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DISCOUNTED OPTIMAL STOPPING FOR MAXIMA OF SOME JUMP-DIFFUSION PROCESSES

PAVEL V. GAPEEV,* *WIAS and Russian Academy of Sciences*

Abstract

In this paper we present closed form solutions of some discounted optimal stopping problems for the maximum process in a model driven by a Brownian motion and a compound Poisson process with exponential jumps. The method of proof is based on reducing the initial problems to integro-differential free-boundary problems, where the normal-reflection and smooth-fit conditions may break down and the latter then replaced by the continuous-fit condition. We show that, under certain relationships on the parameters of the model, the optimal stopping boundary can be uniquely determined as a component of the solution of a two-dimensional system of nonlinear ordinary differential equations. The obtained results can be interpreted as pricing perpetual American lookback options with fixed and floating strikes in a jump-diffusion model.

Keywords: Discounted optimal stopping problem; Brownian motion; compound Poisson process; maximum process; integro-differential free-boundary problem; continuous and smooth fit; normal reflection; change-of-variable formula with local time on surfaces; perpetual American lookback option

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1. Introduction

The main aim of this paper is to present closed form solutions to the discounted optimal stopping problems, (2.4) and (5.1), below, for the running maximum S associated with the process X defined in (2.1)–(2.2). These problems are related to the option pricing theory in mathematical finance, where the process X can describe the price of a risky asset (e.g. a stock) on a financial market. In this case, the values of (2.4) and (5.1) can be interpreted as *fair prices* of *perpetual lookback* options of American type with *fixed* and *floating strikes* in a jump-diffusion model, respectively. For a continuous model, (2.4) and (5.1) were solved by Pedersen [23], Guo and Shepp [15], and Beibel and Lerche [4]; see also [11] for the case of finite time horizon.

Observe that, when $K = 0$, (2.4) and (5.1) turn into the classical Russian option problem introduced and explicitly solved by Shepp and Shiryaev [30] by means of reducing the initial problem to an optimal stopping problem for a (continuous) two-dimensional Markov process and then solving the latter problem using the smooth-fit and normal-reflection conditions. It was further observed in [31] that the change-of-measure theorem enables the Russian option problem to be reduced to a one-dimensional optimal stopping problem; this fact explained the simplicity of the structure of the solution in [30]. Building on the optimal stopping analysis of Shepp and Shiryaev [30], [31], Duffie and Harrison [7] derived a rational economic value for

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* Postal address: Weierstrass Institute for Applied Analysis and Stochastics (WIAS), Mohrenstr. 39, D-10117 Berlin, Germany. Email address: gapeev@wias-berlin.de and p.gapeev@lse.ac.uk

the Russian option and then extended their arbitrage arguments to perpetual lookback options. More recently, Shepp *et al.* [32] proposed a barrier version of the Russian option, where the decision about stopping should be taken before the price process reaches a positive level. Peskir [25] presented a solution to the Russian option problem in the finite horizon case; see also [8] for a numeric algorithm for solving the corresponding free-boundary problem and [10] for a study of asymptotic behavior of the optimal stopping boundary near expiration.

In recent years, the Russian option problem in models with jumps has been studied quite extensively. Gerber *et al.* [14] and then Mordecki and Moreira [22] obtained closed form solutions to the perpetual Russian option problem for diffusions with negative exponential jumps. Asmussen *et al.* [2] derived explicit expressions for the prices of perpetual Russian options in the dense class of Lévy processes with phase-type jumps in both directions by reducing the initial problem to the first passage time problem and solving the latter by martingale stopping and Wiener–Hopf factorization. Avram *et al.* [3] studied exit problems for spectrally negative Lévy processes and applied the results to solving optimal stopping problems associated with perpetual Russian and American put options.

In contrast to the Russian option problem, (2.4) is necessarily *two-dimensional* in the sense that it cannot be reduced to an optimal stopping problem for a one-dimensional (time-homogeneous) Markov process. Some other necessarily two-dimensional optimal stopping problems for continuous processes were earlier considered in [6] and [24]. The main feature of the optimal stopping problem for the maximum process in continuous models is that the normal-reflection condition at the diagonal of the state space of the process (X, S) holds, which implies that the characterization of the optimal boundary is a unique solution of a *one-dimensional* (first-order) nonlinear ordinary differential equation; see, e.g. [6], [15], [23], [24], [30], and [31]. The key point to note when solving optimal stopping problems for the jump processes established in [27] and [28] is that the smooth-fit condition at the optimal boundary may break down and then needs replacing by the continuous-fit condition; see also [1] for necessary and sufficient conditions for the occurrence of the smooth-fit condition and the references therein, and [29] for an extensive overview.

In the present paper we derive closed form solutions to the discounted optimal stopping problems, (2.4) and (5.1), below, in a jump-diffusion model driven by a Brownian motion and a compound Poisson process with exponential jumps. Such a model was considered in [12], [13], [17]–[19], [20], and [21], where some one-dimensional optimal stopping problems were solved. We note that the approach chosen in this paper, which is based on reducing the initial optimal stopping problem to solving the associated free-boundary problem, provides more valuable information on the nature of the solution and its analytic properties than the standard so-called guess-and-verify approach. More precisely, the obtained solution of the equivalent two-dimensional integro-differential free-boundary problem gives the possibility of observing explicitly not only that the smooth-fit condition for the value function on the optimal boundary but that the normal-reflection condition at the diagonal may also break down owing to the occurrence of jumps in the model. It is shown that, under certain relationships on the parameters of the model, the optimal stopping boundary can be uniquely determined as a component of the solution of a *two-dimensional system* of nonlinear (first-order) ordinary differential equations. These properties prove the structural difference between the solutions of the problem given by (2.4) in the continuous and jump-diffusion cases.

The paper is organized as follows. In Section 2, we formulate the optimal stopping problem, (2.4), for a two-dimensional Markov process related to the perpetual American *fixed-strike* lookback option problem and reduce it to an equivalent integro-differential free-boundary problem. In Section 3, we obtain an explicit solution to the free-boundary problem and derive nonlinear ordinary differential equations for the optimal stopping boundary as well as specify asymptotic behavior of the boundary under different relationships on the parameters of the model. In Section 4, using the change-of-variable formula with local time on surfaces, we verify that the solution of the free-boundary problem turns out to be a solution of the initial optimal stopping problem. In Section 5, we give some concluding remarks as well as present an explicit solution to the optimal stopping problem, (5.1), related to the perpetual American *floating-strike* lookback option problem. The main results of the paper are stated in Theorems 4.1 and 5.1.

2. Formulation of the problem

In this section we introduce the setting and notation of the two-dimensional optimal stopping problem, which is related to pricing the perpetual American fixed-strike lookback option, and formulate an equivalent integro-differential free-boundary problem.

For a precise formulation of the problem let us consider a probability space $(\Omega, \mathcal{F}, P)$ with a standard Brownian motion $B = (B_t)_{t\geq 0}$ and a jump process $J = (J_t)_{t\geq 0}$ defined by $J_t = \sum_{i=1}^{N_t} Y_i$, where $N = (N_t)_{t\geq 0}$ is a Poisson process with intensity $\lambda > 0$ and $(Y_i)_{i\in\mathbb{N}}$ is a sequence of independent random variables exponentially distributed with parameter 1 (B, N, and $(Y_i)_{i\in\mathbb{N}}$ are supposed to be independent). Assume that there exists a process $X = (X_t)_{t\geq 0}$ given by

$$X_t = x\exp\biggl(\biggl(r - \delta - \frac{\sigma^2}{2} - \frac{\lambda\theta}{1-\theta}\biggr)t + \sigma B_t + \theta J_t\biggr), \tag{2.1}$$

where $\sigma \geq 0$, $0 \leq \delta < r$, $\theta < 1$, and $\theta \neq 0$. It follows that the process X solves the stochastic differential equation

$$\mathrm{d}X_t = (r-\delta)X_{t-}\,\mathrm{d}t + \sigma X_{t-}\,\mathrm{d}B_t + X_{t-}\int_0^\infty (\mathrm{e}^{\theta y} - 1)(\mu(\mathrm{d}t, \mathrm{d}y) - \nu(\mathrm{d}t, \mathrm{d}y)) \qquad (X_0 = x), \tag{2.2}$$

where $x > 0$ is given and fixed. It can be assumed that the process X describes a stock price on a financial market, where $r > 0$ is the riskless interest rate and the dividend rate paid to stockholders is δ. Here $\mu(\mathrm{d}t, \mathrm{d}y)$ is the measure of the jumps of the process J with the compensator $\nu(\mathrm{d}t, \mathrm{d}y) = \lambda\,\mathrm{d}t\,\mathbf{1}(y > 0)\mathrm{e}^{-y}\,\mathrm{d}y$, which means that we work directly under a martingale measure for X; see, e.g. [34, Chapter VII, Section 3g]. Note that the assumption $\theta < 1$ guarantees that the jumps of X are integrable under the martingale measure, which is no restriction. With the process X let us associate the *maximum* process $S = (S_t)_{t\geq 0}$ defined by

$$S_t = \Bigl(\sup_{0\leq u\leq t} X_u\Bigr) \vee s \tag{2.3}$$

for an arbitrary $s \geq x > 0$. The main purpose of the present paper is to derive a solution to the optimal stopping problem for the time-homogeneous (strong) Markov process $(X, S) = (X_t, S_t)_{t\geq 0}$ given by

$$V_*(x, s) = \sup_\tau \mathrm{E}_{x,s}[\mathrm{e}^{-r\tau}(S_\tau - K)^+], \tag{2.4}$$

where the supremum is taken over all stopping times τ with respect to the natural filtration of X, and $\mathrm{E}_{x,s}$ denotes the expectation under the assumption that the (two-dimensional) process (X, S) defined by (2.1)–(2.3) starts at $(x, s) \in E$. Here $E = \{(x, s) \mid 0 < x \leq s\}$ denotes the state space of the process (X, S). The value of (2.4) coincides with an *arbitrage-free price* of a perpetual American fixed-strike lookback option with the strike price $K > 0$; see, e.g.

[34, Chapter VIII]. It is also seen that if $\sigma = 0$ and $0 < \theta < 1$ with $r - \delta - \lambda\theta/(1-\theta) \geq 0$, then $X_t = S_t$ for all $t \geq 0$ and, thus, (2.4) coincides with the value function of the perpetual American call option problem; see, e.g. [12] for a solution of this problem in the given model. Recall that for the continuous case, in which $\sigma > 0$ and $\theta = 0$, (2.4) was solved in [23] and [15].

2.1. The structure of the optimal stopping time in (2.4)

We determine the structure of the optimal stopping time in (2.4) as follows.

(i) By applying the arguments from [6, Subsection 3.2] and [24, Proposition 2.1] to the optimal stopping problem, (2.4), we see that it is never optimal to stop when $X_t = S_t$ for $t \geq 0$, when either $\sigma > 0, \theta < 0$, or $r - \delta - \lambda\theta/(1-\theta) < 0$ holds (this fact will also be proved independently in (iv), below). It also follows, directly from the structure of (2.4), that it is never optimal to stop when $S_t \leq K$ for $t \geq 0$. In other words, this shows that all points (x, s) from the set

$$C' = \{(x, s) \in E \mid 0 < x \leq s \leq K\} \tag{2.5}$$

and from the diagonal $\{(x, s) \in E \mid x = s\}$ belong to the continuation region

$$C_* = \{(x, s) \in E \mid V_*(x, s) > (s - K)^+\}. \tag{2.6}$$

(From the solution below it is seen that $V_*(x, s)$ is continuous, so that C_* is open.)

(ii) Let us fix $(x, s) \in C_*$ and let $\tau_* = \tau_*(x, s)$ denote the optimal stopping time in (2.4). Then, taking another starting point (y, s) for the process (X, S) such that $0 < x < y \leq s$ and using the fact that the running maximum S from (2.3) of the process X from (2.1) started at the point y is greater or equal to the running maximum S of X started at x, by virtue of the linear structure of the payoff function in the optimal stopping problem, (2.4), we obtain

$$V_*(y, s) \geq \mathrm{E}_{y,s}[\mathrm{e}^{-\lambda\tau_*}(S_{\tau_*} - K)^+] \geq \mathrm{E}_{x,s}[\mathrm{e}^{-\lambda\tau_*}(S_{\tau_*} - K)^+] = V_*(x, s) > (s - K)^+,$$

and, thus, we conclude that $(y, s) \in C_*$. Conversely, we note that the process (X, S) stays at the same level under the fixed second variable until it hits the diagonal $\{(x, s) \in E \mid x = s\}$. Following the lines of [24, Subsection 3.3] we also clearly observe that, due to the discounting in (2.4), we should not let the process (X, S) run too much to the left as it could be 'too expensive' to get back to the diagonal in order to offset the 'cost' spent to travel all the way. These arguments, together with the comments in [6, Subsection 3.3] and the fact that, by the structure of (2.4) and (2.3) with (2.1), the function $V_*(x, s)$ is convex in x on $(0, s)$ for each $s > 0$, show that there exists a function $g_*(s)$ for $s > K$ such that the continuation region, (2.6), is an open set consisting of (2.5) and of the set

$$C''_* = \{(x, s) \in E \mid g_*(s) < x \leq s, s > K\}, \tag{2.7}$$

while the stopping region is the closure of the set

$$D_* = \{(x, s) \in E \mid 0 < x < g_*(s), s > K\}. \tag{2.8}$$

(iii) Let us now show that in (2.7) and (2.8) the function $g_*(s)$ is increasing on (K, ∞); this fact also follows from the solution below. As in (2.4) the function $s - K$ is linear in s on (K, ∞), and by means of standard arguments, it is shown that $V_*(x, s) - (s - K)$ is decreasing in s on (K, ∞). Hence, if, for given $(x, s) \in C''_*$, we take s' such that $K < s' < s$, then $V_*(x, s') - (s' - K) \geq V_*(x, s) - (s - K) > 0$ so that $(x, s') \in C''_*$ and, thus, the assertion follows.

(iv) Let us denote by $V_*'(x, s)$ the value function of the optimal stopping problem related to the corresponding Russian option problem, where the optimal stopping time has the structure $\tau_*' = \inf\{t \geq 0 \mid X_t \leq a_* S_t\}$. It is easily seen that for the case in which $K = 0$, the function $V_*'(x, s)$ coincides with (2.4) and (5.1), while, under different relationships on the parameters of the model, $a_* < 1$ can be uniquely determined by (5.10), (5.12), (5.14), and (5.16). Suppose that $g_*(s) > a_* s$ for some $s > K$. Then, as is clearly seen from (2.4), for any given and fixed $x \in (a_* s, g_*(s))$, we have $V_*'(x, s) - K > s - K = V_*(x, s)$ contradicting the obvious fact that $V_*'(x, s) - K \leq V_*(x, s)$ for all $(x, s) \in E$ with $s > K$. Thus, we may conclude that $g_*(s) \leq a_* s < s$ for all $s > K$.

2.2. The free-boundary problem

By means of standard arguments it can be shown that the infinitesimal operator $\mathbb{L}$ of the process (X, S) acts on a function $F(x, s)$ from the class $C^{2,1}$ on E (or F from $C^{1,1}$ on E when $\sigma = 0$) according to the following rule:

$$(\mathbb{L}F)(x, s) = (r - \delta + \zeta) x F_x(x, s) + \frac{\sigma^2}{2} x^2 F_{xx}(x, s) + \int_0^\infty (F(x\mathrm{e}^{\theta y}, x\mathrm{e}^{\theta y} \vee s) - F(x, s)) \lambda \mathrm{e}^{-y} \, \mathrm{d}y$$

for all $0 < x < s$, where $\zeta = -\lambda\theta/(1 - \theta)$. Using standard arguments based on the strong Markov property it follows that the function $V_*(x, s)$ belongs to the class $C^{2,1}$ on $C_* \equiv C' \cup C_*''$ (or $V_*(x, s)$ belongs to $C^{1,1}$ on C_* when $\sigma = 0$). In order to find analytic expressions for the unknown value function $V_*(x, s)$ in (2.4) and the unknown boundary $g_*(s)$ in (2.7) and (2.8), let us use the results of general theory of optimal stopping problems for Markov processes; see, e.g. [33, Chapter III, Section 8] and [29, Chapter IV, Section 8]. We can reduce the optimal stopping problem, (2.4), to the equivalent free-boundary problem:

$$(\mathbb{L}V)(x, s) = rV(x, s) \quad \text{for } (x, s) \in C \equiv C' \cup C'' \text{ such that } x \neq s, \tag{2.9}$$

$$V(x, s)|_{x=g(s)+} = s - K \quad \text{(continuous-fit condition)}, \tag{2.10}$$

$$V(x, s) = (s - K)^+ \quad \text{for } (x, s) \in D, \tag{2.11}$$

$$V(x, s) > (s - K)^+ \quad \text{for } (x, s) \in C, \tag{2.12}$$

where C'' and D are defined as C_*'' and D_* in (2.7) and (2.8) with $g(s)$ in lieu of $g_*(s)$, respectively, and (2.10) plays the role of the instantaneous-stopping condition which is satisfied for all $s > K$. Observe that the superharmonic characterization of the value function (see [9], [29, Chapter IV, Section 9], and [33]) implies that $V_*(x, s)$ is the smallest function satisfying (2.9)–(2.11) with the boundary $g_*(s)$. Moreover, we further assume that the following conditions are satisfied for all $s > K$:

$$V_x(x, s)|_{x=g(s)+} = 0 \quad \text{if either } \sigma > 0 \text{ or } r - \delta + \zeta < 0 \text{ (smooth-fit condition)}, \tag{2.13}$$

$$V_s(x, s)|_{x=s-} = 0 \quad \text{if either } \sigma > 0 \text{ or } r - \delta + \zeta > 0 \text{ (normal-reflection condition)}. \tag{2.14}$$

Assumption (2.13) can be explained by the fact that in those cases, leaving the continuation region C_*, the process X can pass through the boundary $g_*(s)$ continuously. This property was earlier observed by Peskir and Shiryaev [27, Section 2], [28] when solving some other optimal stopping problems for jump processes. Assumption (2.14) can be explained by the fact that in those cases the process X can hit the diagonal continuously. This property was earlier explained in [6, Section 3.3]. We recall that for the continuous case, in which $\sigma > 0$ and $\theta = 0$, the free-boundary problem given by (2.9)–(2.14) was solved in [23] and [15].

2.3. Asymptotics for the value function $V_*(x, s)$

In order to specify the boundary $g_*(s)$ as a solution of the free-boundary problem, (2.9)–(2.14), we need to observe that, from (2.4), it follows that the inequalities

$$0 \le \sup_\tau \mathrm{E}_{x,s}[\mathrm{e}^{-r\tau} S_\tau] - K \le \sup_\tau \mathrm{E}_{x,s}[\mathrm{e}^{-r\tau}(S_\tau - K)^+] \le \sup_\tau \mathrm{E}_{x,s}[\mathrm{e}^{-r\tau} S_\tau], \tag{2.15}$$

which are equivalent to

$$0 \le V'_*(x, s) - K \le V_*(x, s) \le V'_*(x, s), \tag{2.16}$$

hold for all $(x, s) \in E$ with $s > K$. Thus, by setting $x = s$ into (2.16) we obtain

$$0 \le \frac{V'_*(s, s)}{s} - \frac{K}{s} \le \frac{V_*(s, s)}{s} \le \frac{V'_*(s, s)}{s} \tag{2.17}$$

for all $s > K$, so that by letting s tend to ∞ in (2.17) we obtain

$$\liminf_{s\to\infty} \frac{V_*(s, s)}{s} = \limsup_{s\to\infty} \frac{V_*(s, s)}{s} = \lim_{s\to\infty} \frac{V'_*(s, s)}{s}. \tag{2.18}$$

2.4. Estimating the value function $V_*(x, s)$

In order to estimate the value function $V_*(x, s)$ in (2.4), we observe that, from (2.15) and (2.16), it directly follows that the inequalities

$$0 \le V_*(x, s) - \mathrm{E}_{x,s}[\mathrm{e}^{-r\tau'_*}(S_{\tau'_*} - K)^+] \le K\,\mathrm{E}_{x,s}[\mathrm{e}^{-r\tau'_*}] \le \frac{K V'_*(x, s)}{s}$$

hold for all $(x, s) \in E$ with $s > K$, where $V'_*(x, s)$ and $\tau'_* = \inf\{t \ge 0 \mid X_t \le a_* S_t\}$ are the value function and the optimal stopping time, respectively, in (2.4) and (5.1) for the case in which $K = 0$.

3. Solution to the free-boundary problem

In this section we obtain solutions to the free-boundary problem, (2.9)–(2.14), and derive ordinary differential equations for the optimal boundary under different relationships on the parameters of the model, (2.1) and (2.2).

3.1. Reducing the free-boundary problem

By means of straightforward calculations we reduce (2.9) to the form

$$(r - \delta + \zeta) x V_x(x, s) + \frac{\sigma^2}{2} x^2 V_{xx}(x, s) - \alpha\lambda x^\alpha G(x, s) = (r + \lambda) V(x, s), \tag{3.1}$$

with $\alpha = 1/\theta$ and $\zeta = -\lambda\theta/(1-\theta)$, where taking into account conditions (2.10) and (2.11) we set

$$G(x, s) = -\int_x^s \frac{V(z, s)}{z^{\alpha+1}}\,\mathrm{d}z - \int_s^\infty \frac{V(z, z)}{z^{\alpha+1}}\,\mathrm{d}z \quad \text{if } \alpha = \frac{1}{\theta} > 1, \tag{3.2}$$

$$G(x, s) = \int_{g(s)}^x \frac{V(z, s)}{z^{\alpha+1}}\,\mathrm{d}z - \frac{s - K}{\alpha g(s)^\alpha} \quad \text{if } \alpha = \frac{1}{\theta} < 0, \tag{3.3}$$

for all $0 < g(s) < x \le s$ and $s > K$. Then, using the arguments from [12, Subsection 3.2], we may conclude that the function $G(x, s)$ in (3.2) and (3.3) solves an ordinary (third-order) differential equation, which is equivalent to (3.1), and its general solution is given by

$$G(x, s) = C_1(s)\frac{x^{\beta_1}}{\beta_1} + C_2(s)\frac{x^{\beta_2}}{\beta_2} + C_3(s)\frac{x^{\beta_3}}{\beta_3}, \tag{3.4}$$

where $C_1(s)$, $C_2(s)$, and $C_3(s)$ are some arbitrary functions and $\beta_3 < \beta_2 < \beta_1$, $\beta_i \neq 0$ for $i = 1, 2, 3$ are the real roots of the corresponding (characteristic) equation:

$$\frac{\sigma^2}{2}\beta^3 + \left[\sigma^2\left(\alpha - \frac{1}{2}\right) + r - \delta + \zeta\right]\beta^2 + \left[\alpha\left(\frac{\sigma^2(\alpha-1)}{2} + r - \delta + \zeta\right) - (r+\lambda)\right]\beta - \alpha\lambda = 0. \tag{3.5}$$

Therefore, differentiating both sides of (3.2) and (3.3) we find that the integro-differential equation, (3.1), has the general solution

$$V(x, s) = C_1(s)x^{\gamma_1} + C_2(s)x^{\gamma_2} + C_3(s)x^{\gamma_3}, \tag{3.6}$$

where we set $\gamma_i = \beta_i + \alpha$ for $i = 1, 2, 3$. Observe that if $\sigma = 0$ and $r - \delta + \zeta < 0$ then we can set $C_3(s) \equiv 0$ in (3.4) and (3.6), while the roots of (3.5) are explicitly given by

$$\beta_i = \frac{r+\lambda}{2(r-\delta+\zeta)} - \frac{\alpha}{2} - (-1)^i\sqrt{\left(\frac{r+\lambda}{2(r-\delta+\zeta)} - \frac{\alpha}{2}\right)^2 + \frac{\alpha\lambda}{r-\delta+\zeta}} \tag{3.7}$$

for $i = 1, 2$. Thus, by substituting (3.4) and (3.6) into (3.2) and letting $x = s$ we obtain

$$C_1(s)\frac{s^{\gamma_1}}{\beta_1} + C_2(s)\frac{s^{\gamma_2}}{\beta_2} + C_3(s)\frac{s^{\gamma_3}}{\beta_3} = f(s)s^{\alpha}(s-K), \tag{3.8}$$

where

$$f(s) = -\frac{1}{s-K}\int_s^{\infty}(C_1(z)z^{\beta_1-1} + C_2(z)z^{\beta_2-1} + C_3(z)z^{\beta_3-1})\,\mathrm{d}z \tag{3.9}$$

for $s > K$. Hence, by differentiating both sides of (3.8) and by applying conditions (3.3) and (2.10), and (2.13) and (2.14) to the functions in (3.4) and (3.6), respectively, we find that the following equalities hold for all $s > K$:

$$C_1'(s)\frac{s^{\gamma_1}}{\beta_1} + C_2'(s)\frac{s^{\gamma_2}}{\beta_2} + C_3'(s)\frac{s^{\gamma_3}}{\beta_3} = 0, \tag{3.10}$$

$$C_1(s)\frac{g(s)^{\gamma_1}}{\beta_1} + C_2(s)\frac{g(s)^{\gamma_2}}{\beta_2} + C_3(s)\frac{g(s)^{\gamma_3}}{\beta_3} = -\frac{s-K}{\alpha}, \tag{3.11}$$

$$C_1(s)g(s)^{\gamma_1} + C_2(s)g(s)^{\gamma_2} + C_3(s)g(s)^{\gamma_3} = s - K, \tag{3.12}$$

$$\gamma_1 C_1(s)g(s)^{\gamma_1} + \gamma_2 C_2(s)g(s)^{\gamma_2} + \gamma_3 C_3(s)g(s)^{\gamma_3} = 0, \tag{3.13}$$

$$C_1'(s)s^{\gamma_1} + C_2'(s)s^{\gamma_2} + C_3'(s)s^{\gamma_3} = 0. \tag{3.14}$$

Here, (3.8) and (3.10) hold if $0 < \theta < 1$, (3.11) holds if $\theta < 0$, (3.13) holds if either $\sigma > 0$ or $r - \delta + \zeta < 0$ with $\zeta = -\lambda\theta/(1-\theta)$, and (3.14) holds if either $\sigma > 0$ or $r - \delta + \zeta > 0$. We assume that the functions $C_i(s)$ for $i = 1, 2, 3$, as well as the boundary $g(s)$, are continuously differentiable for $s > K$. Below we determine the unknown functions $C_i(s)$ for $i = 1, 2, 3$ and the optimal boundary $g_*(s)$ under different relationships on the parameters of the model.

3.2. The subcase of negative jumps

Let us consider the subcase of negative jumps, i.e. $\alpha = 1/\theta < 0$. If, in addition, $\sigma > 0$ holds then, by solving the system (3.11)–(3.13) using straightforward calculations, we find that the solution of the system (2.9)–(2.11) and (2.13) is given by

$$V(x, s; g_*(s)) = \frac{\beta_1\gamma_2\gamma_3(s-K)}{\alpha(\gamma_2-\gamma_1)(\gamma_1-\gamma_3)}\left(\frac{x}{g_*(s)}\right)^{\gamma_1} + \frac{\beta_2\gamma_1\gamma_3(s-K)}{\alpha(\gamma_2-\gamma_1)(\gamma_3-\gamma_2)}\left(\frac{x}{g_*(s)}\right)^{\gamma_2} + \frac{\beta_3\gamma_1\gamma_2(s-K)}{\alpha(\gamma_1-\gamma_3)(\gamma_3-\gamma_2)}\left(\frac{x}{g_*(s)}\right)^{\gamma_3} \tag{3.15}$$

for $0 < g_*(s) < x \le s$ and $s > K$. Then, by applying condition (3.14) we find that condition (2.14) implies that the function $g_*(s)$ solves the following (first-order nonlinear) ordinary differential equation:

$$g'(s) = \frac{g(s)}{\gamma_1\gamma_2\gamma_3(s-K)} \times \frac{\beta_1\gamma_2\gamma_3(\gamma_2-\gamma_3)(s/g(s))^{\gamma_1} - \beta_2\gamma_1\gamma_3(\gamma_1-\gamma_3)(s/g(s))^{\gamma_2} + \beta_3\gamma_1\gamma_2(\gamma_1-\gamma_2)(s/g(s))^{\gamma_3}}{\beta_1(\gamma_2-\gamma_3)(s/g(s))^{\gamma_1} - \beta_2(\gamma_1-\gamma_3)(s/g(s))^{\gamma_2} + \beta_3(\gamma_1-\gamma_2)(s/g(s))^{\gamma_3}} \tag{3.16}$$

for $s > K$, with $\gamma_i = \beta_i + \alpha$, $i = 1, 2, 3$, where the β_is are the roots of (3.5).

Observe that if, in addition, $\sigma = 0$ holds, then we can put $C_3(s) \equiv 0$ in (3.4) and (3.6), and omit condition (2.13) which in turn implies condition (3.13). Thus, by solving the system (3.11)–(3.12) using straightforward calculations, we find that the solution of the system (2.9)–(2.11) is given by

$$V(x, s; g_*(s)) = \frac{\beta_1\gamma_2(s-K)}{\alpha(\gamma_1-\gamma_2)}\left(\frac{x}{g_*(s)}\right)^{\gamma_1} - \frac{\beta_2\gamma_1(s-K)}{\alpha(\gamma_1-\gamma_2)}\left(\frac{x}{g_*(s)}\right)^{\gamma_2} \tag{3.17}$$

for $0 < g_*(s) < x \le s$ and $s > K$. Then, by applying condition (3.14) we find that condition (2.14) implies that the function $g_*(s)$ solves the differential equation

$$g'(s) = \frac{g(s)}{\gamma_1\gamma_2(s-K)}\left(\frac{\beta_1\gamma_2(s/g(s))^{\gamma_1} - \beta_2\gamma_1(s/g(s))^{\gamma_2}}{\beta_1(s/g(s))^{\gamma_1} - \beta_2(s/g(s))^{\gamma_2}}\right) \tag{3.18}$$

for $s > K$, with $\gamma_i = \beta_i + \alpha$, $i = 1, 2$, where β_1 and β_2 are given by (3.7). Note that in this case we have $\beta_3 < 0 < \beta_2 < -\alpha < 1 - \alpha < \beta_1$ so that $\gamma_3 < \alpha < \gamma_2 < 0 < 1 < \gamma_1$ with $\gamma_i = \beta_i + \alpha$, where β_i, $i = 1, 2, 3$ are the roots of (3.5). Thus, by means of standard arguments it can be shown that the right-hand sides of (3.16) and (3.18) are positive, so that the function $g_*(s)$ is strictly increasing on (K, ∞).

Let us define $h_*(s) = g_*(s)/s$ for all $s > K$ and set $\underline{h} = \liminf_{s\to\infty} h_*(s)$ and $\bar{h} = \limsup_{s\to\infty} h_*(s)$. In order to specify the solutions of (3.16) and (3.18), which coincide with the optimal stopping boundary $g_*(s)$, we observe that from (3.15) and (3.17) it follows that (2.18) directly implies

$$\begin{aligned}\beta_1\gamma_2\gamma_3(\gamma_3-\gamma_2)\underline{h}^{-\gamma_1} + \beta_2\gamma_1\gamma_3(\gamma_1-\gamma_3)\bar{h}^{-\gamma_2} + \beta_3\gamma_1\gamma_2(\gamma_2-\gamma_1)\bar{h}^{-\gamma_3} \\ = \beta_1\gamma_2\gamma_3(\gamma_3-\gamma_2)\bar{h}^{-\gamma_1} + \beta_2\gamma_1\gamma_3(\gamma_1-\gamma_3)\underline{h}^{-\gamma_2} + \beta_3\gamma_1\gamma_2(\gamma_2-\gamma_1)\underline{h}^{-\gamma_3} \\ = \beta_1\gamma_2\gamma_3(\gamma_3-\gamma_2)a_*^{-\gamma_1} + \beta_2\gamma_1\gamma_3(\gamma_1-\gamma_3)a_*^{-\gamma_2} + \beta_3\gamma_1\gamma_2(\gamma_2-\gamma_1)a_*^{-\gamma_3}\end{aligned} \tag{3.19}$$

when $\sigma > 0$ and

$$\beta_1\gamma_2\bar{h}^{-\gamma_1} - \beta_2\gamma_1\underline{h}^{-\gamma_2} = \beta_1\gamma_2\underline{h}^{-\gamma_1} - \beta_2\gamma_1\bar{h}^{-\gamma_2} = \beta_1\gamma_2 a_*^{-\gamma_1} - \beta_2\gamma_1 a_*^{-\gamma_2} \tag{3.20}$$

when $\sigma = 0$, where a_* is uniquely determined by (5.10) and (5.12) under $K = 0$, respectively. Then, using the fact that $h_*(s) = g_*(s)/s \le a_*$ for $s > K$ and thus $\underline{h} \le \bar{h} \le a_* < 1$, from (3.19) and (3.20) we obtain $\underline{h} = \bar{h} = a_*$. Hence, we find that the optimal boundary $g_*(s)$ should satisfy the property

$$\lim_{s\to\infty} \frac{g_*(s)}{s} = a_*, \tag{3.21}$$

which gives a condition at ∞ for (3.16) and (3.18). By virtue of the results on the existence and uniqueness of solutions for first-order ordinary differential equations, we may therefore conclude that condition (3.21) uniquely specifies the solutions of (3.16) and (3.18), which correspond to the problem given by (2.4). Taking into account (3.15) and (3.17), we also note that from inequalities (2.16) it follows that the optimal boundary $g_*(s)$ satisfies the properties

$$g_*(K+) = 0 \quad \text{and} \quad g_*(s) \sim A_*(s-K)^{1/\gamma_1} \quad \text{under } s \downarrow K \tag{3.22}$$

for some constant $A_* > 0$, which can also be determined by means of condition (3.21).

3.3. The subcase of positive jumps

Let us now consider the subcase of positive jumps, i.e. $\alpha = 1/\theta > 1$. If, in addition, $\sigma > 0$ holds then, by solving the system (3.8), (3.12), and (3.13) using straightforward calculations, we find that the solution of the system (2.9)–(2.11) and (2.13) is given by

$$\begin{aligned}
&V(x, s; g_*(s)) \\
&\quad = \frac{\beta_1(s-K)(\beta_2\beta_3(\gamma_2-\gamma_3)s^\alpha f_*(s) + \beta_3\gamma_3(s/g_*(s))^{\gamma_2} - \beta_2\gamma_2(s/g_*(s))^{\gamma_3})}{\beta_2\beta_3(\gamma_2-\gamma_3)(s/g_*(s))^{\gamma_1} - \beta_1\beta_3(\gamma_1-\gamma_3)(s/g_*(s))^{\gamma_2} + \beta_1\beta_2(\gamma_1-\gamma_2)(s/g_*(s))^{\gamma_3}} \\
&\qquad \times \left(\frac{x}{g_*(s)}\right)^{\gamma_1} \\
&\qquad + \frac{\beta_2(s-K)(\beta_1\beta_3(\gamma_3-\gamma_1)s^\alpha f_*(s) - \beta_3\gamma_3(s/g_*(s))^{\gamma_1} + \beta_1\gamma_1(s/g_*(s))^{\gamma_3})}{\beta_2\beta_3(\gamma_2-\gamma_3)(s/g_*(s))^{\gamma_1} - \beta_1\beta_3(\gamma_1-\gamma_3)(s/g_*(s))^{\gamma_2} + \beta_1\beta_2(\gamma_1-\gamma_2)(s/g_*(s))^{\gamma_3}} \\
&\qquad \times \left(\frac{x}{g_*(s)}\right)^{\gamma_2} \\
&\qquad + \frac{\beta_3(s-K)(\beta_1\beta_2(\gamma_1-\gamma_2)s^\alpha f_*(s) + \beta_2\gamma_2(s/g_*(s))^{\gamma_1} - \beta_1\gamma_1(s/g_*(s))^{\gamma_2})}{\beta_2\beta_3(\gamma_2-\gamma_3)(s/g_*(s))^{\gamma_1} - \beta_1\beta_3(\gamma_1-\gamma_3)(s/g_*(s))^{\gamma_2} + \beta_1\beta_2(\gamma_1-\gamma_2)(s/g_*(s))^{\gamma_3}} \\
&\qquad \times \left(\frac{x}{g_*(s)}\right)^{\gamma_3}
\end{aligned} \tag{3.23}$$

for $0 < g_*(s) < x \le s$, where the function $f_*(s)$ is given by

$$f_*(s) = -\frac{1}{s-K}\int_s^\infty \frac{V(z, z; g_*(s))}{z^{\alpha+1}}\,\mathrm{d}z \tag{3.24}$$

for $s > K$. Then, by applying conditions (3.10) and (3.14) we find that conditions (3.2) and (2.14) imply that the functions $f_*(s)$ and $g_*(s)$ solve the following system of nonlinear

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(first-order) ordinary differential equations:

$$
\begin{aligned}
f'(s) = &-\frac{f(s)}{s-K}+ \\
&\frac{\beta_1\beta_2\beta_3 f(s)((\gamma_2-\gamma_3)(s/g(s))^{\gamma_1}-(\gamma_1-\gamma_3)(s/g(s))^{\gamma_2}+(\gamma_1-\gamma_2)(s/g(s))^{\gamma_3})}{s(\beta_2\beta_3(\gamma_2-\gamma_3)(s/g(s))^{\gamma_1}-\beta_1\beta_3(\gamma_1-\gamma_3)(s/g(s))^{\gamma_2}+\beta_1\beta_2(\gamma_1-\gamma_2)(s/g(s))^{\gamma_3})}+ \\
&\frac{\beta_3\gamma_3(\gamma_1-\gamma_2)(s/g(s))^{\gamma_1+\gamma_2}-\beta_2\gamma_2(\gamma_1-\gamma_3)(s/g(s))^{\gamma_1+\gamma_3}+\beta_1\gamma_1(\gamma_2-\gamma_3)(s/g(s))^{\gamma_2+\gamma_3}}{s^{\alpha+1}(\beta_2\beta_3(\gamma_2-\gamma_3)(s/g(s))^{\gamma_1}-\beta_1\beta_3(\gamma_1-\gamma_3)(s/g(s))^{\gamma_2}+\beta_1\beta_2(\gamma_1-\gamma_2)(s/g(s))^{\gamma_3})}
\end{aligned}
\tag{3.25}
$$

and

$$
\begin{aligned}
g'(s) = &\frac{g(s)}{s-K}\times \\
&\frac{\beta_3\gamma_3(\gamma_1-\gamma_2)(s/g(s))^{\gamma_1+\gamma_2}-\beta_2\gamma_2(\gamma_1-\gamma_3)(s/g(s))^{\gamma_1+\gamma_3}+\beta_1\gamma_1(\gamma_2-\gamma_3)(s/g(s))^{\gamma_2+\gamma_3}}{\beta_3(\gamma_1-\gamma_2)(s/g(s))^{\gamma_1+\gamma_2}-\beta_2(\gamma_1-\gamma_3)(s/g(s))^{\gamma_1+\gamma_3}+\beta_1(\gamma_2-\gamma_3)(s/g(s))^{\gamma_2+\gamma_3}}\times \\
&\frac{\beta_2\beta_3(\gamma_2-\gamma_3)(s/g(s))^{\gamma_1}-\beta_1\beta_3(\gamma_1-\gamma_3)(s/g(s))^{\gamma_2}+\beta_1\beta_2(\gamma_1-\gamma_2)(s/g(s))^{\gamma_3}}{\eta_2\eta_3(\gamma_2-\gamma_3)(s/g(s))^{\gamma_1}-\eta_1\eta_3(\gamma_1-\gamma_3)(s/g(s))^{\gamma_2}+\eta_1\eta_2(\gamma_1-\gamma_2)(s/g(s))^{\gamma_3}-\rho f(s)s^{\alpha}}
\end{aligned}
\tag{3.26}
$$

for $s > K$, with $\eta_i = \beta_i\gamma_i$ and $\gamma_i = \beta_i + \alpha$, $i = 1, 2, 3$, where the β_is are the roots of (3.5) and $\rho = \beta_1\beta_2\beta_3(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)(\gamma_2-\gamma_3)$.

In order to specify the solution of (3.25), by virtue of the inequalities given in (2.16) and using (5.13) we find that the function $f_*(s)$, given in (3.24), should satisfy the property

$$
\begin{aligned}
\lim_{s\to\infty} f_*(s)s^{\alpha} = &\frac{\gamma_2(\gamma_3-1)}{(\gamma_2-\gamma_1)(\beta_1(\gamma_3-1)a_*^{\gamma_1}-\beta_3(\gamma_1-1)a_*^{\gamma_3})} \\
&+\frac{\gamma_3(\gamma_1-1)}{(\gamma_3-\gamma_2)(\beta_2(\gamma_1-1)a_*^{\gamma_2}-\beta_1(\gamma_2-1)a_*^{\gamma_1})} \\
&+\frac{\gamma_1(\gamma_2-1)}{(\gamma_1-\gamma_3)(\beta_3(\gamma_2-1)a_*^{\gamma_3}-\beta_2(\gamma_3-1)a_*^{\gamma_2})},
\end{aligned}
\tag{3.27}
$$

where a_* is uniquely determined by (5.14) under $K = 0$. Hence, from (3.9) and (3.24) it therefore follows that (3.27) gives a condition at ∞ for (3.25).

Observe that if, in addition, $\sigma = 0$ and $\alpha = 1/\theta > 1$ holds with $r - \delta - \lambda\theta/(1-\theta) < 0$, then we can put $C_3(s) \equiv 0$ in (3.4) and (3.6) and omit condition (2.14) which in turn implies condition (3.14). Thus, by solving the system (3.11)–(3.13) using straightforward calculations, we find that the solution of the system (2.9)–(2.11) and (2.13) is given by

$$
V(x,s;g_*(s)) = \frac{\gamma_2(s-K)}{\gamma_2-\gamma_1}\left(\frac{x}{g_*(s)}\right)^{\gamma_1} - \frac{\gamma_1(s-K)}{\gamma_2-\gamma_1}\left(\frac{x}{g_*(s)}\right)^{\gamma_2}
\tag{3.28}
$$

for $0 < g_*(s) < x \le s$ and $s > K$. Then, by applying condition (3.10) we find that condition (3.2) implies that the function $g_*(s)$ solves the differential equation

$$
g'(s) = \frac{g(s)}{\gamma_1\gamma_2(s-K)}\left(\frac{\beta_2\gamma_2(s/g(s))^{\gamma_1}-\beta_1\gamma_1(s/g(s))^{\gamma_2}}{\beta_2(s/g(s))^{\gamma_1}-\beta_1(s/g(s))^{\gamma_2}}\right)
\tag{3.29}
$$

for $s > K$, with $\gamma_i = \beta_i + \alpha$, $i = 1, 2$, where β_1 and β_2 are given by (3.7). Note that in this case, under $\sigma > 0$, we have $\beta_3 < -\alpha < 1-\alpha < \beta_2 < 0 < \beta_1$ so that $\gamma_3 < 0 < 1 < \gamma_2 < \alpha < \gamma_1$

with $\gamma_i = \beta_i + \alpha$, $i = 1, 2, 3$, where the β_is are the roots of (3.5), while, under $\sigma = 0$ and $r - \delta - \lambda\theta/(1-\theta) < 0$, we have $\beta_2 < -\alpha < 1 - \alpha < \beta_1 < 0$ so that $\gamma_2 < 0 < 1 < \gamma_1$ with $\gamma_i = \beta_i + \alpha$, $i = 1, 2$, where β_1 and β_2 are given by (3.7). Thus, by means of standard arguments it can be shown that the right-hand sides of (3.16) and (3.18) are positive, so that the function $g_*(s)$ is strictly increasing on (K, ∞).

Let us recall that $\underline{h} = \liminf_{s\to\infty} h_*(s)$ and $\bar{h} = \limsup_{s\to\infty} h_*(s)$ with $h_*(s) = g_*(s)/s$ for all $s > K$. In order to specify the solutions of (3.26) and (3.29), which coincides with the optimal stopping boundary $g_*(s)$, we observe that from the expressions (3.23) with (3.27) and (3.28) it follows that (2.18) directly implies that

$$\begin{aligned}
&\frac{(\gamma_2-\gamma_3)\bar{h}^{-\gamma_1} - (\gamma_1-\gamma_3)\underline{h}^{-\gamma_2} + (\gamma_1-\gamma_2)\underline{h}^{-\gamma_3}}{\beta_2\beta_3(\gamma_2-\gamma_3)\underline{h}^{-\gamma_1} - \beta_1\beta_3(\gamma_1-\gamma_3)\underline{h}^{-\gamma_2} + \beta_1\beta_2(\gamma_1-\gamma_2)\underline{h}^{-\gamma_3}} \\
&\quad= \frac{(\gamma_2-\gamma_3)\underline{h}^{-\gamma_1} - (\gamma_1-\gamma_3)\bar{h}^{-\gamma_2} + (\gamma_1-\gamma_2)\bar{h}^{-\gamma_3}}{\beta_2\beta_3(\gamma_2-\gamma_3)\bar{h}^{-\gamma_1} - \beta_1\beta_3(\gamma_1-\gamma_3)\bar{h}^{-\gamma_2} + \beta_1\beta_2(\gamma_1-\gamma_2)\bar{h}^{-\gamma_3}} \\
&\quad= \frac{(\gamma_2-\gamma_3)a_*^{-\gamma_1} - (\gamma_1-\gamma_3)a_*^{-\gamma_2} + (\gamma_1-\gamma_2)a_*^{-\gamma_3}}{\beta_2\beta_3(\gamma_2-\gamma_3)a_*^{-\gamma_1} - \beta_1\beta_3(\gamma_1-\gamma_3)a_*^{-\gamma_2} + \beta_1\beta_2(\gamma_1-\gamma_2)a_*^{-\gamma_3}}
\end{aligned} \tag{3.30}$$

when $\sigma > 0$, and (3.2) yields

$$\gamma_2\bar{h}^{-\gamma_1} - \gamma_1\underline{h}^{-\gamma_2} = \gamma_2\underline{h}^{-\gamma_1} - \gamma_1\bar{h}^{-\gamma_2} = \gamma_2 a_*^{-\gamma_1} - \gamma_1 a_*^{-\gamma_2} \tag{3.31}$$

when $\sigma = 0$, where a_* is uniquely determined by (5.14) and (5.16) under $K = 0$, respectively. Then, using the fact that $h_*(s) = g_*(s)/s \le a_*$ for $s > K$ and thus $\underline{h} \le \bar{h} \le a_* < 1$, from (3.30) and (3.31) we obtain $\underline{h} = \bar{h} = a_*$. Hence, we find that the optimal boundary $g_*(s)$ should satisfy (3.21) which gives a condition at ∞ for (3.26) and (3.29). By virtue of the results on the existence and uniqueness of solutions for systems of first-order ordinary differential equations (see also the arguments in [15, pp. 655–656]), we may therefore conclude that conditions (3.27) and (3.21) uniquely specify the solution of the system (3.25), (3.26), and (3.29), which corresponds to the problem given by (2.4). Taking into account (3.23) and (3.28), we also note that from the inequalities in (2.16) it follows that the optimal boundary $g_*(s)$ satisfies (3.22) for some constant $A_* > 0$, which can be determined by means of condition (3.21).

3.4. The value function on the set C'

Observe that the above arguments show that, started at the point $(x, s) \in C'$, the process (X, S) can be stopped optimally only after it passes through the point (K, K). Thus, using standard arguments based on the strong Markov property it follows that

$$V_*(x, s) = U(x; K)V_*(K, K)$$

for all $(x, s) \in C'$ with $V_*(K, K) = \lim_{s\downarrow K} V_*(K, s)$, where we set

$$U(x; K) = \mathrm{E}_x[\mathrm{e}^{-r\theta_*}]$$

and

$$\theta_* = \inf\{t \ge 0 \mid X_t \ge K\}.$$

Here E_x denotes the expectation under the assumption that $X_0 = x$ for some $0 < x \le K$.

By means of straightforward calculations based on solving the corresponding boundary-value problem (see also [2], [3], and [19]) it follows that when $\alpha = 1/\theta < 0$ holds, we have

$$U(x; K) = \left(\frac{x}{K}\right)^{\gamma_1}, \tag{3.32}$$

with $\gamma_1 = \beta_1 + \alpha$, where if $\sigma > 0$ then β_1 is the largest root of (3.5), while if $\sigma = 0$ then β_1 is given by (3.7). It also follows that when $\alpha = 1/\theta > 1$ holds, we have

$$U(x; K) = \frac{\beta_1 \gamma_2}{\alpha(\gamma_1 - \gamma_2)} \left(\frac{x}{K}\right)^{\gamma_1} - \frac{\beta_2 \gamma_1}{\alpha(\gamma_1 - \gamma_2)} \left(\frac{x}{K}\right)^{\gamma_2}, \tag{3.33}$$

with $\gamma_i = \beta_i + \alpha$, where if $\sigma > 0$ then β_1 and β_2 are the two largest roots of (3.5), while if $\sigma = 0$ and $r - \delta - \lambda\theta/(1-\theta) < 0$ then β_1 and β_2 are given by (3.7).

4. Main result and proof

In this section, using the facts proved above, we formulate and prove the main result of the paper.

Theorem 4.1. *Let the process (X, S) be given by (2.1)–(2.3). Then the value function of the optimal stopping problem, (2.4), has the structure*

$$V_*(x, s) = \begin{cases} V(x, s; g_*(s)) & \text{if } g_*(s) < x \le s \text{ and } s > K, \\ U(x; K) V_*(K, K) & \text{if } 0 < x \le s \le K, \\ s - K & \text{if } 0 < x \le g_*(s) \text{ and } s > K, \end{cases} \tag{4.1}$$

with $V_*(K, K) = \lim_{s \downarrow K} V_*(K, s)$, and the optimal stopping time has the structure

$$\tau_* = \inf\{t \ge 0 \mid X_t \le g_*(S_t)\}, \tag{4.2}$$

where the functions $V(x, s; g_*(s))$ and $U(x; K)$, as well as the increasing boundary $g_*(s) \le a_* s < s$ for $s > K$ satisfying $g_*(K+) = 0$ and $g_*(s) \sim A_*(s - K)^{1/\gamma}$ under $s \downarrow K$ (see Figure 1), are specified as follows.

(i) *If $\sigma > 0$ and $\theta < 0$ then $V(x, s; g_*(s))$ is given by (3.15), $U(x; K)$ is given by (3.32), and $g_*(s)$ is uniquely determined from the differential equation (3.16) and condition (3.21), where $\gamma_i = \beta_i + 1/\theta$, $i = 1, 2, 3$ and the β_is are the roots of (3.5), while a_* is the unique solution of (5.10) under $K = 0$.*

(ii) *If $\sigma = 0$ and $\theta < 0$ then $V(x, s; g_*(s))$ is given by (3.17), $U(x; K)$ is given by (3.32), and $g_*(s)$ is uniquely determined from the differential equation (3.18) and condition (3.21), where $\gamma_i = \beta_i + 1/\theta$, $i = 1, 2$ and β_1 and β_2 are given by (3.7), while a_* is the unique solution of (5.12) under $K = 0$.*

(iii) *If $\sigma > 0$ and $0 < \theta < 1$ then $V(x, s; g_*(s))$ is given by (3.23), $U(x; K)$ is given by (3.33), and $g_*(s)$ is uniquely determined from the system of differential equations (3.25) and (3.26) and conditions (3.27) and (3.21), where $\gamma_i = \beta_i + 1/\theta$, $i = 1, 2, 3$ and the β_is are the roots of (3.5), while a_* is the unique solution of (5.14) under $K = 0$.*

(iv) *If $\sigma = 0$ and $0 < \theta < 1$ with $r - \delta - \lambda\theta/(1-\theta) < 0$ then $V(x, s; g_*(s))$ is given by (3.28), $U(x; K)$ is given by (3.33), and $g_*(s)$ is uniquely determined from the differential equation (3.29) and condition (3.21), where $\gamma_i = \beta_i + 1/\theta$, $i = 1, 2$ and β_1 and β_2 are given by (3.7), while a_* is the unique solution of (5.16) under $K = 0$.*

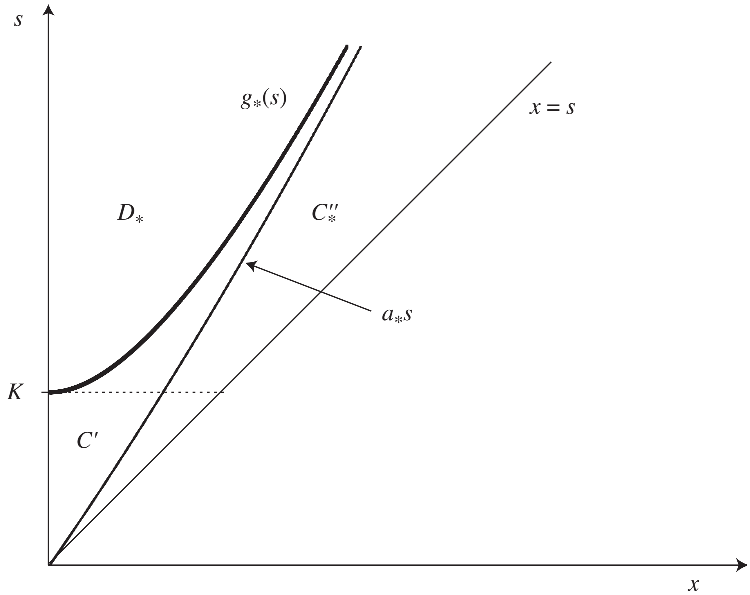

FIGURE 1: A computer drawing of the optimal stopping boundary $g_*(s)$.

Proof. In order to verify the assertions stated above, it remains to show that the functions given in (4.1) and (2.4) coincide and that the stopping time τ_* given in (4.2), with the boundary $g_*(s)$ specified above, is optimal. For this, let us denote by $V(x, s)$ the right-hand side of (4.1). In this case, by means of straightforward calculations and the assumptions above, it follows that the function $V(x, s)$ solves the system (2.9)–(2.11), and the smooth-fit condition (2.13) is satisfied when either $\sigma > 0$ or $r - \delta - \lambda\theta/(1-\theta) < 0$ holds, while the normal-reflection condition (2.14) is satisfied when either $\sigma > 0$ or $r - \delta - \lambda\theta/(1-\theta) > 0$ holds. Hence, taking into account the fact that the function $V(x, s)$ is continuous and the boundary $g_*(s)$ is assumed to be continuously differentiable for all $s > K$, by applying the change-of-variable formula [26, Theorem 3.1] to $\mathrm{e}^{-rt} V(X_t, S_t)$ we obtain

$$\begin{aligned} \mathrm{e}^{-rt} V(X_t, S_t) &= V(x, s) + \int_0^t \mathrm{e}^{-ru} (\mathbb{L}V - rV)(X_u, S_u)\, \mathbf{1}(X_u \neq g_*(S_u), X_u \neq S_u)\, \mathrm{d}u \\ &\quad + \int_0^t \mathrm{e}^{-ru} V_s(X_{u-}, S_{u-})\, \mathrm{d}S_u - \sum_{0<u\le t} \mathrm{e}^{-ru} V_s(X_{u-}, S_{u-}) \Delta S_u + M_t, \end{aligned} \tag{4.3}$$

where the process $(M_t)_{t\ge0}$ given by

$$\begin{aligned} M_t &= \int_0^t \mathrm{e}^{-ru} V_x(X_u, S_u)\, \mathbf{1}(X_u \neq g_*(S_u), X_u \neq S_u) \sigma X_u\, \mathrm{d}B_u \\ &\quad + \int_0^t \int_0^\infty \mathrm{e}^{-ru} (V(X_{u-}\mathrm{e}^{\theta y}, X_{u-}\mathrm{e}^{\theta y} \vee S_{u-}) - V(X_{u-}, S_{u-}))(\mu(\mathrm{d}u, \mathrm{d}y) - \nu(\mathrm{d}u, \mathrm{d}y)) \end{aligned} \tag{4.4}$$

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is a local martingale with respect to $\mathrm{P}_{x,s}$, a probability measure under which the process (X, S) defined by (2.1)–(2.3) starts at $(x, s) \in E$. We remark that, when $\sigma > 0$, the smooth-fit condition (2.13) holds, so that there is no local time term in (4.3). Note that when $\sigma = 0$ and $r - \delta - \lambda\theta/(1-\theta) = 0$, the indicators in (4.3) and (4.4) can be set to one. Observe that when either $\sigma > 0$ or $\theta < 0$, the process S increases only continuously, so that the sum with respect to ΔS_u in (4.3) is equal to 0; the integral with respect to $\mathrm{d}S_u$ is also equal to 0 as at the diagonal $\{(x, s) \in E \mid x = s\}$ we assume (2.14). When $\sigma = 0$ and $0 < \theta < 1$ with $r - \delta - \lambda\theta/(1-\theta) < 0$, the process S increases only by jumping; thus, in (4.3), the integral with respect to $\mathrm{d}S_u$ is deleted by the sum with respect to ΔS_u.

Using straightforward calculations and the arguments from the previous section, it can be verified that $(\mathbb{L}V - rV)(x, s) \le 0$ for all $(x, s) \in E$ such that $x \ne g_*(s)$ and $x \ne s$. Moreover, by means of standard arguments it can be shown that the function $V(x, s)$ is increasing in both variables and, thus, property (2.12) holds and together with (2.10) and (2.11) yields $V(x, s) \ge (s-K)^+$ for all $(x, s) \in E$. Observe that from (2.1) it is seen that, when either $\sigma > 0$ or $r - \delta - \lambda\theta/(1-\theta) \ne 0$, the time spent by the process X at the diagonal $\{(x, s) \in E \mid x = s\}$ and at the boundary $g_*(s)$ is of Lebesgue measure 0. Thus, in these cases the indicators appearing in (4.3) and (4.4) can also be ignored. Hence, from (4.3) it therefore follows that the inequalities

$$\mathrm{e}^{-r\tau}(S_\tau - K)^+ \le \mathrm{e}^{-r\tau} V(X_\tau, S_\tau) \le V(x, s) + M_\tau \tag{4.5}$$

hold for any finite stopping time τ with respect to the natural filtration of X.

Let $(\tau_n)_{n\in\mathbb{N}}$ be an arbitrary localizing sequence of stopping times for the process $(M_t)_{t\ge 0}$. Taking the expectation with respect to $\mathrm{P}_{x,s}$ in (4.5), by means of the optional sampling theorem (see, e.g. [16, Chapter I, Theorem 1.39]) we obtain

$$\begin{aligned}\mathrm{E}_{x,s}[\mathrm{e}^{-r(\tau\wedge\tau_n)}(S_{\tau\wedge\tau_n} - K)^+] &\le \mathrm{E}_{x,s}[\mathrm{e}^{-r(\tau\wedge\tau_n)} V(X_{\tau\wedge\tau_n}, S_{\tau\wedge\tau_n})] \\ &\le V(x, s) + \mathrm{E}_{x,s}[M_{\tau\wedge\tau_n}] \\ &= V(x, s)\end{aligned}$$

for all $(x, s) \in E$. Hence, letting n tend to ∞ and using Fatou's lemma, for any finite stopping time τ we find that the following inequalities are satisfied for all $(x, s) \in E$:

$$\mathrm{E}_{x,s}[\mathrm{e}^{-r\tau}(S_\tau - K)^+] \le \mathrm{E}_{x,s}[\mathrm{e}^{-r\tau} V(X_\tau, S_\tau)] \le V(x, s). \tag{4.6}$$

By virtue of the fact that the function $V(x, s)$ together with the boundary $g_*(s)$ satisfy the system (2.9)–(2.12) and taking into account the structure of τ_* given in (4.2), it follows, from (4.3), that the following equalities hold for all $(x, s) \in E$ and any localizing sequence $(\tau_n)_{n\in\mathbb{N}}$ of $(M_t)_{t\ge 0}$:

$$\mathrm{e}^{-r(\tau_*\wedge\tau_n)}(S_{\tau_*\wedge\tau_n} - K)^+ = \mathrm{e}^{-r(\tau_*\wedge\tau_n)} V(X_{\tau_*\wedge\tau_n}, S_{\tau_*\wedge\tau_n}) = V(x, s) + M_{\tau_*\wedge\tau_n}. \tag{4.7}$$

Observe that by virtue of (2.15) and (2.16), and taking into account the integrability of jumps of the process X, applying the same arguments as in [30, pp. 635–636] and using the independence of the processes B and J in (2.1), it can be shown that the following property holds for all $(x, s) \in E$:

$$\mathrm{E}_{x,s}\Big[\sup_{t\ge 0} \mathrm{e}^{-r(\tau_*\wedge t)} S_{\tau_*\wedge t}\Big] = \mathrm{E}_{x,s}\Big[\sup_{t\ge 0} \mathrm{e}^{-r(\tau_*\wedge t)} X_{\tau_*\wedge t}\Big] < \infty,$$

where the variable $\mathrm{e}^{-r\tau_*} S_{\tau_*}$ is bounded on the set $\{\tau_* = \infty\}$. We also note that, by using the asymptotic behavior of $g_*(s)$ at ∞, it can be verified that $\mathrm{P}_{x,s}[\tau_* < \infty] = 1$ for all $(x, s) \in E$.

Hence, letting n tend to ∞ and using conditions (2.10) and (2.11), we can apply the Lebesgue dominated convergence theorem for (4.7) to obtain

$$\mathrm{E}_{x,s}[\mathrm{e}^{-r\tau_*}(S_{\tau_*} - K)^+] = V(x, s)$$

for all $(x, s) \in E$, which together with (4.6) directly implies the desired assertion.

Remark 4.1. Observe that when $\sigma = 0$ and $\theta < 0$ the smooth-fit condition, (2.13), fails to hold. This property can be explained by the fact that, in this case, leaving the continuation region $g_*(s) < x \leq s$, the process X can pass through the boundary $g_*(s)$ only by jumping. Such an effect was earlier observed and explained in [27, Section 2] and [28] by solving other optimal stopping problems for jump processes.

Remark 4.2. Note that when $\sigma = 0$ and $0 < \theta < 1$ with $r - \delta - \lambda\theta/(1-\theta) < 0$ the normal-reflection condition, (2.14), fails to hold. This property can be explained by the fact that, in this case, the process X can hit the diagonal $\{(x, s) \in E \mid x = s\}$ only by jumping.

According to the results of [1], we may conclude that the properties described in Remarks 4.1 and 4.2 appear because of the finite intensity of jumps and the exponential distribution of jump sizes of the compound Poisson process J.

5. Conclusions

In this section we give some concluding remarks and present an explicit solution to the optimal stopping problem which is related to pricing the perpetual American floating-strike lookback option.

5.1. Some comments

We have considered the two-dimensional American fixed-strike lookback option optimal stopping problem in a jump-diffusion model with infinite time horizon. In order to be able to derive nonlinear (first-order) ordinary differential equations for the optimal boundary that separates the continuation and stopping regions, we have let the jumps of the driving compound Poisson process be exponentially distributed. We have proved that, under certain relationships on the parameters of the model, the optimal boundary can be determined as a component of the solution of a two-dimensional system of nonlinear ordinary differential equations. This is in contrast with the structure of solutions of optimal stopping problems for maxima of continuous diffusion processes, where the optimal boundaries are determined by one-dimensional nonlinear ordinary differential equations. We have also derived some special conditions which uniquely specify the solution corresponding to the initial optimal stopping problem in the family of solutions of the related system of differential equations. The existence and uniqueness of such a solution is obtained by means of standard methods of first-order ordinary differential equations.

Note that the arguments presented above show that the structure of the optimal exercise time in the American fixed-strike lookback option problem does not change under extensions of the driving process from Brownian motion to a compound Poisson process with mixed-exponentially distributed jumps as well as to a more general Lévy process. The same phenomenon holds in the case of standard American put and call as well as Russian option problems; see, e.g. [2], [3], [20], and [21]. We also remark that, from the arguments above, it can be seen that the following structural properties of the solution should be observed under certain extensions of the considered jump-diffusion model. If the driving compound Poisson process had only negative mixed-exponential jumps, then the nonlinear (first-order) ordinary differential equation for the optimal exercise boundary would remain one-dimensional. In contrast to this case, if the driving process had positive or both-sided mixed-exponential jumps, then the dimension of the system of nonlinear ordinary differential equations for the boundary would increase to one plus the number of independent positive exponential jump components in the given mixture. If the driving process had jumps of more general probability distribution or were even a more general Lévy process, then the solution of the free-boundary problem would not be determined in a closed form and the boundary would be characterized only by nonlinear integral equations.

In the rest of the paper we derive a solution to the perpetual American floating-strike lookback option problem in the jumps-diffusion model, (2.1)–(2.3). In contrast to the fixed-strike case, by means of the change-of-measure theorem, the related two-dimensional optimal stopping problem can be reduced to an optimal stopping problem for a one-dimensional strong Markov process, $(S_t/X_t)_{t\geq 0}$, which explains the simplicity of the structure of the solution in (5.9)–(5.16); see [31] and [4] for a solution of the problem in the continuous model case.

5.2. Formulation of the floating-strike problem

Let us now consider the optimal stopping problem

$$W_*(x, s) = \sup_{\tau} \mathrm{E}_{x,s}[\mathrm{e}^{-r\tau}(S_\tau - KX_\tau)^+], \tag{5.1}$$

where the supremum is taken over all stopping times τ with respect to the natural filtration of X. The value of (2.4) coincides with an *arbitrage-free price* of a perpetual American floating-strike lookback option (or 'partial lookback' as it is called in [5]) with the strike price $K > 0$. Note that for the continuous case, in which $\sigma > 0$ and $\theta = 0$, the problem given by (5.1) was solved in [4]. It is also seen that if $\sigma = 0$ and $0 < \theta < 1$ with $r - \delta - \lambda\theta/(1-\theta) \geq 0$ then $X_t = S_t$ for all $t \geq 0$ and, thus, the optimal stopping time in (5.1) is trivial. By means of the same arguments as above (see also [4]) it can be shown that the optimal stopping time in (5.1) has the structure

$$\sigma_* = \inf\{t \geq 0 \mid X_t \leq b_* S_t\}. \tag{5.2}$$

In order to find analytic expressions for the unknown value function $W_*(x, s)$ from (5.1) and the unknown boundary $b_* s$ from (5.2), we can formulate the following free-boundary problem:

$$(\mathbb{L}W)(x, s) = rW(x, s) \quad \text{for } bs < x < s, \tag{5.3}$$

$$W(x, s)|_{x=bs+} = s(1 - Kb) \quad \text{(continuous-fit condition)}, \tag{5.4}$$

$$W(x, s) = (s - Kx)^+ \quad \text{for } 0 < x < bs, \tag{5.5}$$

$$W(x, s) > (s - Kx)^+ \quad \text{for } bs < x \leq s, \tag{5.6}$$

where (5.4) plays the role of the instantaneous-stopping condition and, in addition, we have

$$W_x(x, s)|_{x=bs+} = -K \quad \text{if either } \sigma > 0 \text{ or } r - \delta + \zeta < 0 \text{ (smooth-fit condition)}, \tag{5.7}$$

$$W_s(x, s)|_{x=s-} = 0 \quad \text{if either } \sigma > 0 \text{ or } r - \delta + \zeta > 0 \text{ (normal-reflection condition)}, \tag{5.8}$$

satisfied for all $s > 0$. Note that, by virtue of the structure of (5.1) and (5.2), it is easily seen that $b_* \leq 1/K$. Recall that for the continuous case, in which $\sigma > 0$ and $\theta = 0$, the free-boundary problem (5.3)–(5.8) was solved in [4].

5.3. Solution of the floating-strike problem

Following the schema of arguments from the previous section and using straightforward calculations, it can be shown that when $\sigma > 0$ and $\alpha = 1/\theta < 0$ the solution of system (5.3)–(5.7) takes the form

$$\begin{aligned} W(x, s; b_* s) = {} & \frac{\beta_1((1-\alpha)\gamma_2\gamma_3 + \alpha(\gamma_2 - 1)(\gamma_3 - 1)Kb_*)s}{\alpha(1-\alpha)(\gamma_2-\gamma_1)(\gamma_1-\gamma_3)}\left(\frac{x}{b_* s}\right)^{\gamma_1} \\ & + \frac{\beta_2((1-\alpha)\gamma_1\gamma_3 + \alpha(\gamma_1 - 1)(\gamma_3 - 1)Kb_*)s}{\alpha(1-\alpha)(\gamma_2-\gamma_1)(\gamma_3-\gamma_2)}\left(\frac{x}{b_* s}\right)^{\gamma_2} \\ & + \frac{\beta_3((1-\alpha)\gamma_1\gamma_2 + \alpha(\gamma_1 - 1)(\gamma_2 - 1)Kb_*)s}{\alpha(1-\alpha)(\gamma_1-\gamma_3)(\gamma_3-\gamma_2)}\left(\frac{x}{b_* s}\right)^{\gamma_3} \end{aligned} \tag{5.9}$$

for $0 < b_* s < x \le s$, and from condition (5.8) it follows that b_* solves the equation

$$\begin{aligned} & \frac{\beta_1(\gamma_1 - 1)((1-\alpha)\gamma_2\gamma_3 + \alpha(\gamma_2 - 1)(\gamma_3 - 1)Kb)}{(\gamma_2-\gamma_1)(\gamma_1-\gamma_3)b^{\gamma_1}} \\ & + \frac{\beta_2(\gamma_2 - 1)((1-\alpha)\gamma_1\gamma_3 + \alpha(\gamma_1 - 1)(\gamma_3 - 1)Kb)}{(\gamma_2-\gamma_1)(\gamma_3-\gamma_2)b^{\gamma_2}} \\ & \quad = \frac{\beta_3(\gamma_3 - 1)((1-\alpha)\gamma_1\gamma_2 + \alpha(\gamma_1 - 1)(\gamma_2 - 1)Kb)}{(\gamma_3-\gamma_1)(\gamma_3-\gamma_2)b^{\gamma_3}}, \end{aligned} \tag{5.10}$$

while, when $\sigma = 0$ and $\alpha = 1/\theta < 0$, the solution of system (5.3)–(5.6) takes the form

$$\begin{aligned} W(x, s; b_* s) = {} & \frac{\beta_1((1-\alpha)\gamma_2 + \alpha(\gamma_2 - 1)Kb_*)s}{\alpha(1-\alpha)(\gamma_1-\gamma_2)}\left(\frac{x}{b_* s}\right)^{\gamma_1} \\ & - \frac{\beta_2((1-\alpha)\gamma_1 + \alpha(\gamma_1 - 1)Kb_*)s}{\alpha(1-\alpha)(\gamma_1-\gamma_2)}\left(\frac{x}{b_* s}\right)^{\gamma_2} \end{aligned} \tag{5.11}$$

for $0 < b_* s < x \le s$, and from condition (5.8) it follows that b_* solves the equation

$$b^{\gamma_1-\gamma_2} = \frac{\beta_2(\gamma_2 - 1)}{\beta_1(\gamma_1 - 1)}\left(\frac{(1-\alpha)\gamma_1 + \alpha(\gamma_1 - 1)Kb}{(1-\alpha)\gamma_2 + \alpha(\gamma_2 - 1)Kb}\right). \tag{5.12}$$

It can be shown that when $\sigma > 0$ and $\alpha = 1/\theta > 1$, the solution of system (5.3)–(5.6) and (5.8) takes the form

$$\begin{aligned} W(x, s; b_* s) = {} & \frac{\beta_1(\gamma_3 - 1)(\gamma_2 - (\gamma_2 - 1)Kb_*)b_*^{\gamma_1} s}{(\gamma_2-\gamma_1)(\beta_1(\gamma_3 - 1)b_*^{\gamma_1} - \beta_3(\gamma_1 - 1)b_*^{\gamma_3})}\left(\frac{x}{b_* s}\right)^{\gamma_1} \\ & + \frac{\beta_2(\gamma_1 - 1)(\gamma_3 - (\gamma_3 - 1)Kb_*)b_*^{\gamma_2} s}{(\gamma_3-\gamma_2)(\beta_2(\gamma_1 - 1)b_*^{\gamma_2} - \beta_1(\gamma_2 - 1)b_*^{\gamma_1})}\left(\frac{x}{b_* s}\right)^{\gamma_2} \\ & + \frac{\beta_3(\gamma_2 - 1)(\gamma_1 - (\gamma_1 - 1)Kb_*)b_*^{\gamma_3} s}{(\gamma_1-\gamma_3)(\beta_3(\gamma_2 - 1)b_*^{\gamma_3} - \beta_2(\gamma_3 - 1)b_*^{\gamma_2})}\left(\frac{x}{b_* s}\right)^{\gamma_3} \end{aligned} \tag{5.13}$$

for $0 < b_* s < x \le s$, and from condition (5.7) it follows that b_* solves the equation

$$\frac{\beta_1(\gamma_1-1)(\gamma_3-1)(\gamma_2-(\gamma_2-1)Kb)}{(\gamma_2-\gamma_1)(\beta_1(\gamma_3-1)b^{\gamma_1}-\beta_3(\gamma_1-1)b^{\gamma_3})} + \frac{\beta_2(\gamma_1-1)(\gamma_2-1)(\gamma_3-(\gamma_3-1)Kb)}{(\gamma_3-\gamma_2)(\beta_2(\gamma_1-1)b^{\gamma_2}-\beta_1(\gamma_2-1)b^{\gamma_1})} = \frac{\beta_3(\gamma_2-1)(\gamma_3-1)(\gamma_1-(\gamma_1-1)Kb)}{(\gamma_3-\gamma_1)(\beta_3(\gamma_2-1)b^{\gamma_3}-\beta_2(\gamma_3-1)b^{\gamma_2})}, \tag{5.14}$$

while, when $\sigma = 0$ and $\alpha = 1/\theta > 1$ with $r - \delta - \lambda\theta/(1-\theta) < 0$, the solution of system (5.3)–(5.6) takes the form

$$W(x, s; b_* s) = \frac{(\gamma_2-(\gamma_2-1)Kb_*)s}{\gamma_2-\gamma_1}\left(\frac{x}{b_* s}\right)^{\gamma_1} - \frac{(\gamma_1-(\gamma_1-1)Kb_*)s}{\gamma_2-\gamma_1}\left(\frac{x}{b_* s}\right)^{\gamma_2} \tag{5.15}$$

for $0 < b_* s < x \le s$, and from condition (5.7) it follows that b_* solves the equation

$$b^{\gamma_1-\gamma_2} = \frac{\beta_2}{\beta_1}\frac{\gamma_2(\gamma_1-1)+(\gamma_1-\gamma_2(\gamma_1-1))Kb}{\gamma_1(\gamma_2-1)+(\gamma_2-\gamma_1(\gamma_2-1))Kb}. \tag{5.16}$$

Summarizing the facts proved above, we formulate the following result.

Theorem 5.1. *Let the process (X, S) be defined by (2.1)–(2.3). Then the value function of the problem given by (5.1) takes the structure*

$$W_*(x, s) = \begin{cases} W(x, s; b_* s) & \text{if } b_* s < x \le s, \\ s - Kx & \text{if } 0 < x \le b_* s, \end{cases}$$

and the optimal stopping time is explicitly given by (5.2), where the function $W(x, s; b_* s)$ and the boundary $b_* s \le s/K$ for $s > 0$ are specified as follows.

(i) *If $\sigma > 0$ and $\theta < 0$ then $W(x, s; b_* s)$ is given by (5.9) and b_* is the unique solution of (5.10), where $\gamma_i = \beta_i + 1/\theta$, $i = 1, 2, 3$ and the β_is are the roots of (3.5).*

(ii) *If $\sigma = 0$ and $\theta < 0$ then $W(x, s; b_* s)$ is given by (5.11) and b_* is the unique solution of (5.12), where $\gamma_i = \beta_i + 1/\theta$, $i = 1, 2$ and β_1 and β_2 are given by (3.7).*

(iii) *If $\sigma > 0$ and $0 < \theta < 1$ then $W(x, s; b_* s)$ is given by (5.13) and b_* is the unique solution of (5.14), where $\gamma_i = \beta_i + 1/\theta$, $i = 1, 2, 3$ and the β_is are the roots of (3.5).*

(iv) *If $\sigma = 0$ and $0 < \theta < 1$ with $r - \delta - \lambda\theta/(1-\theta) < 0$ then $W(x, s; b_* s)$ is given by (5.15) and b_* is the unique solution of (5.16), where $\gamma_i = \beta_i + 1/\theta$, $i = 1, 2$ and β_1 and β_2 are given by (3.7).*

Theorem 5.1 can be proved by means of the same arguments as used in the proof of Theorem 4.1, above.

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