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CARATHÉODORY'S THEOREM WITH LINEAR CONSTRAINTS

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Carathéodory has shown that if $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$, ($m$ finite) are points of R^n and if $\mathbf{x}_0 = \sum_{i=1}^m \lambda_i \mathbf{x}_i$ for some

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \Omega = \left\{\boldsymbol{\lambda} \in R^m \,\middle|\, \sum_{i=1}^m \lambda_i = 1, \lambda_i \geq 0 \ \forall\, i\right\},$$

then $\exists \boldsymbol{\mu} \in \Omega$ with at most $n+1$ nonzero components and for which $\mathbf{x}_0 = \sum_i^m \mu_i \mathbf{x}_i$. (See [5]). The authors of [2] have extended this result to include the case where $m = +\infty$. In theorems 1 and 2 below we establish somewhat similar results for the case in which Ω is further restricted by a finite system of linear inequalities (or equalities).

THEOREM 1. *Let D be a $k \times m$ matrix, $\mathbf{d} \in R^k$ and $D\boldsymbol{\lambda} \leq \mathbf{d}$ be a system of k linear inequalities. Define $\bar{\Omega}$ by*

$$\bar{\Omega} = \left\{\boldsymbol{\lambda} \in R^m \,\middle|\, \sum_{i=1}^m \lambda_i = 1, \lambda_i \geq 0 \ \forall\, i,\ D\boldsymbol{\lambda} \leq \mathbf{d}\right\}.$$

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ be a finite set of points from R^n and let

$$\mathbf{x}_0 = \sum_{i=1}^m \lambda_i^0 \mathbf{x}_i$$

for some $\boldsymbol{\lambda}^0 \in \bar{\Omega}$. Then $\exists \boldsymbol{\mu}^0 \in \bar{\Omega}$ with at most $n+k+1$ nonzero components and for which

$$\mathbf{x}_0 = \sum_{i=1}^m \mu_i^0 \mathbf{x}_i.$$

Proof. Let $\mathbf{0}$ denote the $1 \times m$ vector each element of which is zero, and consider the linear programming problem

$$\max \mathbf{0} \cdot \boldsymbol{\lambda}$$

$$\text{s.t.} \sum_{i-1}^m \lambda_i \mathbf{x}_i = \mathbf{x}_0$$

$$(*) \qquad D\boldsymbol{\lambda} \leq \mathbf{d}$$

$$\sum_{i=1}^m \lambda_i = 1$$

$$\lambda_i \geq 0 \ \forall\, i.$$

Since (*) is feasible ($\boldsymbol{\lambda}^0$ is a feasible solution) then, from [4], Chapter 3, there exists an optimal basic feasible solution $\boldsymbol{\mu}^0$ which, by definition, has at most as many nonzero components as there are constraints. Hence $\exists \boldsymbol{\mu}^0 \in \bar{\Omega}$ with at most $n+k+1$ nonzero components and satisfying

$$\mathbf{x}_0 = \sum_{i=1}^{m} \mu_i^0 \mathbf{x}_i. \quad \text{Q.E.D.}$$

THEOREM 2. *Let D be a $k \times \infty$ matrix whose columns form a closed sequence in R^k, $\mathbf{d} \in R^k$ and $D\boldsymbol{\lambda} \leq \mathbf{d}$ be a system of k linear inequalities. Define $\bar{\Omega}$ by*

$$\bar{\Omega} = \left\{ \boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_m, \ldots) \;\middle|\; \sum_{i=1}^{\infty} \lambda_i = 1, \lambda_i \geq 0 \;\forall\, i,\; D\boldsymbol{\lambda} \leq \mathbf{d} \right\}.$$

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m, \ldots$ be a closed sequence in R^n and let

$$\mathbf{x}_0 = \sum_{i=1}^{\infty} \lambda_i^0 \mathbf{x}_i$$

for some $\boldsymbol{\lambda}^0 \in \bar{\Omega}$. Then $\exists \boldsymbol{\mu}^0 \in \bar{\Omega}$ with at most $n+k+1$ nonzero components and for which

$$\mathbf{x}_0 = \sum_{i=1}^{\infty} \mu_i^0 \mathbf{x}_i.$$

Proof. Let $\mathbf{0}$ denote the $1 \times \infty$ vector of zeros, and consider the "semi-infinite" programming problem (see [1])

$$\max \mathbf{0} \cdot \boldsymbol{\lambda}$$

$$\text{s.t.} \sum_{i=1}^{\infty} \lambda_i \mathbf{x}_i = \mathbf{x}_0$$

$$(**) \qquad D\boldsymbol{\lambda} \leq \mathbf{d}$$

$$\sum_{i=1}^{\infty} \lambda_i = 1$$

$$\lambda_i \geq 0 \;\forall\, i.$$

Since the objective function coefficients in (**) are all zeros then by Haar's theorem ([1] and [3]) on homogeneous inequalities, and the dual theorem of semi-infinite programming [1], there exists an optimal feasible solution $\boldsymbol{\mu}^0$ with at most $n+k+1$ nonzero components. Hence $\exists \boldsymbol{\mu}^0 \in \bar{\Omega}$ with at most $n+k+1$ nonzero components and satisfying

$$\mathbf{x}_0 = \sum_{i=1}^{\infty} \mu_i^0 \mathbf{x}_i. \quad \text{Q.E.D.}$$

It is worth noting that in case $k=0$, theorems 1 and 2 provide alternative proofs respectively of Carathéodory's theorem and its infinite extension as given in [2].

References

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