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# A CLASSIFICATION OF 2-VARIETIES

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**1. Introduction.** The purpose of this paper is to give a classification of those varieties $\mathscr{V}_F$ of power-associative algebras over a field $F$ which satisfy the condition

(1.1) For each $A$ in $\mathscr{V}_F$ and each ideal $I$ of $A$, $I^2$ is an ideal of $A$.

Such varieties have been called 2-varieties by Zwier [**17**]. It is well-known that the varieties of associative, alternative and Lie algebras have property (1.1), as do the $(\gamma, \delta)$ algebras of Albert [**1**]. Moreover it has been shown in [**3**] that condition (1.1) is equivalent to the requirement that all algebras in $\mathscr{V}_F$ satisfy a pair of identities of the following type:

$$(1.2)\qquad (x_1x_2)x_3 = \alpha_1(x_3x_1)x_2 + \alpha_2(x_1x_3)x_2 + \alpha_3x_2(x_3x_1) + \alpha_4x_2(x_1x_3) + \alpha_5(x_3x_2)x_1 + \alpha_6(x_2x_3)x_1 + \alpha_7x_1(x_3x_2) + \alpha_8x_1(x_2x_3).$$

$$(1.3)\qquad x_3(x_1x_2) = \beta_1(x_3x_1)x_2 + \beta_2(x_1x_3)x_2 + \beta_3x_2(x_3x_1) + \beta_4x_2(x_1x_3) + \beta_5(x_3x_2)x_1 + \beta_6(x_2x_3)x_1 + \beta_7x_1(x_3x_2) + \beta_8x_1(x_2x_3).$$

The $\alpha$'s and $\beta$'s are assumed to be in $F$.

In 1949 Albert [**1**] gave a classification of those 2-varieties $\mathscr{V}_F$ which satisfy the further condition

(1.4) There exists in $\mathscr{V}_F$ a non-commutative algebra with identity.

By complicated arguments involving the notion of quasi-equivalence, Albert showed that the principal algebras of this type were the $(\gamma, \delta)$ algebras. Subsequent investigations into the structure of $(\gamma, \delta)$ algebras were made by Kleinfeld, Kokoris, Maneri, Hentzel and others, and there exists a satisfactory structure theory for these algebras (see [**6**; **7**; **8**; **9**; **10**; **11**; **12**; **13**; **14**]). The reason we return to the classification of algebras satisfying (1.2) and (1.3) is twofold. First of all, in view of the fact that associator dependent algebras have already been classified [**12**], we may restrict ourselves to those algebras which are not associator dependent, thereby avoiding lengthy calculations involving quasi-equivalence. Secondly, condition (1.4) eliminates from the very beginning any consideration of Lie algebras and so is too restrictive. Ideally a survey of 2-varieties should explain which of these varieties are of known type, for instance Lie or alternative, and which are uninteresting, so that future investigations can concentrate on the rest. In view of the large number of

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parameters, namely 16, which define a 2-variety it seems surprising that substantial progress is possible. Thus we show that most 2-varieties are not very interesting. Guided by the examples of alternative and Lie algebras we first consider those 2-varieties satisfying a condition similar to Albert's. Then we consider a much different condition, namely

(1.5) There exists in $\mathscr{V}_F$ a non-zero, finite dimensional, semi-simple nil algebra.

In the first instance the algebras in the variety turn out to be associator dependent, or commutative, or they satisfy one of the following identities:

$$(1.6) \quad \alpha_5((y, z), x) = (x, z, y) - (x, y, z),$$

$$(1.7) \quad \alpha_1((z, x), y) = (x, y, z),$$

$$(1.8) \quad \alpha_1((z, x), y) = (x, y, z) + (y, z, x) + (x, z, y)$$

$$(1.9) \quad ((z, x), y) = \frac{1}{2\delta + 1}[(x, z, y) - (z, x, y) - (z, y, x) + (x, y, z)] + (y, z, x) - (y, x, z), \text{ where } \delta \neq 1, -1, -\tfrac{1}{2},$$

$$(1.10) \quad \alpha_1((z, x), y) = \frac{-\delta}{\delta^2 - 1}[(z, y, x) - (x, y, z)] + \frac{1}{1 - \delta}[(x, z, y) - (z, x, y)] \text{ and } (x, y, z) + (y, z, x) + (z, x, y) = 0, \text{ where } \delta \neq 1, -1, -\tfrac{1}{2}.$$

With the exception of the algebras which satisfy (1.7), these being non-commutative Jordan and in fact quasi-associative, these algebras have not been previously studied. We intend to remedy this situation in subsequent papers. Identity (1.6) is of particular interest in the case $\alpha_5 = -\frac{1}{4}$, because the algebras satisfying such an identity are not quasi-equivalent to associator dependent algebras, much less $(\gamma, \delta)$ algebras. Thus it would appear that Albert's classification of 2-varieties satisfying (1.4) is incomplete. For varieties satisfying (1.5) the classification is not so crisp as above. Nevertheless we show that an important class of algebras of this kind are the algebras of type $\delta$. These satisfy the identity

$$(1.11) \quad (1 + \delta)x_3(x_1 \circ x_2) + (1 - \delta)(x_1 \circ x_2)x_3 = x_2(x_1 \circ x_3) + x_1(x_2 \circ x_3), \text{ where } a \circ b = 1/2(ab + ba).$$

We show that the finite dimensional, semi-simple, algebras of type $\delta \neq -\frac{1}{2}$ are direct sums of fields and semi-simple nil algebras. This part of the survey shows that the following classes deserve further attention.

(1.12) Semi-simple, nil algebras of type $\delta$,

(1.13) Algebras satisfying (1.2) and (1.3), where the following relations hold among the $\alpha$'s and $\beta$'s:

$$(1.14) \quad 1 = -\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 = \alpha_5 - \alpha_6 - \alpha_7 + \alpha_8 = \beta_1 - \beta_2 - \beta_3 + \beta_4 = -\beta_5 + \beta_6 + \beta_7 - \beta_8 = \beta_1 + \beta_5 - \alpha_1 - \alpha_5 = \beta_2 + \beta_6 - \alpha_2 - \alpha_6 = \textstyle\sum_{i=1}^{8} \alpha_i = \sum_{i=1}^{8} \beta_i.$$

*Remark.* Since submitting this paper, we have proved the finite dimensional semi-simple nil algebras of type $\delta \neq -\frac{1}{2}$ to be Lie algebras. Details will appear elsewhere.

**2. Preliminaries.** Throughout this paper $F$ will denote a field of characteristic $\neq 2, 3$, and unless otherwise stated, $A$ will denote a finite dimensional algebra over $F$. For elements $a, b, c$ in $A$ the associator and commutator are defined by $(a, b, c) = (ab)c - a(bc)$ and $(a, b) = ab - ba$. The algebra $A$ is power-associative if each element of $A$ generates an associative subalgebra. In particular, for power-associative algebras we have the identity

(2.1) $\quad (x, x, x) = 0.$

Linearizing (2.1) yields

(2.2) $\quad (x, y, z) + (y, z, x) + (z, x, y) + (x, z, y) + (z, y, x) + (y, x, z) = 0.$

A power-associative algebra is said to be nil if each of its elements generates a nilpotent (associative) subalgebra. This nil algebras are radical in the general sense of Kurosch - Amitsur [**5**]. However the nil radical is too large to be of any value in investigations which may involve Lie algebras. Consequently we shall restrict ourselves to the solvable radical in this paper. This is defined as follows: For an algebra $A$, let $A^n$ denote the linear span of all products of $n$ elements of $A$. Inductively define $A^{(n)}$ by $A^{(0)} = A$, $A^{(n)} = [A^{(n-1)}]^2$, for $n > 0$. Then $A$ is said to be solvable if $A^{(n)} = 0$, for some $n$. The (solvable) radical of an arbitrary algebra $A$ is its maximal solvable ideal $\beta(A)$, and $A$ is called semi-simple if $\beta(A) = 0$. Finally $A$ is called nilpotent if for some $n$, $A^n = 0$.

The right and left multiplications, $R_a$ and $L_a$, determined by an element $a \in A$, are the maps defined by $R_a : x \to xa$ and $L_a : x \to ax$.

For elements $a, b$ in $A$ define $a \circ n = \frac{1}{2}(ab + ba)$. We have the following Peirce decomposition relative to an idempotent $e$ of an algebra $A$.

(2.3) *Proposition. If $A$ is power-associative then $A$ is a vector space direct sum $A = A_1(e) + A_{1/2}(e) + A_0(e)$, where $A_\lambda(e) = \{x \in A | ex = xe = \lambda x, \lambda = 1, 0\}$ and $A_{1/2}(e) = \{x \in A | e \circ x = \frac{1}{2}x\}$. In addition $A_1(e)A_0(e) = A_0(e)A_1(e) = 0$, $A_{1/2}(e) \circ A_{1/2}(e) \subseteq A_1(e) + A_0(e)$, $A_\lambda(e) \circ A_{1/2}(e) \subseteq A_{1/2}(e) + A_{1-\lambda}(e)$, and $A_\lambda(e) \circ A_\lambda(e) \subseteq A_\lambda(e)$, for $\lambda = 0, 1$. Moreover if $A$ satisfies the identities* (1.2) *and* (1.3) *then $A_0(e)$ is a subalgebra of $A$.*

*Proof.* All but the last assertion are well known [**16**]. Choosing $x_1, x_2 \in A_0(e)$ and $x_3 = e$ and substituting in (1.2) and (1.3) yields $(x_1x_2)e = 0 = e(x_1x_2)$, so that $x_1x_2 \in A_0(e)$, and hence $A_0(e)$ is a subalgebra. This completes the proof of the proposition.

We remark here that if $A$ is a power-associative algebra which is not nil then one of its elements generates an associative subalgebra which is not nilpotent and hence contains an idempotent. This makes the above proposition applicable. Furthermore if $e$ is an idempotent and $f$ an idempotent in $A_0(e)$, then $e + f$ is an idempotent and $A_1(e + f) \supset A_1(e)$. Hence if $e$ is a principal idempotent in the sense that dim $A_1(e)$ is maximal among all idempotents, then $A_0(e)$ must be a nil subalgebra of $A$. This observation will be quite useful in the sequel.

An arbitrary algebra $A$ can be turned into an anti-commutative algebra $A^{(-)}$ by replacing the product $ab$ of $A$ with the commutator $(a, b)$. If $A^{(-)}$ is a Lie algebra the $A$ is said to be Lie admissible. Direct calculations shows $A$ to be Lie admissible if and only if

$$(2.4) \qquad (x, y, z) + (y, z, x) + (z, x, y) - (x, z, y) - (z, y, x) - (y, x, z) = 0.$$

Comparison with (2.2) shows that (2.4) is equivalent to

$$(2.5) \qquad (x, y, z) + (y, z, x) + (z, x, y) = 0.$$

In [**12**] one finds a fairly satisfactory theory of associator dependent algebras. These are defined as satisfying the following identity:

$$(2.6) \qquad \textstyle\sum_\sigma \alpha_\sigma(\sigma(x), \sigma(y), \sigma(z)) = 0,$$

where the summation runs through all $\sigma$ in the symmetric group on 3 things and $\alpha_\sigma \in F$, with the $\alpha_\sigma$'s not all equal. This assures one a stronger identity than third power associativity. It is clear that Lie admissible algebras are associator dependent, as are algebras of $(\gamma, \delta)$ type [**11**]. An important class of associator dependent algebras includes the flexible algebras, which satisfy the identity

$$(2.7) \qquad (x, y, x) = 0,$$

or equivalently

$$(2.8) \qquad (x, y, z) + (z, y, x) = 0.$$

We wish finally to introduce some useful notation. If $\mathscr{V}_F$ is the 2-variety defined by (1.2) and (1.3), then we may identify $\mathscr{V}_F$ with the point $(\alpha_1, \ldots, \alpha_8, \beta_1, \ldots, \beta_8)$ of $F^{16}$. For a subset $\Gamma$ of $F^{16}$ we denote by $\mathscr{V}_F(\Gamma)$ the set of all 2-varieties defined by (1.2) and (1.3) for which $(\alpha_1, \ldots, \alpha_8, \beta_1, \ldots, \beta_8) \in \Gamma$. If $\mathscr{V}_F \in \mathscr{V}_F(\Gamma)$ we shall say that the variety $\mathscr{V}_F$ is of type $\mathscr{V}_F(\Gamma)$. Occasionally we shall abuse notation and write $A \in \mathscr{V}_F(\Gamma)$, where $A$ is an algebra. What this means is that $A \in \mathscr{V}_F$ for some $\mathscr{V}_F \in \mathscr{V}_F(\Gamma)$. Finally $\Gamma'$ will denote the complement of $\Gamma$.

**3. Radicals and the elimination of uninteresting varieties.** The main result of this section is that for most 2-varieties the properties of being nil, nilpotent or solvable are all equivalent provided one assumes finite dimensionality. This fact permits a considerable reduction in the classification of the varieties.

We introduce now the following two algebraic manifolds of $F^{16}$. $\Gamma$ will denote the set of all points $(\alpha_1, \ldots, \alpha_8, \beta_1, \ldots, \beta_8)$ of $F^{16}$ such that

$$(3.1) \qquad 1 = -\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 = \alpha_5 - \alpha_6 - \alpha_7 + \alpha_8 \\ = \beta_1 - \beta_2 - \beta_3 + \beta_4 = -\beta_5 + \beta_6 + \beta_7 - \beta_8,$$

and $\Delta$ consists of those points satisfying

$$(3.2) \qquad 1 = \sum_{i=1}^{8} \alpha_i = \sum_{i=1}^{8} \beta_i.$$

(3.3) THEOREM. *Let $A \in \mathscr{V}_F(\Gamma')$ be a finite dimensional algebra and suppose $B$ is a solvable subalgebra of $A$. Then*

(1) *The algebra $B^*$ generated by $\{R_b, L_b | b \in B\}$ is nilpotent.*

(2) *If $A$ is solvable then $A$ is nilpotent.*

*Proof.* The results of [**3**] and [**4**] show the defining properties of $\Gamma'$ are sufficient to insure the nilpotency of the enveloping algebra $\mathscr{U}(B)$ of $B$. Therefore, as $B^*$ is a homomorphic image of $\mathscr{U}(B)$, $B^*$ is nilpotent. This proves (1). For (2) we take $B = A$ to conclude that $A^*$ is nilpotent. It is well-known [**16**] that this implies the nilpotence of $A$.

(3.4) THEOREM. *Each finite dimensional nil algebra $A \in \mathscr{V}_F(\Gamma')$ is nilpotent.*

*Proof.* We use induction on the dimension of $A$, the result being trivial for one dimensional algebras. Let $B$ be a maximal nilpotent subalgebra of $A$. From part (1) of (3.3) it follows that $B^*$ is nilpotent. Thus $AB^{*n} = 0$, for some integer $n$. It follows that there exists an integer $k$ for which $AB^{*k} \subseteq B$, and we choose $k$ to be the smallest positive such integer. Assume $k > 1$ and choose $u \in AB^{*k-1}$, such that $u \notin B$. Then for each $b \in B$, $uR_b$ and $uL_b$ belong to $AB^{*k}$. Thus $uB + Bu \subseteq B$. From this inclusion and identities (1.2) and (1.3) it follows that $u^nB + Bu^n \subseteq B$, for every integer $n$. Therefore if $C = B + Fu + Fu^2 + \ldots$ is the subalgebra of $A$ generated by $B$ and $u$, then $B$ is an ideal of $C$. Moreover since $u$ is nilpotent, $C/B$ is nilpotent, so that $C$ is solvable. Then part (2) of (3.3) shows that $C$ is nilpotent, contrary to the maximality of $B$. Thus we have $k = 1$ and $AB^* \subseteq B$. In particular $AR_b + AL_b \subseteq B$, for every $b$ in $B$, which shows that $B$ is an ideal of $A$. Clearly $B \neq 0$, hence $\dim (A/B) < \dim (A)$. Thus by the induction hypothesis $A/B$ is nilpotent, and this implies the solvability and nilpotence of $A$, as required. This completes the proof of the theorem.

*Remark.* Hentzel [**8**] earlier proved that finite dimensional $(-1, 1)$ nil algebras are nilpotent. Theorem (3.4) is a generalization of this result.

If we set $x_1 = x_2 = x_3 = x$, in (1.2) and (1.3), we find that $0 = (-1 + \sum_{i=1}^{8} \alpha_i)x^3 = (-1 + \sum_{i=1}^{8} \beta_i)x^3$. Thus the varieties $\mathscr{V}_F(\Delta')$ consist solely of nil algebras. An immediate consequence of Theorem (3.4) is that the varieties $\mathscr{V}_F(\Gamma' \cap \Delta')$ contain no nonnilpotent finite dimensional algebras. We consider these varieties to be of little interest. It should be noted however that $\Gamma' \cap \Delta'$

is a Zariski open subset of $F^{16}$, and in the case $|F| = \infty$, it is also a dense subset. Thus most varieties are of type $\mathscr{V}_F(\Gamma' \cap \Delta')$. The set of all 2-varieties is the union of $\mathscr{V}_F(\Delta)$ and $\mathscr{V}_F(\Delta')$, and $\mathscr{V}_F(\Delta') = \mathscr{V}_F(\Delta' \cap \Gamma) \cup \mathscr{V}_F(\Delta' \cap \Gamma')$. Thus with the exception of the varieties $\mathscr{V}_F(\Delta' \cap \Gamma)$, which are studied in Section 7, we may safely assume that the relations (3.2) hold for the parameters defining the varieties. What this means is that the identities (1.2) and (1.3) defining $\mathscr{V}_F$ are not inconsistent with the existence of an algebra $A \in \mathscr{V}_F$ with identity element.

**4. Reduction to associator dependent algebras.** We study now in depth the consequences which may be derived from identities that are compatible with the existence of an identity element. We ascertain as much as possible the strength or weakness of every identity of degree three, which is to say every identity of the form

$$\begin{aligned}(4.1)\qquad 0 = {} & \alpha_1(xy)z + \alpha_2(yz)x + \alpha_3(zx)y + \alpha_4(yx)z + \alpha_5(xz)y + \alpha_6(zy)x \\ & + \alpha_7 x(yz) + \alpha_8 y(zx) + \alpha_9 z(xy) + \alpha_{10} y(xz) + \alpha_{11} x(zy) + \alpha_{12} z(yx).\end{aligned}$$

The proof of the following theorem gives a systematic procedure for analyzing all identities of degree three.

(4.2) THEOREM. *If $A$ satisfies an identity of degree three and if the existence of an identity element does not lead to a contradiction then either* (1) *commutativity is implied, or* (2) *$A$ is associator dependent, or* (3) *the identity satisfied by $A$ is equivalent to one of the form* $((x, y), z) = \beta_1(x, y, z) + \beta_2(y, z, x) + \beta_3(z, x, y) + \beta_4(x, z, y) + \beta_5(z, y, x) + \beta_6(y, x, z)$.

*Proof.* By a sequence of commuting and reassociating it becomes clear that every term of (4.1) may be reduced to a scalar multiple of $(xy)z$. Moreover, since $(xy, z) = x(y, z) + (x, z)y + (x, y, z) + (z, x, y) - (x, z, y)$, one can rewrite (4.1) in the form.

$$\begin{aligned}(4.3)\qquad \gamma(xy)z = {} & \delta_1 x(y, z) + \delta_2 y(z, x) + \delta_3 z(x, y) + \delta_4((x, y), z) \\ & + \delta_5((y, z), x) + \delta_6((z, x), y) + \delta_7(x, y, z) + \delta_8(y, z, x) \\ & + \delta_9(z, x, y) + \delta_{10}(x, z, y) + \delta_{11}(z, y, x) + \delta_{12}(y, x, z),\end{aligned}$$

where $\gamma = \delta_1 + \delta_2 + \ldots + \delta_{12}$. Setting $x = y = z = 1$, in (4.3) implies $\gamma = 0$. Next, setting just one of the variables equal to 1, one finds that $\delta_1(y, z) = \delta_2(z, x) = \delta_3(x, y) = 0$. So unless commutativity is to be implied, we must have $\delta_1 = \delta_2 = \delta_3 = 0$. What remains of (4.3) is the identity

$$\begin{aligned}(4.4)\qquad 0 = {} & \delta_4((x, y), z) + \delta_5((y, z), x) + \delta_6((z, x), y) + \delta_7(x, y, z) \\ & + \delta_8(y, z, x) + \delta_9(z, x, y) + \delta_{10}(x, z, y) + \delta_{11}(z, y, x) + \delta_{12}(y, x, z).\end{aligned}$$

This may be worked against the identity

$$\begin{aligned}(4.5)\qquad ((x, y), z) + ((y, z), x) + ((z, x), y) = {} & (x, y, z) + (y, z, x) \\ & + (z, x, y) - (x, z, y) - (z, y, x) - (y, x, z),\end{aligned}$$

which holds in every algebra. If $\delta_4 = \delta_5 = \delta_6$, then comparison of (4.4) with (4.5) either leads to an associator dependent relation or else (4.4) is equivalent to (4.5) and hence is true in every algebra. Thus we may assume $\delta_4$, $\delta_5$ and $\delta_6$ are not all equal. Then we can eliminate one of the double commutators between (4.4) and (4.5) to obtain, say,

$$((x, y), z) + \epsilon((y, z), x) = \sigma_1(x, y, z) + \sigma_2(y, z, x) + \sigma_3(z, x, y) + \sigma_4(x, z, y) + \sigma_5(z, y, x) + \sigma_6(y, x, z). \tag{4.6}$$

By interchanging $z$ and $x$ in (4.6) we get

$$-\epsilon((x, y), z) - ((y, z), x) = \sigma_1(z, y, x) + \sigma_2(y, x, z) + \sigma_3(x, z, y) + \sigma_4(z, x, y) + \sigma_5(x, y, z) + \sigma_6(y, z, x). \tag{4.7}$$

Unless $\epsilon = 1$ or $\epsilon = -1$, we can eliminate one of the double commutators between (4.6) and (4.7). If $\epsilon = 1$, then comparison of (4.6) with (4.5) yields an identity which expresses one double commutator as a linear combination of associators and we are done. Finally if $\epsilon = -1$, then adding (4.5) and (4.6) yields an identity similar to (4.6), but with $\epsilon = 2$, and so the previous reduction can be carried out. This completes the proof of the theorem.

In general we cannot say too much about identities which are weaker than the commutative law, but sometimes this is not the case, as the following result shows.

(4.8) THEOREM. *If $A$ is an algebra satisfying the identity $((x, y), z) = (x, y, z) + (z, x, y) - (x, z, y)$, and if $A$ has no non-zero nilpotent elements, then $A$ must be commutative.*

*Proof.* Permuting the three variables cyclically twice and adding results in an identity, which when compared (4.5), implies that $A$ is Lie admissible, hence satisfies

$$(x, y, z) + (y, z, x) + (z, x, y) = 0. \tag{4.9}$$

But then the defining identity may be rewritten as

$$((x, y), z) = -(y, z, x) - (x, z, y). \tag{4.10}$$

Setting $x = y$ in (4.10), we have

$$(x, z, x) = 0. \tag{4.11}$$

Linearizing (4.11) and comparing that with (4.10) then yields

$$((x, y), z) = 0. \tag{4.12}$$

Since in every algebra $(xy, z) = x(y, z) + (x, z)y + (x, y, z) + (z, x, y) - (x, z, y)$ it follows from (4.9) and the linearization of (4.11) that $(xy, z) = x(y, z) + (x, z)y$. From this last relation and (4.12) we find that $0 = ((a^2, b), c)$

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$= (a(a, b), c) + ((a, b)a, c) = 2(a(a, b), c) = 2(a, c)(a, b)$. Thus

(4.13) $(a, c)(a, b) = 0.$

Now setting $b = c$ in (4.13) completes the proof of the theorem.

Next we consider an application of Theorem (4.2) to 2-varieties. In fact only identity (1.2) is required. We assume the $\alpha$'s are such that the existence of an identity element forces neither commutativity nor anti-commutativity. This results in the following conditions on the $\alpha$'s.

(4.14) $$\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 = 0 = \alpha_1 + \alpha_3 + \alpha_5 + \alpha_6 = \alpha_1 + \alpha_2 + \alpha_5 + \alpha_7, \quad \text{and} \quad \sum_{i=1}^{8} \alpha_i = 1.$$

One may note by the way that (4.14) is equivalent to the following condition: Adjoining an identity element to an algebra satisfying (1.2) produces an an algebra which again satisfies (1.2).

From the relations (4.14) one can reduce the eight parameters to four free ones, since

(4.15) $\alpha_4 = \alpha_1, \alpha_6 = -\alpha_1 - \alpha_3 - \alpha_5, \alpha_7 = -\alpha_1 - \alpha_2 - \alpha_5$, and $\alpha_8 = 1 + \alpha_5$.

Revising (1.2) in the light of (4.15), we find that

(4.16) $$(x, y, z) = \alpha_1((z, x), y) - \alpha_5((y, z), x) + (\alpha_1 + \alpha_2)(x, z, y) + (-\alpha_1 - \alpha_3)(y, z, x).$$

Permuting cyclically and using (4.5), we get $J = (2\alpha_1 - 2\alpha_5 - 2\alpha_1 - \alpha_2 - \alpha_3)J$, where $J = (x, y, z) + (y, z, x) + (z, x, y)$. This shows

(4.17) An algebra satisfying (1.2) and (4.14) is Lie admissible if $2\alpha_5 + \alpha_2 + \alpha_3 + 1 \neq 0.$

Interchanging $x$ and $y$ in (4.16) we have

(4.18) $$(y, x, z) = \alpha_5((z, x), y) - \alpha_1((y, z), x) + (-\alpha_1 - \alpha_3)(x, z, y) + (\alpha_1 + \alpha_2)(y, z, x).$$

Multiplying (4.16) by $\alpha_1$ and (4.18) by $\alpha_5$ and then subtracting yields

(4.19) $$(\alpha_1^2 - \alpha_5^2)((z, x), y) = \alpha_1(x, y, z) - \alpha_5(y, x, z) + (\alpha_1^2 + \alpha_1\alpha_3 + \alpha_1\alpha_5 + \alpha_2\alpha_5)(y, z, x) + (-\alpha_1^2 - \alpha_1\alpha_2 - \alpha_1\alpha_5 - \alpha_3\alpha_5)(x, z, y).$$

It will be fruitful now to examine special cases. If $\alpha_1 = \alpha_5 = 0$, then (4.16) reduces to a form of associator dependence which does not follow from Lie admissibility, and as such algebras have already been studied in detail [**12**], we shall ignore this class here. Let us suppose then that $\alpha_1 = 0$, and $\alpha_5 \neq 0$. Then (4.16) becomes $\alpha_5((y, z), x) = \alpha_2(x, z, y) - \alpha_3(y, z, x) - (x, y, z)$. Substituting $z = y$, we get an associator dependence relation which does not follow from Lie admissibility, unless $\alpha_2 - 1 = -\alpha_3 = 0$. In the latter case we

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get the identity

$$(4.20) \quad \alpha_5((y, z), x) = (x, z, y) - (x, y, z).$$

If $\alpha_5 = 0$ and $\alpha_1 \neq 0$ then (4.16) reduces to $\alpha_1((z, x), y) = (\alpha_1 + \alpha_3)(y, z, x) + (x, y, z) + (-\alpha_1 - \alpha_2)(x, z, y)$. Substituting $z = x$, leads to either flexibility, if $\alpha_1 + \alpha_3 = 0 = -\alpha_1 - \alpha_2$, or to nothing if $\alpha_1 + \alpha_3 = 1 = -\alpha_1 - \alpha_2$, or to associator dependency. In the first case,

$$(4.21) \quad \alpha_1((z, x), y) = (x, y, z), \text{ and } (a, b, a) = 0.$$

Such algebras are non-commutative Jordan algebras and quasi-associative in the sense of Albert [**2**] as well and need not be considered further here. In the second case we obtain

$$(4.22) \quad \alpha_1((z, x), y) = (x, y, z) + (y, z, x) + (x, z, y).$$

If $\alpha_1 \neq \frac{1}{2}$, then (4.22) implies Lie admissibility and so (4.22) may be changed to the identity $\alpha_1((z, x), y) = (x, z, y) - (z, x, y)$. In this form the identity is a counterpart to (4.20).

From now on we shall assume $\alpha_1 \neq 0$, and $\alpha_5 \neq 0$. If $\alpha_5 = -\alpha_1$ then the identity $((x, y), z) + (y, z), x) + ((z, x), y) = 2J$, which holds in any power-associative algebra, when compared with (4.16), leads to an associator dependent relation. If $\alpha_5 = \alpha_1$, then comparison of (4.16) and (4.18) again leads to an associator dependence relation. Thus we may assume $\alpha_5 \neq \pm\alpha_1$. Now set $z = x$ in (4.19). This results in either flexibility or associator dependence unless certain relations among the $\alpha$'s hold. However flexibility in conjunction with Lie admissibility leads to an identity of the form (4.21). The same is true otherwise, since we have already considered (4.20) and (4.22). If we let $\alpha_5 = p\alpha_1$, then we have $\alpha_1^2 + \alpha_1\alpha_3 + \alpha_1\alpha_5 + \alpha_2\alpha_5 - \alpha_5 = \alpha_1 = -\alpha_1^2 - \alpha_1\alpha_2 - \alpha_1\alpha_5 - \alpha_3\alpha_5$. Cancelling $\alpha_1$, we obtain the equations

$$(p + 1)\alpha_1 + p\alpha_2 + \alpha_3 = p + 1$$
$$(p + 1)\alpha_1 + \alpha_2 + p\alpha_3 = -1$$

which have the solution

$$(4.23) \quad \alpha_5 = p\alpha_1, \quad \alpha_3 = -\alpha_1 - \frac{2p + 1}{p^2 - 1}, \quad \alpha_2 = -\alpha_1 + \frac{p^2 + p + 1}{p^2 - 1}.$$

Now (4.19) can be rewritten as $(\alpha_1^2 - \alpha_5^2)((z, x), y) = \alpha_1(x, y, z) - \alpha_5(y, x, z) + (\alpha_1 + \alpha_5)(y, z, x) + \alpha_1(x, z, y)$, or

$$(4.24) \quad (1 - p^2)\alpha_1((z, x), y) = p[(y, z, x) - (y, x, z)] + (x, z, y) - (z, x, y) + J.$$

At this point we can take two distinct paths, for either $J = 0$ is a consequence of our identity or it is not. If it is not, we can use (4.17) and (4.23) to get a

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complete solution for all the $\alpha$'s in terms of $p$ alone. That solution is

$$(4.25)\quad \alpha_1 = -\frac{2p+1}{2(p^2-1)}, \quad \alpha_2 = \frac{2p^2+4p+3}{2(p^2-1)}, \quad \alpha_3 = -\frac{2p+1}{2(p^2-1)},$$
$$\alpha_5 = -\frac{2p^2+p}{2(p^2-1)}, \quad p \neq 1, -1, -\frac{1}{2}.$$

In the light of (4.25) we can rewrite (4.19) as

$$(4.26)\quad ((z, x), y) = \frac{1}{2p+1}[(x, y, z) - (z, x, y) - (z, y, x) + (x, y, z)] + (y, z, x) - (y, x, z).$$

It turns out that if one substitutes (4.26) back into (4.16), that a tautology results. Thus the two identities are equivalent for those values of the coefficients. Rewriting (4.19) in the light of (4.23) and using Lie admissibility yields

$$(4.27)\quad \alpha_1((z, x), y) = -\frac{p}{p^2-1}[(z, y, x) - (x, y, z)] + \frac{-1}{p-1}[(x, z, y) - (z, x, y)],$$

and

$$(x, y, z) + (y, z, x) + (z, x, y) = 0, \quad \text{and } p \neq 1, -1, -\frac{1}{2}.$$

We have completed an exhaustive study of all the consequences of (1.2) when subject to the condition (4.14). Aside from associator dependent algebras, which have already been classified, the new identities this focuses on are (4.20), (4.22), (4.26), and (4.27). These new classes will be investigated elsewhere.

Since Albert used quasi-equivalence in his classification of 2-varieties, perhaps a remark about these new classes of algebras via quasi-equivalence is in order. For any algebra $A$ and scalar $\lambda \neq \frac{1}{2}$, in some extension of the ground field of $A$ one can form an algebra $A(\lambda)$ by defining a new product $a * b$ on the vector space $A$ as $a * b = \lambda ab + (1 - \lambda)ba$. Such algebras $A$ and $A(\lambda)$ are said to be quasi-equivalent. Now if $A$ satisfies the identity (4.20) then we might try to choose $\lambda$ so that $A(\lambda)$ is associator dependent, although if it turns out to be Lie-admissibility that is not very helpful. In case $\alpha_5 = -\frac{1}{4}$, for any $\lambda$, $A(\lambda)$ satisfies exactly the same identity (4.20) as $A$. Thus such an algebra $A$ is not in general quasi-equivalent to an associator dependent algebra, much less an algebra of type $(\gamma, \delta)$. Hence the variety defined by (4.20) with $\alpha_5 = -\frac{1}{4}$ is apparently an important, new class of algebras. It is not difficult to verify, by the way, that under the assumption of third power associativity this variety is a 2-variety. Consequently Albert's classification [**1**] would appear to be incomplete.

**5. Reduction to algebras of type $\mathfrak{d}$.** We turn now to the study of varieties $\mathcal{V}_F$ satisfying the condition

(5.1) $\mathcal{V}_F$ contains a nonzero, finite dimensional, semi-simple nil algebra.

It follows immediately from Theorem (3.4) that a variety $\mathcal{V}_F$ satisfying (5.1) is of type $\mathcal{V}_F(\Gamma)$. That means the following relations must hold in the identities (1.2) and (1.3) defining $\mathcal{V}_F$.

(5.2) $$1 = -\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 = \alpha_5 - \alpha_6 - \alpha_7 + \alpha_8 = \beta_1 - \beta_2 - \beta_3 + \beta_4 = -\beta_5 + \beta_6 + \beta_7 - \beta_8.$$

Furthermore $\mathcal{V}_F(\Gamma) = \mathcal{V}_F(\Gamma \cap \Delta) \cup \mathcal{V}_F(\Gamma \cap \Delta')$, where $\Delta$ consists of those $(\alpha_1, \ldots, \alpha_8, \beta_1, \ldots, \beta_8)$ for which

(5.3) $$1 = \sum_{i=1}^{8} \alpha_i = \sum_{i=1}^{8} \beta_i.$$

We now introduce new parameters $p, q, r, s, p', q', r', s'$ as follows:

(5.4) $$p = \alpha_1 + \alpha_5,\ q = \alpha_2 + \alpha_6,\ r = \alpha_3 + \alpha_7,\ s = \alpha_4 + \alpha_8,$$
$$p' = \beta_1 + \beta_5,\ q' = \beta_2 + \beta_6,\ r' = \beta_3 + \beta_7,\ s' = \beta_4 + \beta_8.$$

(5.5) PROPOSITION. *The algebras of the varieties $\mathcal{V}_F(\Gamma)$ satisfy the identities*

(5.6) $$(x_1 \circ x_2)x_3 = p[(x_3x_1) \circ x_2 + (x_3x_2) \circ x_1] + s[x_2(x_1 \circ x_3) + x_1(x_2 \circ x_3)] + q/2[(x_1, x_3, x_2) + (x_2, x_3, x_1)],$$

(5.7) $$x_3(x_1 \circ x_2) = p'[(x_3x_1) \circ x_2 + (x_3x_2) \circ x_1] + s'[x_2(x_1 \circ x_3) + x_1(x_2 \circ x_3)] + q'/2[(x_1, x_3, x_2) + (x_2, x_3, x_1)].$$

*Moreover, the algebras in $\mathcal{V}_F(\Gamma \cap \Delta)$ satisfy the identities*

(5.8) $$(x_1 \circ x_2)x_3 = \tfrac{1}{2}[x_2(x_1 \circ x_3) + x_1(x_2 \circ x_3)] + p[x_3(x_1 \circ x_2) - (x_1 \circ x_2)x_3] + (q - p)/2[(x_1, x_3, x_2) + (x_2, x_3, x_1)],$$

(5.9) $$x_3(x_1 \circ x_2) = \tfrac{1}{2}[x_2(x_1 \circ x_3) + x_1(x_2 \circ x_3)] + p'[x_3(x_1 \circ x_2) - (x_1 \circ x_2)x_3) + (q' - p')/2[(x_1, x_3, x_2) + (x_2, x_3, x_1)].$$

*Proof.* From equations (5.2) and (5.4) we have

(5.10) $r = p - q + s$, and $r' = p' - q' + s'$.

Interchanging $x_1$ and $x_2$ in (1.2) and adding we get $2(x_1 \circ x_2)x_3 = (\alpha_1 + \alpha_5)[(x_3x_1)x_2 + (x_3x_2)x_1] + (\alpha_2 + \alpha_6)[(x_1x_3)x_2 + (x_2x_3)x_1] + (\alpha_3 + \alpha_7)[x_2(x_3x_1) + x_1(x_3x_2)] + (\alpha_4 + \alpha_8)[x_2(x_1x_3) + x_1(x_2x_3)]$. Hence from (5.4) and (5.10) we get $2(x_1 \circ x_2)x_3 = p[(x_3x_1)x_2 + (x_3x_2)x_1] + q[(x_1x_3)x_2 + (x_2x_3)x_1) + (p - q + s)[x_2(x_3x_1) + x_1(x_3x_2)] + s[x_2(x_1x_3) + x_1(x_2x_3)] = p[2(x_3x_1) \circ x_2 + 2(x_3x_2) \circ x_1] + s[2x_2(x_1 \circ x_3) + 2x_1(x_2 \circ x_3)] + q[(x_1x_3)x_2 + x_1(x_3x_2) + (x_2x_3)x_1 - x_2(x_3x_1)]$ Then dividing by 2 gives the identity (5.6). Identity (5.7) may be proved in exactly the same way. For varieties of the type $\mathcal{V}_F(\Gamma \cap \Delta)$

we have the further relation (5.3), which implies $p + q + r + s = 1$, and $p' + q' + r' + s' = 1$. Combining these with (5.10), we find that

$$(5.11) \quad s = \tfrac{1}{2} - p, r = \tfrac{1}{2} - q, s' = \tfrac{1}{2} - p', r' = \tfrac{1}{2} - q'.$$

In view of (5.11), the identity (5.6) becomes $(x_1 \circ x_2)x_3 = p[(x_3x_1) \circ x_2 + (x_3x_2) \circ x_1] + (\frac{1}{2} - p)[x_2(x_1 \circ x_3) + x_1(x_2 \circ x_3)] + q/2[(x_1, x_3, x_2) + (x_2, x_3, x_1)]$. Moreover since

$$(5.12) \quad (x_3x_1) \circ x_2 + (x_3x_2) \circ x_1 = (x_1 \circ x_3)x_2 + (x_2 \circ x_3)x_1 - \tfrac{1}{2}[(x_1, x_3, x_2) + (x_2, x_3, x_1)]$$

is an identity in any algebra, we may change the term proportional to $p$ in the previous identity, so that $(x_1 \circ x_2)x_3 = p[(x_1 \circ x_3)x_2 + (x_2 \circ x_3)x_1 - \frac{1}{2}(x_1, x_3, x_2) - \frac{1}{2}(x_2, x_3, x_1)] + (\frac{1}{2} - p)[x_2(x_1 \circ x_3) + x_1(x_2 \circ x_3)] + q/2[(x_1, x_3, x_2) + (x_2, x_3, x_1)] = p[(x_1 \circ x_3)x_2 + (x_2 \circ x_3)x_1 - x_2(x_1 \circ x_3) - x_1(x_2 \circ x_3)] + \frac{1}{2}[x_2(x_1 \circ x_3) + x_1(x_2 \circ x_3)] + (q - p)/2[(x_1, x_3, x_2) + (x_2, x_3, x_1)]$. However, because of third power associativity or (2.2), $(x_1 \circ x_3)x_2 + (x_2 \circ x_3)x_1 - x_2(x_1 \circ x_3) - x_1(x_2 \circ x_3) = -(x_1 \circ x_2)x_3 + x_3(x_1 \circ x_2)$. Thus changing again the term proportional to $p$, we find that $(x_1 \circ x_2)x_3 = p[-(x_1 \circ x_2)x_3 + x_3(x_1 \circ x_2)] + \frac{1}{2}[x_2(x_1 \circ x_3) + x_1(x_2 \circ x_3)] + (q - p)/2[(x_1, x_3, x_2) + (x_2, x_3, x_1)]$, which is (5.8). The proof of (5.9) is similar and will be omitted.

Evidently $\mathscr{V}_F(\Gamma) = \mathscr{V}_F(\Gamma \cap \Delta) \cup \mathscr{V}_F(\Gamma \cap \Delta')$. Varieties of type $\mathscr{V}_F(\Gamma \cap \Delta')$ will be considered in Section 7. For varieties of the type $\mathscr{V}_F(\Gamma \cap \Delta)$ we have $\mathscr{V}_F(\Gamma \cap \Delta) = \mathscr{V}_F(\Gamma \cap \Delta \cap \Omega) \cup \mathscr{V}_F(\Gamma \cap \Delta \cap \Omega')$, where $\Omega$ consists of those $(\alpha_1, \ldots, \alpha_8, \beta_1, \ldots, \beta_8)$ in $F^{16}$ for which

$$(5.13) \quad p' = 1 + p, \quad \text{and} \quad q' = 1 + q.$$

We may recall that $p = \alpha_1 + \alpha_5$, $q = \alpha_2 + \alpha_6$, $p' = \beta_1 + \beta_5$, and $q' = \beta_2 + \beta_6$. We have no results whatsoever about algebras of the type $\mathscr{V}_F(\Gamma \cap \Delta \cap \Omega)$. However for the much larger class $\mathscr{V}_F(\Gamma \cap \Delta \cap \Omega')$ the situation is quite different.

(5.14) THEOREM. *The algebras $A$ in $\mathscr{V}_F(\Gamma \cap \Delta \cap \Omega')$ satisfy an identity of the type*

$$(5.15) \quad \lambda x_3(x_1 \circ x_2) + \mu(x_1 \circ x_2)x_3 = \tfrac{1}{2}(\lambda + \mu)[x_2(x_1 \circ x_3) + x_1(x_2 \circ x_3)],$$

*where $\lambda$, $\mu$ are in $F$ and not both equal to zero.*

*Proof*. For an algebra $A$ in $\mathscr{V}_F(\Gamma \cap \Delta)$ the relations (5.8) and (5.9) hold. We eliminate the associators from these identities by multiplying (5.8) by $q' - p'$, multiplying (5.9) by $q - p$, and then subtracting. We get $0 = \frac{1}{2}[(q' - p') - (q - p)][x_2(x_1 \circ x_3) + x_1(x_2 \circ x_3)] + [-(1 + p)(q' - p') + p'(q - p)][(x_1 \circ x_2)x_3] + [p(q' - p') - (p' - 1)(q - p)][x_3(x_1 \circ x_2)]$. Then if we set $\lambda' = p(q' - p') - (p' - 1)(q - p)$ and $\mu' = -(1 + p)(q' - p') + p'(q - p)$, we find that $\lambda' + \mu' = -[(q' - p') - (q - p)]$, so that the above identity becomes $\lambda' x_3(x_1 \circ x_2) + \mu'(x_1 \circ x_2)x_3 = \frac{1}{2}(\lambda' + \mu')[x_2(x_1 \circ x_3) +$

$x_1(x_2 \circ x_3)]$. If either $\lambda' \neq 0$ or $\mu' \neq 0$, there is nothing further to be proved, for we set $\lambda = \lambda'$ and $\mu = \mu'$. Let us suppose therefore that $\lambda' = \mu' = 0$. Then $0 = \lambda' + \mu'$, hence $q' - p' = q - p$. Thus $0 = \lambda' = (p + 1 - p')(q - p)$. Now if $p + 1 - p' = 0$, then from the relation $q' - p' = q - p$, we find $q' = (q - p) + p' = 1 + q$. But then $A \notin \mathscr{V}_F(\Gamma \cap \Delta \cap \Omega')$. Therefore we may assume that $q - p = 0$. However in this case the identity (5.8) reduces to $-p[x_3(x_1 \circ x_2)] + (1 + p)[(x_1 \circ x_2)x_3] = \frac{1}{2}[x_2(x_1 \circ x_3) + x_1(x_2 \circ x_3)]$, and we may set $\lambda = -p$, and $\mu = 1 + p$, in this situation. We have succeeded in deriving a 2-parameter identity for the varieties $\mathscr{V}_F(\Gamma \cap \Delta \cap \Omega')$. Our aim is to reduce this further to a 1-parameter identity. However we first require the following result, the proof of which is obvious.

(5.16) THEOREM. *If $A \in \mathscr{V}_F(\Gamma)$ is anti-commutative then $A$ is a Lie algebra. If $A \in \mathscr{V}_F(\Gamma \cap \Delta)$ is commutative then $A$ is associative.*

(5.17) THEOREM. *Let $A \in \mathscr{V}_F(\Gamma \cap \Delta \cap \Omega')$ be a finite dimensional semi-simple algebra. If the parameters $\lambda$ and $\mu$ of* (5.15) *are such that $\lambda + \mu = 0$, then $A$ is a direct sum of fields and a semi-simple Lie algebra.*

*Proof.* When $\lambda + \mu = 0$, the identity (5.15) reduces to

$$(5.18) \qquad (x_1 \circ x_2)x_3 = x_3(x_1 \circ x_2).$$

Using (5.18) we show that $A$ is the direct sum $A = A_1 \oplus A_0$, where $A_1$ is a semi-simple algebra with identity and $A_0$ is a semi-simple nil algebra. If $A$ itself is nil then we choose $A_1 = 0$, and $A_0 = A$. Assume therefore that $A$ is not a nil algebra. Then $A$ contains at least one idempotent. Let $e$ be a principal idempotent of $A$. Relative to $e$ we have the decomposition $A = A_1(e) + A_{1/2}(e) + A_0(e)$, where $A_0(e)$ is a nil subalgebra of $A$ (see Proposition (2.3)). Choosing $x_1 = x_2 = e$, in (5.18) we have $x_3e = ex_3$, for all $x_3$ in $A$. Therefore $A_{1/2}(e) = \{x|ex = xe = \frac{1}{2}x\}$. Setting $x_1 = x_2 = e$, and $x_3$ in $A_{1/2}(e)$ in (1.2), we find that $\frac{1}{2}x_3 = \frac{1}{4}(\alpha_1 + \alpha_2 + \ldots + \alpha_8)x_3$. But the sum of the eight $\alpha$'s equals 1, so that $x_3 = 0$. Therefore $A_{1/2}(e) = 0$, and we have a vector space decomposition $A = A_1(e) \oplus A_0(e)$.

Furthermore, if we choose $x_1 = e$, and $x_2$, $x_3$ in $A_1(e)$ and substitute in (5.18), we find that $x_2x_3 = x_3x_2$. Since $x_2x_3 + x_3x_2 \in A_1(e)$, it follows that $A_1(e)$ is a subalgebra of $A$. Clearly $A_1(e)$ and $A_0(e)$ are semi-simple, while $A_1(e)$ has an identity element. Now let $Z = \{z|za = az$, for all $a$ in $A\}$. Using (5.18) we see that for $z$ in $Z$ and $x_2$, $x_3$ in $A$, $(zx_2)x_3 = x_3(zx_2)$. Thus $zA = Az \subseteq Z$, which shows that $Z$ is an ideal of $A$. Also $Z \supseteq A \circ A$, because of (5.18). Since $A_1(e) = A_1(e) \circ e$, it follows that $A_1(e) \subseteq Z$. In particular, $A_1(e)$ must therefore be commutative and hence associative, using Theorem (5.16). Thus $A_1(e)$ must be a direct sum of fields. Let $Z' = Z \cap A_0(e)$. Clearly $Z'$ is an ideal of $A_0(e)$. As an algebra $Z'$ must be associative because of Theorem (5.16). Since $Z'$ is nil, it must be solvable. Then the semi-simplicity of $A_0(e)$ forces $Z' = 0$. But $Z' \supseteq A_0(e) \circ A_0(e)$. Hence $A_0(e) \circ A_0(e) = 0$, and $A_0(e)$

is anti-commutative. Therefore $A_0(e)$ is a Lie algebra by Theorem (5.16). This completes the proof of the theorem.

Having disposed of those algebras satisfying (5.15) with $\lambda + \mu = 0$, we now consider those in which $\lambda + \mu \neq 0$. We then divide both sides of (5.15) by $\frac{1}{2}(\lambda + \mu)$, to obtain the identity

$$\frac{2\lambda}{\lambda + \mu} x_3(x_1 \circ x_2) + \frac{2\mu}{\lambda + \mu}(x_1 \circ x_2)x_3 = x_2(x_1 \circ x_3) + x_1(x_2 \circ x_3).$$

Setting $\delta = (\lambda - \mu)/(\lambda + \mu)$, this identity becomes

(5.19) $\quad (1 + \delta)[x_3(x_1 \circ x_2)] + (1 - \delta)[(x_1 \circ x_2)x_3] = x_2(x_1 \circ x_3) + x_1(x_2 \circ x_3).$

Accordingly, we define an algebra $A$ to be of type $\delta$ if (1) $A \in \mathscr{V}_F(\Gamma \cap \Delta)$, and (2) $A$ satisfies the identity (5.19). In summary, we have shown in this section that the classification of the varieties $\mathscr{V}_F(\Gamma)$ is reduced to the study of the varieties $\mathscr{V}_F(\Gamma \cap \Delta')$ and $\mathscr{V}_F(\Gamma \cap \Delta \cap \Omega)$, and the classification of algebras of type $\delta$.

**6. Semi-simple algebras of type δ.** Throughout this section $A$ will denote an algebra satisfying identities (1.2), (1.3) and

(6.1) $\quad (1 + \delta)[x_3(x_1 \circ x_2)] + (1 - \delta)[(x_1 \circ x_2)x_3] = x_1(x_2 \circ x_3) + x_2(x_1 \circ x_3).$

We study first the Peirce decomposition $A = A_1(e) + A_{1/2}(e) + A_0(e)$, relative to an idempotent $e$.

(6.2) LEMMA. *If $\delta = -1/2$ then $A$ satisfies the identity*

$$(x_1 \circ x_2) \circ x_3 = x_1 \circ (x_1 \circ x_3), \quad \textit{and } A_{1/2}(e) = 0.$$

*Proof.* When $\delta = -1/2$ the identity (6.1) becomes $0 = 1/2\, x_3(x_1 \circ x_2) + 3/2(x_1 \circ x_2)x_3 - x_1(x_2 \circ x_3) - x_2(x_1 \circ x_3)$. Interchanging $x_1$ and $x_3$, $0 = 1/2\, x_1(x_3 \circ x_2) + 3/2(x_3 \circ x_2)x_1 - x_3(x_2 \circ x_1) - x_2(x_3 \circ x_1)$. Subtracting these two relations yields $0 = 3/2[x_3(x_1 \circ x_2) + (x_1 \circ x_2)x_3 - x_1(x_2 \circ x_3) - (x_2 \circ x_3)x_1] = 3[(x_1 \circ x_2) \circ x_3 - x_1 \circ (x_2 \circ x_3)]$. Thus $(x_1 \circ x_2) \circ x_3 = x_1 \circ (x_2 \circ x_3)$. Now setting $x_1 = x_2 = e$, and $x_3 \in A_{1/2}(e)$ in this identity yields $1/2\, x_3 = 1/4\, x_3$, so that $x_3 = 0$. Thus $A_{1/2}(e) = 0$. This completes the proof of the lemma.

(6.3) LEMMA. *If $\delta \neq -\frac{1}{2}$, then $eA_{1/2}(e) \subseteq A_{1/2}(e)$, and $A_{1/2}(e)e \subseteq A_{1/2}(e)$.*

*Proof.* In (6.1) choose $x_3 = a \in A_{1/2}(e)$, and $x_1 = x_2 = e$. Then $(1 + \delta)ae + (1 - \delta)ea = ea$, or $(1 + \delta)ae - (\delta)ea = 0$. But $ea = a - ae$, so that $0 = (1 + \delta)ae - \delta\,(a - ae) = (1 + 2\delta)ae - (\delta)a$. As $1 + 2\delta \neq 0$, it follows that $ae \in A_{1/2}(e)$. From this it follows that $ea = a - ae \in A_{1/2}(e)$. This completes the proof of the lemma.

(6.4) LEMMA. *If $\delta \neq -\frac{1}{2}$, then $[A_{1/2}(e)]^2 = 0$.*

*Proof.* Let $u$, $v$ be elements of $A_{1/2}(e)$. In (6.1) set $x_1 = u$, $x_2 = v$, $x_3 = e$. Then $(1 + \delta)e(u \circ v) + (1 - \delta)(u \circ v)e = u \circ v$. On the other hand $u \circ v \in A_1(e) + A_0(e)$, because of Proposition (2.3). Thus $u \circ v = a_1 + a_0$, where $a_1$ is in $A_1(e)$ and $a_0$ is in $A_0(e)$ and $(1 + \delta)a_1 + (1 - \delta)a_1 = a_0 + a_1$. Thus $a_0 = a_1 = 0$, and so $u \circ v = 0$. Now choose $x_1 = e$, $x_2 = u$, and $x_3 = v$ in (6.1). Then $\frac{1}{2}(1 + \delta)vu + \frac{1}{2}(1 - \delta)uv = e(u \circ v) + \frac{1}{2}uv$, whence $(1 + \delta)vu - (\delta)uv = 0$. Since $uv = -vu$, $(1 + 2\delta)vu = 0$, so that $vu = 0$. This completes the proof of the lemma.

(6.5) Lemma. *If* $\delta \neq -1/2$, *then* $A_{1/2}(e)A_0(e) + A_0(e)A_{1/2}(e) \subseteq A_{1/2}(e)$.

*Proof.* Let $u$ be an element of $A_{1/2}(e)$ and $v$ an element of $A_0(e)$. In (6.1) set $x_1 = e$, $x_2 = v$, and $x_3 = u$. Then $0 = e(u \circ v) + \frac{1}{2}vu$. Next set $x_3 = e$, $x_1 = u$, and $x_2 = v$ in (6.1). Then $(1 + \delta)e(u \circ v) + (1 - \delta)(u \circ v)e = \frac{1}{2}vu$. Comparing these we find that $(2 + \delta)e(u \circ v) + (1 - \delta)(u \circ v)e = 0$. However $u \circ v \in A_{1/2}(e) + A_1(e)$ because of Proposition (2.3). Thus $u \circ v = a_{1/2} + a_1$, where $a_{1/2} \in A_{1/2}(e)$, and $a_1 \in A_1(e)$. Therefore $(2 + \delta)(ea_{1/2} + a_1) + (1 - \delta)(a_{1/2}e + a_1) = 0$. Equating $A_1(e)$ components in this equation and using the fact that $ea_{1/2}$ and $a_{1/2}e$ are elements of $A_{1/2}(e)$, we find that $a_1 = 0$. Thus $u \circ v \in A_{1/2}(e)$. Then it follows from our previous relation, $0 = e(u \circ v) + \frac{1}{2}vu$, that $vu \in A_{1/2}(e)$. Hence $uv = -vu + 2(u \circ v)$ is an element of $A_{1/2}(e)$. This completes the proof of the lemma.

(6.6) Lemma. *If* $\delta \neq -\frac{1}{2}$, *then* $A_{1/2}(e)A_1(e) + A_1(e)A_{1/2}(e) \subseteq A_{1/2}(e)$.

*Proof.* Let $u \in A_{1/2}(e)$, $v \in A_1(e)$. Substituting $x_1 = e$, $x_2 = v$, and $x_3 = u$ in (6.1) yields $(1 + \delta)uv + (1 - \delta)vu = e(u \circ v) + \frac{1}{2}vu$, so that $(1 + \delta)uv + (\frac{1}{2} - \delta)vu = e(u \circ v)$. However $u \circ v \in A_{1/2}(e) + A_0(e)$. Hence Lemma (6.3) implies that $e(u \circ v) \in A_{1/2}(e)$. Therefore

$$(6.7)\qquad (1 + \delta)uv + (\tfrac{1}{2} - \delta)vu \in AZ_{1/2}(e).$$

Next set $x_1 = e$, $x_2 = u$, and $x_3 = v$, in (6.1). Then $\frac{1}{2}(1 + \delta)vu + \frac{1}{2}(1 - \delta)uv = e(u \circ v) + uv$. Hence

$$(6.8)\qquad -(1 + \delta)uv + (1 + \delta)vu \in A_{1/2}(e).$$

From (6.7) and (6.8) it follows that $vu \in A_{1/2}(e)$. Then choose $x_3 = e$, $x_1 = u$, $x_2 = v$ in (6.1). Then $(1 + \delta)e(u \circ v) + (1 - \delta)(u \circ v)e = uv + \frac{1}{2}vu$. As $e(u \circ v)$, $(u \circ v)e$ and $vu$ all belong to $A_{1/2}(e)$, we conclude that $uv$ must also. This completes the proof of the lemma.

(6.9) Theorem. *If $A$ is a semi-simple algebra of type $\delta$, then $A$ is the direct sum $A = A_1 \oplus A_0$, where $A_1$ is a semi-simple algebra with identity and $A_0$ is a semi-simple nil algebra. Moreover if $\delta \neq -\frac{1}{2}$ then $A_1$ is a direct sum of fields.*

*Proof.* If $A$ is nil then we choose $A_1 = 0$, $A_0 = A$, and we are done. If $A$ is not nil then it contains a principal idempotent $e$, so that we can decompose $A = A_1(e) + A_{1/2}(e) + A_0(e)$, where $A_0(e)$ is a nil subalgebra of $A$. It follows from the preceding lemmas that $A_{1/2}(e)$ is an ideal of $A$ and that $[A_{1/2}(e)]^2 = 0$. Since $A$ is semi-simple, this implies $A_{1/2}(e) = 0$, and $A = A_1(e) + A_0(e)$. Certainly $A_0(e)$ is a semi-simple ideal of $A$. It will suffice to show that $A_1(e)$ is a subalgebra of $A$, in order to complete the first part.

Using (1.2) and (1.3) we see that for $x_1, x_2 \in A_1(e)$ and $x_3 \in A_0(e)$ that $0 = (x_1x_2)x_3 = x_3(x_1x_2)$. Moreover $x_1x_2 = a_1 + a_0$, where $a_1 \in A_1(e)$ and $a_0 \in A_0(e)$. Then from the fact that $(x_1x_2)A_0(e) = A_0(e)(x_1x_2) = A_1(e)A_0(e) = A_0(e)A_1(e) = 0$, it follows that $a_0A_0(e) = A_0(e)a_0 = 0$. But $A_0(e)$ is semi-simple. Therefore $a_0 = 0$, and $x_1x_2 = a_1 \in A_1(e)$. Thus $A_1(e)$ is a subalgebra of $A$.

Finally, suppose $\delta \neq -\frac{1}{2}$, and let $x_2, x_3 \in A_1(e)$, and $x_1 = e$, and substitute this in (6.1). This yields $0 = (\frac{1}{2} + \delta)(x_3x_2 - x_2x_3) = x_3x_2 - x_2x_3$. Thus $A_1(e)$ is a commutative algebra and hence associative, because of Theorem (5.16). Now the semi-simplicity of $A_1$ forces $A_1$ to be a direct sum of fields. This completes the proof of the theorem.

**7. Varieties of the type $\mathscr{V}_F(\Delta' \cap \Gamma)$.** As indicated in Section 3, an algebra $A \in \mathscr{V}_F(\Delta' \cap \Gamma)$ satisfies the identity

(7.1) $\quad x^2x = xx^2 = 0.$

Linearizing, (7.1) becomes

(7.2) $\quad (a \circ b)c + (b \circ c)a + (c \circ a)b = 0.$

As in Section 5, we introduce the parameters $p$, $q$, $p'$, $q'$, where

(7.3) $\quad p = \alpha_1 + \alpha_5, q = \alpha_2 + \alpha_6, p' = \beta_1 + \beta_5, q' = \beta_2 + \beta_6.$

(7.4) THEOREM. *If $A \in \mathscr{V}_F(\Delta' \cap \Gamma)$ is simple then $A$ is a Lie algebra, unless $p = q$, and $p' = q'$.*

*Proof.* Let $A^{(+)}$ denote the commutative algebra which results from replacing the product $ab$ of $A$ by the symmetrical product $a \circ b$. Then $A^{(+)}$ is a commutative algebra of nil index 3, and by a result of Albert's [**2**, p. 557], $A^{(+)}$ must be nilpotent. Therefore the set $T = \{t \in A | t \circ A = 0\}$ is non-zero. From the identity (7.2) it follows that $(A \circ A)T = T(A \circ A) = 0$. Set $x_1 = t \in T$ in (5.6). Then $0 = p[(x_3t) \circ x_2] + \frac{1}{2}q[(t, x_3, x_2) + (x_2, x_3, t)]$, for all $x_2, x_3$ in $A$. But $\frac{1}{2}q[(t, x_3, x_2) + (x_2, x_3, t)] = \frac{1}{2}q[(tx_3)x_2 - t(x_3x_2) + (x_2x_3)t - x_2(x_3t)] = \frac{1}{2}q[(tx_3)x_2 + (x_3x_2)t + (x_2x_3)t + x_2(tx_3)] = q[(tx_3) \circ x_2 + (x_2 \circ x_3)t] = q[(tx_3) \circ x_2]$. Thus $0 = p[(x_3t) \circ x_2] + q[(tx_3) \circ x_2] = (p - q)[(x_3t) \circ x_2]$. Similarly from (5.7) we obtain $(p' - q')[(x_3t) \circ x_2] = 0$, for all $t$ in $T$, and $x_2, x_3$ in $A$. Thus if $p \neq q$, or if $p' \neq q'$, then $x_3t$ is in $T$. This shows that $T$ is a left ideal of $A$. Since $T \circ A = 0$, $T$ is a right ideal as well. The simplicity of $A$ forces $T = A$. But then $A \circ A = T \circ A = 0$. Thus $A$ is anti-commutative and hence a Lie algebra because of (5.16). This completes the proof of the theorem.

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