le a(F) \le b(F) \le \rho(F_r)$$

it follows that

$$\rho(F_l) = a(F) = b(F) = \rho(F_r).$$

If $\rho(F_l) < \rho(F_r)$, then, by proposition 5.1, for every rational number $\alpha \in (\rho(F_l), \rho(F_r))$, F has a periodic mod 1 point of rotation number α. By theorem A(a), $\alpha \in [a(F), b(F)]$. Hence, $(\rho(F_l), \rho(F_r)) \subset [a(F), b(F)]$. But $\rho(F_l) \le a(F) \le b(F) \le \rho(F_r)$, and hence $\rho(F_l) = a(F)$ and $\rho(F_r) = b(F)$. □

6. *Dependence of $a(F)$ and $b(F)$ on F*

In this section, we prove theorems B and C.

THEOREM B. *Let $t \mapsto F_t$ be a map from an interval into the space of old heavy maps, such that the maps $t \mapsto (F_t)_l$ and $t \mapsto (F_t)_r$ are continuous ($(F_t)_l$ and $(F_t)_r$ are regarded as elements of the space of maps of $\mathbb{R}$ into itself with the topology of uniform convergence). Then the maps $t \mapsto a(F_t)$ and $t \mapsto b(F_t)$ are also continuous.*

Proof. By corollary 5.2, $a(F_t) = \rho((F_t)_l)$ and $b(F_t) = \rho((F_t)_r)$. Since $(F_t)_l$ and $(F_t)_r$ are continuous old maps and $\rho((F_t)_l) = a((F_t)_l)$, $\rho((F_t)_r) = a((F_t)_r)$, we obtain (see [**8**], [**6**]) that the maps $t \mapsto \rho((F_t)_l)$ and $t \mapsto \rho((F_t)_r)$ are continuous. □

It is clear that if F is an old heavy map and $\theta \in \mathbb{R}$ then $F + \theta$ (defined by $(F + \theta)(X) = F(X) + \theta$) is also an old heavy map.

THEOREM C. *Let $F:\mathbb{R}\to\mathbb{R}$ be an old heavy map. If $a(F)$ $(b(F))$ is irrational then for all $\theta>0$ we have $a(F+\theta)>a(F)$ (respectively $b(F+\theta)>b(F)$).*

Proof. Clearly, we have $(F+\theta)_l=F_l+\theta$ and $(F+\theta)_r=F_r+\theta$. Hence, by corollary 5.2 and theorem 2 of [**5**], we obtain $a(F+\theta)>a(F)$ for $\theta>0$ if $a(F)$ is irrational and $b(F+\theta)>b(F)$ if $b(F)$ is irrational. □

Remark. In the above proof, instead of using theorem 2 of [**5**], we can use the following simple lemma.

LEMMA 6.1. *Let G be a continuous non-decreasing old map with $\rho(G)$ irrational, and let $\theta>0$. Then $\rho(G+\theta)>\rho(G)$.*

Proof. By lemma 3.4, there exists a minimal set B for G. Since $\rho(G)$ is irrational, the set $B\cap[0,1]$ is infinite, and therefore there exist points $X, Y, Z\in B$ such that $X<Z<Y<X+\theta$. By the minimality of B, there exist $n, k\in\mathbb{Z}$ such that $n>0$ and $X+k<G^n(Y)<Y+k$. Therefore, by lemma 3.3, $\rho(G)<k/n$. But

$$(G+\theta)^n(Y)=(G+\theta)((G+\theta)^{n-1}(Y))=G((G+\theta)^{n-1}(Y))+\theta$$
$$\geq G^n(Y)+\theta>X+k+\theta>Y+k,$$

and therefore, again by lemma 3.3, $\rho(G+\theta)>k/n$. Hence, $\rho(G+\theta)>\rho(G)$. □

7. *Behaviour of the sequences* $((1/n)(F^n(X)-X))_{n=1}^{\infty}$

In this section, we prove theorem D and derive a corollary to it.

THEOREM D. *Let $F:\mathbb{R}\to\mathbb{R}$ be an old heavy map and let $a(F)\leq\alpha\leq\beta\leq b(F)$. Then there exists $T\in\mathbb{R}$ such that*

$$\liminf_{n\to\infty}\frac{1}{n}(F^n(T)-T)=\alpha,$$

$$\limsup_{n\to\infty}\frac{1}{n}(F^n(T)-T)=\beta.$$

Proof. Assume first that $a(F)=b(F)$. Then, by corollary 5.2, $\alpha=\beta=\rho(F_l)=\rho(F_r)$. Since for all $n\geq 0$ and all $t\in\mathbb{R}$ we have

$$F_l^n(T)-T\leq F^n(T)-T\leq F_r^n(T)-T,$$

we obtain

$$\lim_{n\to\infty}\frac{1}{n}(F^n(T)-T)=\alpha=\beta \qquad \text{for all } T\in\mathbb{R}.$$

Assume now for the rest of the proof that $a(F)<b(F)$. We fix $n_0>1/(b(F)-a(F))$. Then there exist two sequences of integers, $(p_n)_{n=n_0}^{\infty}$ and $(r_n)_{n=n_0}^{\infty}$ such that p_n/n, $r_n/n\in(a(F), b(F))$ for all $n\geq n_0$, $\lim_{n\to\infty}p_n/n=\alpha$ and $\lim_{n\to\infty}r_n/n=\beta$.

We define inductively positive integers i_n, j_n, m_n, v_n and integers k_n, u_n $(n=n_0, n_0+1,\ldots)$:

$i_{n_0}=1,\quad k_{n_0}=p_{n_0},\quad m_{n_0}=n_0;$

j_n is such that

$$\left|\frac{u_n}{v_n}-\frac{r_n}{n}\right|<\frac{1}{n},$$

where

$$u_n = k_n + j_n r_n, \quad v_n = m_n + j_n n;$$

i_{n+1} is such that

$$\left|\frac{k_{n+1}}{m_{n+1}} - \frac{p_{n+1}}{n+1}\right| < \frac{1}{n+1},$$

where

$$k_{n+1} = u_n + i_{n+1} p_{n+1}, \quad m_{n+1} = v_n + i_{n+1}(n+1).$$

By lemma 3.4, there exist sets B_l and B_r, minimal for F_l and F_r respectively and such that $B_l \cap \text{Const}(F_l) = \emptyset$ and $B_r \cap \text{Const}(F_r) = \emptyset$. We choose $Z_l \in B_l$ and $Z_r \in B_r$ in such a way that $Z_l < Z_r$, and if $B_l \cap B_r \neq \emptyset$ then $Z_l \in B_l \cap B_r$, $Z_r = Z_l + 1$. We set:

$X_t = F_l^t(Z_l)$, $\quad Y_t = F_r^t(Z_r)$ $\quad$ for $t = 0, 1, \ldots, n_0 - 1$;

$X_{m_n + jn + t} = F_l^t(Z_l + k_n + jr_n)$, $\quad Y_{m_n + jn + t} = F_r^t(Z_r + k_n + jr_n)$,

for $j = 0, 1, \ldots, j_n - 1$ and $t = 0, 1, \ldots, n-1$;

$X_{v_n + i(n+1) + t} = F_l^t(Z_l + u_n + ip_{n+1})$, $\quad Y_{v_n + i(n+1) + t} = F_r^t(Z_r + u_n + ip_{n+1})$,

for $i = 0, 1, \ldots, i_{n+1} - 1$ and $t = 0, 1, \ldots, n$.

Clearly, all points X_q belong to B_l and all points Y_q belong to B_r. From the definition it follows that if q is not of the form $m_n + jn - 1$ or $v_n + i(n+1) - 1$ then $F_l(X_q) = X_{q+1}$ and $F_r(Y_q) = Y_{q+1}$. By lemma 2.3, we then have $F(X_q+) = F_l(X_q) = X_{q+1}$ and $F(Y_q-) = F_r(Y_q) = Y_{q+1}$.

Let $q = m_n + jn - 1$, $j \in \{1, 2, \ldots, j_n\}$. Then

$$F_l(X_q) = F_l^n(X_{m_n + (j-1)n}).$$

Since $\rho(F_l) = a(F) < r_n/n$, we have (by lemma 3.3) $F_l^n(X) < X + r_n$ for all $X \in \mathbb{R}$. Hence

$$\begin{aligned} F_l^n(X_{m_n+(j-1)n}) &= F_l^n(Z_l + k_n + (j-1)r_n) \\ &< Z_l + k_n + jr_n = X_{m_n+jn} = X_q. \end{aligned}$$

Therefore $F_l(X_q) < X_{q+1}$, and by lemma 2.3, $F(X_q+) < X_{q+1}$. Analogously, $F(Y_q-) > Y_{q+1}$. In the same way one can prove that $F(X_q+) < X_{q+1}$ and $F(Y_q-) > Y_{q+1}$ if $q = v_n + i(n+1) - 1$, $i \in \{1, 2, \ldots, i_{n+1}\}$.

If q is of the form $m_n + jn$ or $v_n + i(n+1)$, then clearly $X_q < Y_q$. To prove that this inequality holds also for other q, it is enough to show that if $t \geq 0$ then $F_l^t(Z_l) < F_l^t(Z_r)$. Since $F_l \leq F_r$ and both are non-decreasing, we have by lemma 3.1, $F_l^t \leq F_r^t$ and therefore $F_l^t(Z_l) \leq F_r^t(Z_r)$. If equality holds, then $B_l \cap B_r \neq \emptyset$ and by the definition of Z_l and Z_r, we have $Z_r = Z_l + 1$. Then

$$F_r^t(Z_r) = F_r^t(Z_l + 1) = F_r^t(Z_l) + 1 \geq F_l^t(Z_l) + 1 > F_l^t(Z_l),$$

so equality cannot hold. Hence, $F_l^t(Z_l) < F_r^t(Z_r)$.

Thus, the hypotheses of lemma 2.4 are satisfied and $F(X_q+) < X_{q+1}$, $F(Y_q-) > Y_{q+1}$ for infinitely many q. Therefore there exist increasing maps ψ_i^j: $(X_j, Y_j) \to (X_i, Y_i)$ for all i, j with $0 \leq i \leq j$, such that the conditions (i)-(iii) of lemma 2.4 are satisfied and there exists an increasing sequence $(l_n)_{n=1}^{\infty}$ of non-negative integers such that for all n

(iv″) $\alpha_{l_n} < \alpha_{l_{n+1}}$,

(v″) $\beta_{l_n} > \beta_{l_{n+1}}$,

where $\alpha_q = \inf A_q$, $\beta_q = \sup A_q$, $A_q = \psi_0^q(X_q, Y_q)$ for $q = 0, 1, 2, \ldots$.

Clearly, $A_0 \supset A_1 \supset A_2 \supset \ldots$, and therefore

$$\alpha_0 \le \alpha_1 \le \alpha_2 \le \cdots \le \beta_2 \le \beta_1 \le \beta_0.$$

By (iv″) and (v″), $\alpha_q < \beta_q$ for all q.

We set $T = \lim_{q\to\infty} \beta_q$. We claim that $T \in \bigcap_{q=0}^{\infty} A_q$. By (v″), for each n there exists $T_n \in A_{l_n}$ such that $\beta_{l_{n+1}} < T_n < \beta_{l_n}$. Hence, $T_1 > T_2 > T_3 > \cdots$, and $\lim_{n\to\infty} T_n = T$. For a fixed q, there exists n, such that if $n \ge 1$ then $l_n \ge q$, and consequently $T_n \in A_q$. Since ψ_0^q is increasing, the sequence $((\psi_0^q)^{-1}(T_n))_{n=n_1}^{\infty}$ is decreasing. Since $(\psi_0^q)^{-1}(T_n) \in (X_q, Y_q)$, it converges to some $Z \in [X_q, Y_q)$ as $n \to \infty$. If $Z = X_q$, then $\alpha_q = \lim_{n\to\infty} T_n = T$, which contradicts (iv″). Hence, $Z \in (X_q, Y_q)$. By (iii), we have $\psi_0^q(Z) = \lim_{n\to\infty} T_n = T$, and therefore $T \in A_q$. Hence, indeed $T \in \bigcap_{q=0}^{\infty} A_q$.

Thus, by (ii), we get $F^q(T) \in (X_q, Y_q)$ for $q = 0, 1, 2, \ldots$. Write

$$\begin{aligned} P = \{m_n + jn &: n = n_0, n_0+1, \ldots ; j = 0, 1, \ldots, j_n - 1\} \\ &\cup \{v_n + i(n+1) : n = n_0, n_0+1, \ldots ; i = 0, 1, \ldots, i_{n+1} - 1\}. \end{aligned}$$

If $q = m_n + jn$ then both T and $F^q(T) - (k_n + jr_n)$ are in (X_0, Y_0) and consequently their distance is at most $Z_r - Z_l$. If $q = v_n + i(n+1)$ then both T and $F^q(T) - (u_n + ip_{n+1})$ are in (X_0, Y_0) and also their distance is at most $Z_r - Z_l$. Therefore for q of one of the above forms, $(1/q)(F^q(T) - T)$ differs from $(k_n + jr_n)/(m_n + jn)$ or $(u_n + ip_{n+1})/(v_n + i(n+1))$ respectively by at most $(1/q)(Z_r - Z_l)$. The number $(k_n + jr_n)/(m_n + jn)$ lies between k_n/m_n and r_n/n and the number $(u_n + ip_{n+1})/(v_n + i(n+1))$ lies between u_n/v_n and $p_{n+1}/(n+1)$. Therefore, in view of the fact that $\lim_{q\to\infty} (1/q)(Z_r - Z_l) = 0$, we obtain

$$\left.\begin{aligned} \liminf_{n\to\infty} \left(\min \left(\frac{k_n}{m_n}, \frac{u_n}{v_n}, \frac{p_n}{n}, \frac{r_n}{n} \right) \right) &\le \liminf_{\substack{q\to\infty \\ q\in P}} \frac{1}{q} (F^q(T) - T), \\ \limsup_{n\to\infty} \left(\max \left(\frac{k_n}{m_n}, \frac{u_n}{v_n}, \frac{p_n}{n}, \frac{r_n}{n} \right) \right) &\ge \limsup_{\substack{q\to\infty \\ q\in P}} \frac{1}{q} (F^q(T) - T). \end{aligned}\right\} \tag{7.1}$$

Since

$$\left| \frac{u_n}{v_n} - \frac{r_n}{n} \right| < \frac{1}{n} \quad \text{and} \quad \left| \frac{k_n}{m_n} - \frac{p_n}{n} \right| < \frac{1}{n},$$

we have

$$\left.\begin{aligned} \liminf_{n\to\infty} \left(\min \left(\frac{p_n}{n}, \frac{r_n}{n} \right) \right) &= \liminf_{n\to\infty} \left(\min \left(\frac{k_n}{m_n}, \frac{u_n}{v_n} \right) \right), \\ \limsup_{n\to\infty} \left(\max \left(\frac{p_n}{n}, \frac{r_n}{n} \right) \right) &= \limsup_{n\to\infty} \left(\max \left(\frac{k_n}{m_n}, \frac{u_n}{v_n} \right) \right). \end{aligned}\right\} \tag{7.2}$$

For $q = m_n$ we have

$$\left|\frac{1}{q}(F^q(T)-T)-\frac{k_n}{m_n}\right| < \frac{1}{q}(Z_r - Z_l),$$

and for $q = v_n$,

$$\left|\frac{1}{q}(F^q(T)-T)-\frac{u_n}{v_n}\right| < \frac{1}{q}(Z_r - Z_l).$$

Therefore,

$$\left.\begin{aligned} \liminf_{\substack{q\to\infty\\ q\in P}} \frac{1}{q}(F^q(T)-T) &\le \liminf_{n\to\infty}\left(\min\left(\frac{k_n}{m_n},\frac{u_n}{v_n}\right)\right) \\ \limsup_{\substack{q\to\infty\\ q\in P}} \frac{1}{q}(F^q(T)-T) &\ge \limsup_{n\to\infty}\left(\max\left(\frac{k_n}{m_n},\frac{u_n}{v_n}\right)\right). \end{aligned}\right\} \tag{7.3}$$

Since $\lim_{n\to\infty} p_n/n = \alpha$, $\lim_{n\to\infty} r_n/n = \beta$ and $\alpha \le \beta$, we obtain from (7.1), (7.2) and (7.3),

$$\left.\begin{aligned} \liminf_{\substack{q\to\infty\\ q\in P}} \frac{1}{q}(F^q(T)-T) &= \alpha, \\ \limsup_{\substack{q\to\infty\\ q\in P}} \frac{1}{q}(F^q(T)-T) &= \beta. \end{aligned}\right\} \tag{7.4}$$

Now we have to see what happens if $q \notin P$. Then $q = s + t$ for some $s \in P$ and $t \le n$ (where $s = m_n + jn$ or $s = v_n + i(n+1)$). Since all i_ν and j_ν are positive for $\nu = n_0, n_0+1, \ldots, n-1$ we have

$$q > s \ge 2(n_0 + (n_0+1) + \cdots + (n-1)) = (n-n_0)(n_0+n-1).$$

There exists an integer γ such that $|\rho(F_l)| < \gamma$ and $|\rho(F_r)| < \gamma$. Then for every $Z \in \mathbb{R}$ we have

$$Z - \gamma < F_l(Z) \le F(Z) \le F_r(Z) < Z + \gamma,$$

and consequently

$$Z - \nu\gamma < F^\nu(Z) < Z + \nu\gamma \qquad \text{for } \nu = 1, 2, \ldots.$$

Hence $|F^q(T) - F^s(T)| < t\gamma \le n\gamma$ and $|(1/s)(F^s(T)-T)| < \gamma$. Therefore

$$\begin{aligned} &\left|\frac{1}{q}(F^q(T)-T)-\frac{1}{s}(F^s(T)-T)\right| \\ &\le \left|\frac{1}{q}(F^q(T)-T)-\frac{1}{q}(F^s(T)-T)\right| + \left|\frac{1}{q}(F^s(T)-T)-\frac{1}{s}(F^s(T)-T)\right| \\ &= \frac{1}{q}|F^q(T)-F^s(T)| + \frac{q-s}{q}\left|\frac{1}{s}(F^s(T)-T)\right| \\ &< \frac{2n\gamma}{(n-n_0)(n+n_0-1)} \xrightarrow[n\to\infty]{} 0. \end{aligned}$$

In view of (7.4), this ends the proof of the theorem. □

COROLLARY 7.1. *Let F be an old heavy map, and let $a(F) \leq \alpha \leq b(F)$. Then there exists $T \in \mathbb{R}$ such that* $\lim_{n\to\infty} (1/n)(F^n(T) - T) = \alpha$.

The above corollary means that in a case of old heavy maps the rotation set is closed.

8. *Examples of old heavy maps*

Clearly, all continuous maps are heavy. Hence, if $f: S^1 \to S^1$ is a continuous map of degree one, then its lifting (in a usual sense) is an old heavy map. Therefore all results of this paper are generalizations of the corresponding results for maps of degree one of the circle into itself.

Another important class of old heavy maps arises by taking liftings of some monotone mod 1 maps. A map $f: [0, 1) \to [0, 1)$ is called monotone mod 1 if there is a monotone, continuous bounded map $g: [0, 1) \to \mathbb{R}$ such that $f(X) = g(X) \pmod 1$ for all $X \in [0, 1)$. Such maps were studied e.g. by Hofbauer [3]. If g is non-decreasing and $g(1-) - g(0) > 1$ then the map F defined by

$$F(X+k) = g(X) + k \qquad \text{for } X \in [0, 1),\ k \in \mathbb{Z},$$

is an old heavy map, and is a lifting of f (regarded as a map of a circle into itself). In particular, we may take as f a so called β-transformation (defined for $\beta > 1$ by $g(X) = \beta X$ or $g(X) = \beta X + \alpha$).

Other examples can be obtained when studying the Newton's method of determining zeros of certain functions If $\varphi: \mathbb{R} \to \mathbb{R}$ is a differentiable function then we define a map N by

$$N(x) = x - \frac{\varphi(x)}{\varphi'(x)}.$$

Since the one point compactification of $\mathbb{R}$ is homeomorphic to a circle, we may regard N as a map of a circle into itself. To avoid complications caused by the fact that N is not defined at ∞ and at the zeros of φ', we assume that φ' has finitely many zeros and the limits $N(-\infty) = \lim_{x\to-\infty} N(x)$ and $N(+\infty) = \lim_{x\to+\infty} N(x)$ (finite or infinite) exist. Notice that in our notation $N(\infty-) = N(+\infty)$ and $N(\infty+) = N(-\infty)$. If $\varphi'(x) = 0$ then $N(x-) = N(x+) = \infty$ (on the circle, therefore $+\infty$ and $-\infty$ are identified) and we set $N(x) = \infty$; we set also $N(\infty) = N(+\infty)$. If both $N(-\infty)$ and $N(+\infty)$ are infinite or $N(-\infty) = N(+\infty)$ then N as a map of the circle is continuous. This case was studied in [9].

The case of finite $N(-\infty)$ and $N(+\infty)$ takes place usually if φ has asymptotes. Assume that $a < b < c < d$, $\varphi'(x) < 0$ for $x < c$, $\varphi'(c) = 0$, $\varphi'(x) > 0$ for $x > c$, $\varphi(c) < 0$, $N(d) = c$, $N(x) \neq c$ for $x \neq d$, $N(-\infty) = b$, $N(+\infty) = a$ and $N(b) < N(a)$. Figure 1 shows the graph of such φ, figure 2 the graph of the corresponding N. The map N (as a map of the circle) has no old heavy lifting, but its second iterate, N^2, has. To show the last statement notice that

1° N^2 is discontinuous only at ∞ and c and we have $N^2(\infty-) = N^2(+\infty) = N(a) > N(b) = N^2(-\infty) = N^2(\infty+)$, $N^2(c-) = b > a = N^2(c+)$;

2° at d we have $N^2(d-) = +\infty$, $N^2(d+) = -\infty$, which gives degree one.

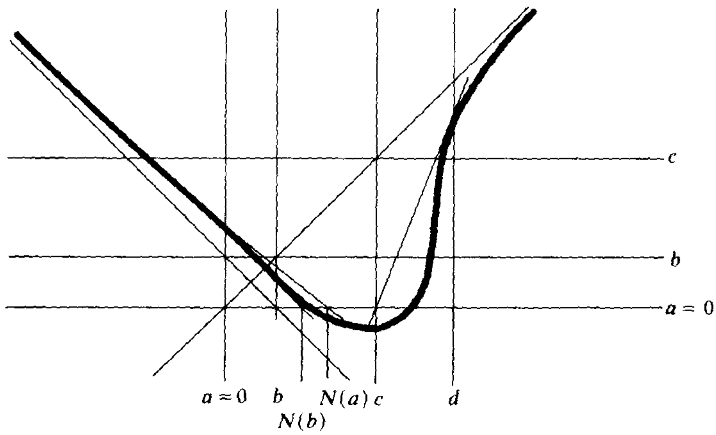

FIGURE 1

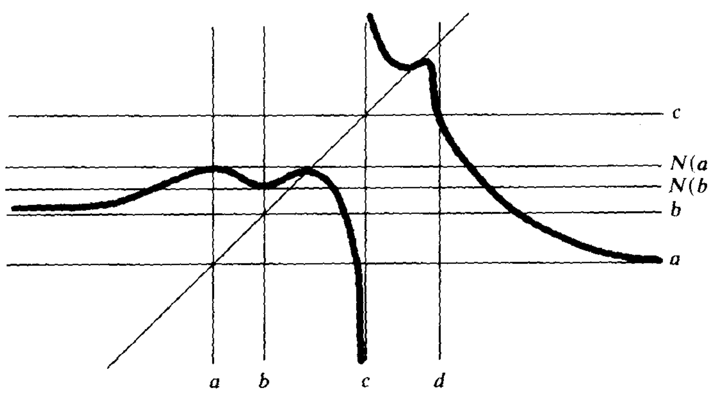

FIGURE 2

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