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ON MODULARITY OF RIGID AND NONRIGID CALABI-YAU VARIETIES ASSOCIATED TO THE ROOT LATTICE A_4

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Abstract. We prove the modularity of four rigid and three nonrigid Calabi-Yau threefolds associated with the A_4 root lattice.

§1. Introduction

In this paper we investigate the geometry and arithmetic of a family of Calabi-Yau threefolds $X_{\boldsymbol{a}}$, $\boldsymbol{a} = (a_1 : \cdots : a_6) \in \mathbb{P}^5$, birational to the projective hypersurface in $T := \mathbb{P}^4 \setminus \{X_1 \cdots X_5 = 0\}$ given by

$$X_{\boldsymbol{a}} \cap T:\ (X_1 + \cdots + X_5)\left(\frac{a_1}{X_1} + \cdots + \frac{a_5}{X_5}\right) = a_6.$$

Our motivation is to find further examples of modular Calabi-Yau varieties, i.e., of Calabi-Yau varieties which are defined over the rationals and whose L-series can be described in terms of modular forms. This is motivated by the Fontaine-Mazur conjecture on the modularity of two dimensional ℓ-adic Galois representations coming from geometry [FM], which is a generalization of the Taniyama-Shimura-Weil conjecture on the modularity of elliptic curves, proved by Wiles et al, [Wi], [BCDT]. More precisely, Fontaine and Mazur [FM] define the notions of a “geometric Galois representation”, and a “Galois representation coming from geometry”. They conjecture that geometric Galois representations are precisely the Galois representations coming from geometry (such as the ones we consider) ([FM, Conjecture 1]), and combining this with classical conjectures (see e.g. [Se2]) leads them to the conjecture that two-dimensional irreducible geometric Galois representations are modular up to a Tate twist ([FM, Conjecture 3c]). From these two conjectures, one obtains the conjecture that two dimensional irreducible Galois representations coming from geometry are modular up to a

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Tate twist. We will use the term "modular" to mean "modular up to a Tate twist", and even to denote direct sums of modular Galois representations.

Rigid Calabi-Yau threefolds (defined over $\mathbb{Q}$) are expected to be modular, since they have 2-dimensional middle cohomology. One expects the L-series of the Galois action on the middle ℓ-adic cohomology to be the Mellin transform of a weight 4 elliptic modular form. Although recently Dieulefait and Manoharmayum [DM] proved that a rigid Calabi-Yau threefold is modular provided it has good reduction at 3 and 7, or at 5 and another suitable prime (in fact most of our examples have bad reduction at 3 and 5, so this result does not apply, and in general does not determine the exact modular form), it is still the case that relatively few examples of modular Calabi-Yau threefolds are explicitly known. Most currently known examples are given in Yui's survey articles [Y1], [Y2]. Other recent examples are given by [CM].

Modularity has been also conjectured for certain nonrigid examples, e.g., [CM]. What is new in this paper is the proof of modularity for several nonrigid examples. Note that we mean modularity in the sense of Fontaine-Mazur, i.e., the semisimplification of the Galois representation is a sum of 2 dimensional pieces. There are few examples of other kinds of modularity of nonrigid Calabi-Yau threefolds known. Consani and Scholten [CS] consider an example corresponding to a Hilbert modular form for which they provide evidence for the modularity and Livné and Yui [LY] very recently gave some cases involving weight 2 and 3 forms. Their examples and techniques are quite different from ours.

We shall study a certain 5-dimensional family $X_{\boldsymbol{a}}$, $\boldsymbol{a} \in \mathbb{P}^5$, of (singular) Calabi-Yau threefolds, associated to the root lattice A_4, by means of Batyrev's construction [Ba] of Calabi-Yau varieties as toric hypersurfaces.

The rigid cases are X_1, X_9, $X_{(1:1:1:1:4:4)}$, and $X_{(1:1:1:4:4:9)}$, where $X_t := X_{(1:1:1:1:1:t)}$. These have 40, 35, 37 and 35 nodes respectively, and their (big) resolutions have 2-dimensional middle cohomology. We will show that the L-series is the Mellin transform of a modular form of weight 4, and level 6, 6, 12 and 60 respectively. The first few terms of their q-expansions are

(1) $$f_6 = q - 2q^2 - 3q^3 + 4q^4 + 6q^5 + 6q^6 - 16q^7 - 8q^8 + \cdots,$$

(2) $$f_{12} = q + 3q^3 - 18q^5 + 8q^7 + 9q^9 + 36q^{11} - 10q^{13} + \cdots,$$

(3) $$f_{60} = q - 3q^3 - 5q^5 - 28q^7 + 9q^9 - 24q^{11} - 70q^{13} + 15q^{15} + \cdots,$$

where f_N has level N. Although the middle cohomology of X_1 and X_9 have the same L-series, we will see that they are not birational to each other, though by a conjecture of Tate, one expects a correspondence between them. The nonrigid examples we consider are X_{25}, $X_{(1:1:1:9:9:9)}$ and $X_{(1:1:4:4:4:16)}$. In these cases we show that L-series of the middle cohomology of the big resolutions are

$$(4)\quad L(f_{30}, s)L(g_{30}, s-1)^4,\ \ L(f'_{30}, s)L(g_{30}, s-1)^2,\ \ L(f_{90}, s)L(g_{30}, s-1),$$

respectively, where $L(h)$ denotes the Mellin transform of the function h, and the functions g_{30}, f_{30}, f'_{30} and f_{90} are cuspidal Hecke eigen newforms, g_{30} having weight 2, the others weight 4, and f_{90} having level 90, the others level 30. (The level 30 has, to our knowledge, not previously appeared in examples of this kind.) The first few terms of the q-expansions are

$$(5)\quad g_{30}(q) = q - q^2 + q^3 + q^4 - q^5 - q^6 - 4q^7 - q^8 + q^9 + q^{10} + q^{12} + \cdots$$

$$(6)\quad f_{30}(q) = q - 2q^2 + 3q^3 + 4q^4 + 5q^5 - 6q^6 + 32q^7 - 8q^8 + 9q^9 + \cdots$$

$$(7)\quad f'_{30}(q) = q + 2q^2 + 3q^3 + 4q^4 - 5q^5 + 6q^6 - 4q^7 + 8q^8 + 9q^9 + \cdots$$

$$(8)\quad f_{90}(q) = q - 2q^2 + 4q^4 - 5q^5 - 4q^7 - 8q^8 + 10q^{10} - 12q^{11} + \cdots$$

Given expression (4), one would expect, by the Tate conjecture, that there is a geometric reason for the occurrence of the weight 2 modular form g_{30}, which is the Mellin transform of the L-series of a certain elliptic curve. We will see that this is indeed the case.

In both rigid and nonrigid cases, we use the powerful theorem due to Faltings, Serre and Livné [Li], which permits one to determine 2-dimensional Galois representations from a finite set of data. In practice this means counting the number of points modulo p for a given finite number of primes, a task which can be done easily by computer.

In Section 2 we consider the toric geometry set up. In Section 3 we discuss the resolution of singularities of the singular subfamily $X_{\boldsymbol{a}}$. In Section 4 we show that $X_{\boldsymbol{a}}$ is birational to a fibre product of families of elliptic curves, which allows us to apply results of Schoen [Sc]. In Section 5 we study a certain elliptic surfaces contained in $X_{\boldsymbol{a}}$, and in Section 6 we count points and apply Livné's method to determine the L-series of the 7 cases of $X_{\boldsymbol{a}}$ mentioned above.

Figure 1 gives a schematic diagram of the 5-dimensional family $X_{\boldsymbol{a}}$ which we shall study, and some of its subfamilies (a complete list is given in Table 1). The diagram gives the dimension of these subfamilies and the value of h^{12} of the big resolution $\widetilde{X}_{\boldsymbol{a}}$ of the general member of the

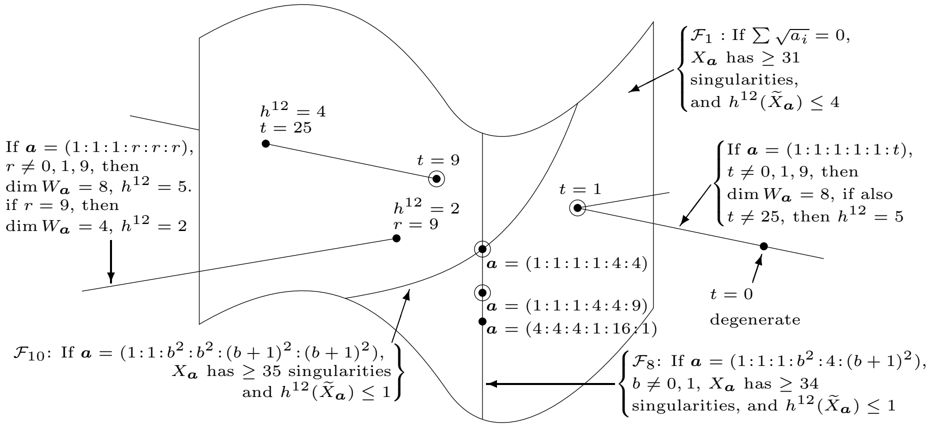

Figure 1: Values of the parameter $\boldsymbol{a}$ for certain members and subfamilies of the family of Calabi-Yau threefolds $X_{\boldsymbol{a}}$, with $\boldsymbol{a}$ for modular $X_{\boldsymbol{a}}$ marked by a point, which is circled if $X_{\boldsymbol{a}}$ is rigid.

subfamily. Values of $\boldsymbol{a}$ where $\widetilde{X}_{\boldsymbol{a}}$ is modular are marked with points, and those which are rigid with circled points. In Section 5 we will see that for two equal indices $a_i = a_j$, there is a corresponding elliptic surface in $X_{\boldsymbol{a}}$. We call the piece of H^3 corresponding to these elliptic surfaces $W_{\boldsymbol{a}}$. When $\dim H^3 - \dim W_{\boldsymbol{a}} = 2$, i.e., $2h^{12} = \dim W_{\boldsymbol{a}}$, one expects $X_{\boldsymbol{a}}$ to be modular, which we will see is the case for all 7 marked points in the diagram.

Finally we would like to point out that the family of Calabi-Yau varieties which we are considering in this paper has recently also appeared in a different context. C. Borcea has studied these varieties in the context of configuration spaces of planar polygons (see [Bo] where these varieties are called *Darboux varieties*).

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Notation. In this paper we consider projective Calabi-Yau varieties defined by polynomial equations with coefficients in $\mathbb{Z}$. We work over the field k, where $k = \mathbb{C}$, $\mathbb{Q}$, $\overline{\mathbb{Q}}$, $\mathbb{F}_p$ or $\overline{\mathbb{F}}_p$. Further notation is as follows.

M_{A_4}	The A_4 root lattice, as a sublattice of $\mathbb{Z}^5$.
N_{A_4}	$(M_{A_4})^\vee$, identified with a sublattice of $M_{A_4} \otimes \mathbb{Q}$.
ε_{ij}	Point in M_{A_4} at $e_i - e_j$.
Δ_{A_4}	Polytope in $M_{A_4} \otimes \mathbb{R}$ with vertices at ε_{ij}.
$\widetilde{\Sigma}_{A_4}$	Fan in $N_{A_4} \otimes \mathbb{R}$ given by all faces of the Weyl chambers.
$\widetilde{\Sigma}^3$	3-dimensional cones in $\widetilde{\Sigma}_{A_4}$.
$\widetilde{P}$	Smooth toric variety defined by $\widetilde{\Sigma}_{A_4}$.
$\mathbb{A}^4_\sigma$	Affine piece of $\widetilde{P}$ corresponding to $\sigma \in \widetilde{\Sigma}^3$.
T	Torus $(k^*)^4 \subset \widetilde{P}$.
T_σ	Orbit under the torus action corresponding to $\sigma \in \widetilde{\Sigma}_{A_4}$; $\widetilde{P} = \coprod_{\sigma\in\widetilde{\Sigma}} T_\sigma$.
X_u	For $u \in \mathbb{P}^{20}$, a Calabi-Yau threefold defined in $\widetilde{P}$ defined by Δ_{A_4}.
$X_{\boldsymbol{a}}$	For $\boldsymbol{a} \in \mathbb{P}^5$, a member of a 5 dimensional family of singular Calabi-Yaus in $\widetilde{P}$, with 30 nodes on $X_{\boldsymbol{a}} \setminus T$ if $\prod_{i=1}^5 a_i \neq 0$.
$\overline{X}_{\boldsymbol{a}}$	A Calabi-Yau given by taking a specific choice of small projective resolution of the 30 singularities on $X_{\boldsymbol{a}} \setminus T$.
$\widetilde{\overline{X}}_{\boldsymbol{a}}$	Big resolution of remaining singularities on $\overline{X}_{\boldsymbol{a}}$.
X_t	$X_{(1:1:1:1:1:t)}$.
$\widetilde{X}$	Big resolution of $X = X_u$, $X_{\boldsymbol{a}}$, $\overline{X}_{\boldsymbol{a}}$ or X_t, for X irreducible.
$\widehat{X}$	A choice of small projective resolution of X, if one exists.

§2. Toric varieties

Batyrev [Ba] constructs Calabi-Yau varieties as hypersurfaces in a toric variety defined by a reflexive polytope Δ ([Ba, p. 510]), in a lattice M, as follows.

The pair (M, Δ) gives rise to a fan Σ in the dual space $N_\mathbb{R} = M_\mathbb{R}^\vee$, and a strictly convex support function h on $N_\mathbb{R}$. Let $(P, \mathcal{O}_P(1))$ be the corresponding polarized toric variety. In general P is a singular Fano variety with Gorenstein singularities and $\mathcal{O}_P(1)$ is the anticanonical bundle.

Batyrev shows that the general element in the anticanonical system is a Calabi-Yau variety, with canonical singularities exactly at the singular points of P. A desingularization $\widetilde{P}$ of P can be constructed by taking the maximal projective triangulation $\widetilde{\Delta^*}$ of the dual polytope Δ^*. We denote the corresponding fan by $\widetilde{\Sigma}$. The strict transforms of the anticanonical

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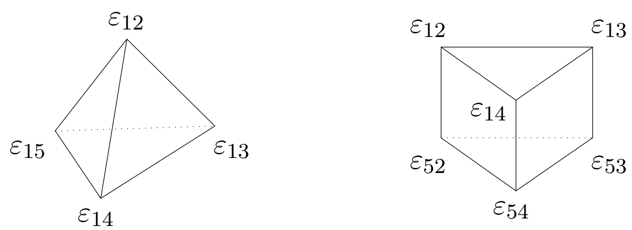
Tetrahedral face $F_1^+ := (x_1 = 1) \cap \Delta$ | Prism face $F_{15}^+ := (x_1 + x_5 = 1) \cap \Delta$



there are 10 such faces | there are 20 such faces

Figure 2: Three dimensional faces of Δ.

divisors on P define a family of Calabi-Yau threefolds on $\widetilde{P}$ whose general element X_u is smooth.

2.1. The polytopes Δ_{A_4}, $\Delta^*_{A_4}$ and $\widetilde{\Delta^*_{A_4}}$

We now describe the lattice and polytope to which we apply Batyrev's construction. More generally, in the following construction one may replace A_4 by A_n, defined similarly; more details can be found in [DL], [P] and [Lu].

Let M be the A_4 root lattice, given as the following sublattice of $\mathbb{Z}^5$:

$$M = M_{A_4} = \left\{ (x_1, x_2, x_3, x_4, x_5) \,\middle|\, x_i \in \mathbb{Z}, \sum x_i = 0 \right\} \subset \mathbb{Z}^5.$$

The inner product on M is induced by the standard inner product on $\mathbb{Z}^5$, and we identify the dual lattice $N := (M_{A_4})^\vee$ with a sublattice of $M_\mathbb{R} := M \otimes \mathbb{R}$.

Definition 2.1. The polytope $\Delta = \Delta_{A_4}$ in $M_\mathbb{R}$ is defined to be the convex hull of the roots $\varepsilon_{ij} := e_i - e_j$, $1 \le i, j \le 5$, $i \ne j$ of M, where $e_1, \ldots, e_5$ is the standard basis for $\mathbb{Z}^5$. In [V, p. 427] it is shown that Δ is reflexive.

It is a simple combinatorial exercise to enumerate the faces of Δ_{A_4}, $\Delta^*_{A_4}$ and $\widetilde{\Delta^*_{A_4}}$. We have the following results.

Lemma 2.2. *The polytope* Δ *has* 20 *vertices,* 60 *edges,* 30 *square faces,* 40 *triangular faces, and* 30 *three dimensional faces, given by*

$$F_i^\varepsilon := (x_i = \varepsilon 1) \cap \Delta, \quad F_{ij}^\varepsilon := (x_i + x_j = \varepsilon 1) \cap \Delta,$$

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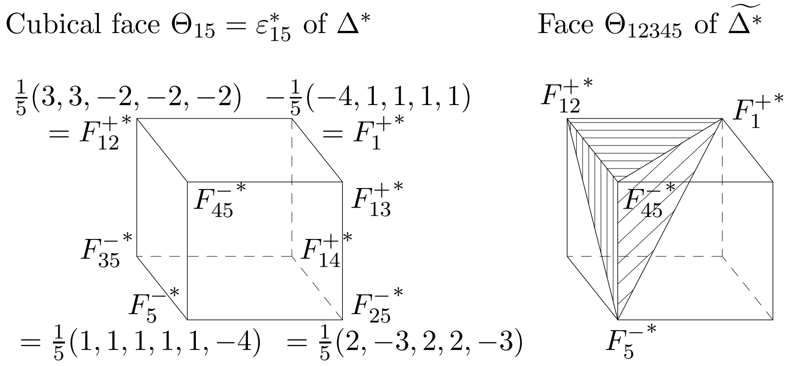

Figure 3: Three dimensional faces of Δ^* and $\widetilde{\Delta^*}$.

for $1 \le i,j \le 5$, $i \ne j$, *and* $\varepsilon = \pm$. *Two of these faces are shown in Figure* 2.

Proof. See [Lu, Lemma 1.18 and Korollar 1.19]. ∎

LEMMA 2.3. *The dual polytope* Δ^* *has* 30 *vertices and* 20 *three dimensional cubical faces,* Θ_{ij} *for* $1 \le i,j \le 5$, $i \ne j$. *The vertices of* Θ_{15} *are shown in Figure* 3, *and* $\Theta_{ij} = \sigma\Theta_{15}$, *where* $\sigma \in S_5$ *with* $\sigma : 1,5 \mapsto i,j$.

Proof. See [Lu, Lemma 1.46]. ∎

LEMMA 2.4. *The subdivided polytope* $\widetilde{\Delta^*}$ *has* 120 *three dimensional faces, which are translations under* S_5 *of* Θ_{12345} *given in Figure* 3.

Proof. See [Lu, Beispiel 2.34]. ∎

In order to apply Batyrev's formulae for h^{11} and h^{12} we need to count the number of lattice points in the interior of the faces of Δ and Δ^*. We make use of the following easy result.

LEMMA 2.5. *If* $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, $\mathbf{w}_4$ *is a basis for a lattice* L, *and* Θ *is a polytope with vertices* $\mathbf{0}, \mathbf{w}_1, \ldots, \mathbf{w}_4$, *then the only lattice points in* Θ *are its vertices.*

By taking appropriate subdivisions of the faces F^{ε}_{ij} and Θ_{ij}, we have

LEMMA 2.6. *No proper face of* Δ *or* Δ^* *contains an interior lattice point, and the only lattice point in the interior of* Δ *or* Δ^* *is the origin.*

2.2. The toric variety $\widetilde{P}$ defined by the fan $\widetilde{\Sigma}$

The fan $\widetilde{\Sigma}$ in $N_{\mathbb{R}}$ consists of cones given by the 120 Weyl chambers

$$\sigma_{ijklm} = \left\{(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \in \mathbb{R}^5 \;\Big|\; \sum_{v=1}^{5} \alpha_v = 0,\ \alpha_i \geq \alpha_j \geq \alpha_k \geq \alpha_l \geq \alpha_m\right\},$$

where $\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}$, together with all their subfaces. E.g., σ_{12345} is the cone on Θ_{12345} (see Figure 3). The dual cones are given by

$$\sigma^{\vee}_{ijklm} := \mathbb{R}_{\geq 0}(e_i - e_j) + \mathbb{R}_{\geq 0}(e_j - e_k) + \mathbb{R}_{\geq 0}(e_k - e_l) + \mathbb{R}_{\geq 0}(e_l - e_m).$$

We will consider Calabi-Yau threefold hypersurfaces in the toric variety $\widetilde{P}$ defined by $\widetilde{\Sigma}$. We first fix choices of local and global coordinates for $\widetilde{P}$.

We identify the torus $T \cong (k^*)^4 \subset \widetilde{P}$ with $\mathbb{P}^4 \setminus \left(\prod_{i=1}^{5} X_i = 0\right)$, and use the projective coordinates $X_1, \dots, X_5$ of $\mathbb{P}^4$ when considering points in T.

For the affine piece $\mathbb{A}^4_\varsigma := \operatorname{Spec}(k[\varsigma(\sigma^{\vee}_{12345})]) \subset \widetilde{P}$, where $\varsigma \in S_5$, we use coordinates x_ς, y_ς, z_ς, w_ς corresponding to the basis $\varepsilon_{\varsigma(12)}$, $\varepsilon_{\varsigma(23)}$, $\varepsilon_{\varsigma(34)}$, $\varepsilon_{\varsigma(45)}$ of $\varsigma(\sigma^{\vee}_{12345})$. Usually we just write x, y, z, w.

The identification of T and $\mathbb{A}^4_\varsigma \setminus (x_\varsigma y_\varsigma z_\varsigma w_\varsigma = 0)$ is given by

$$x_\varsigma = X_{\varsigma(1)}/X_{\varsigma(2)}, \qquad y_\varsigma = X_{\varsigma(2)}/X_{\varsigma(3)},$$
$$z_\varsigma = X_{\varsigma(3)}/X_{\varsigma(4)}, \qquad w_\varsigma = X_{\varsigma(4)}/X_{\varsigma(5)}.$$

This relationship between the coordinates of $\mathbb{P}^4$ and of the affine pieces of $\widetilde{P}$ is explained by the following lemma.

LEMMA 2.7. *The variety $\widetilde{P}$ is the graph of the Cremona transformation $X_i \mapsto 1/X_i$ of $\mathbb{P}^4$. Thus $\widetilde{P}$ is obtained from $\mathbb{P}^4$ by blowing up successively the* (*strict transforms of the*) *points* $(1:0:0:0), (0:1:0:0), \dots, (0:0:0:1)$, *lines and planes spanned by any subset of these points.*

Proof. See [DL, Lemma 5.1]. ∎

2.2.1. Toric orbits in $\widetilde{P}$

There is a decomposition $\widetilde{P} = \bigsqcup_{\sigma \in \widetilde{\Sigma}} T_\sigma$, where T_σ is the toric orbit of $\widetilde{P}$ corresponding to $\sigma \in \widetilde{\Sigma}$. Since $\widetilde{\Sigma}$ is given by taking cones on the faces of $\widetilde{\Delta}^*$, we use the notation $T_\Theta := T_{\mathbb{R}_+\Theta}$ where Θ is a face of $\widetilde{\Delta}^*$. By standard methods of toric geometry we have

$$\overline{T_{F_i^{\varepsilon *}}} \cong \widetilde{P}_{A_3},$$
$$\overline{T_{F_{ij}^{\varepsilon *}}} \cong \widetilde{P}_{A_2} \times \widetilde{P}_{A_1} \cong \widetilde{\mathbb{P}}^2 \times \mathbb{P}^1,$$

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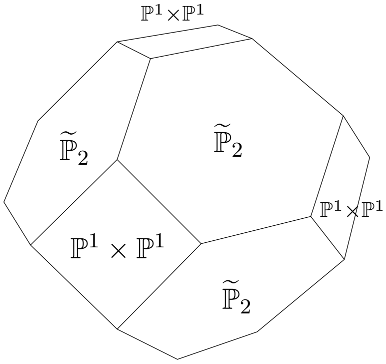
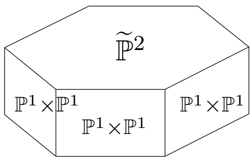


There are 10 copies of $\overline{T_{F_i^{\varepsilon *}}}$ in $\widetilde{P} \setminus T$, two being $\overline{\{x=0\}}$ and $\overline{\{w=0\}}$. The closures of T_{Θ^*}, for Θ a two dimensional face of F_j^{ε}, consists of 8 copies of $\widetilde{\mathbb{P}}^2$ and 6 copies of $\mathbb{P}^1 \times \mathbb{P}^1$. These intersect as indicated in this figure; hexagonal faces correspond to $\widetilde{\mathbb{P}}^2$, and square faces correspond to $\mathbb{P}^1 \times \mathbb{P}^1$.

Figure 4: The threefold $\overline{T_{F_i^{\varepsilon *}}}$ in $\widetilde{P} \setminus T$.



There are 20 copies of $\overline{T_{F_{ij}^{\varepsilon *}}}$ in $\widetilde{P}$, two being $\overline{\{y=0\}}$ and $\overline{\{z=0\}}$. The closures of T_{Θ^*}, for Θ a two dimensional face of F_{ij}^{ε}, consists of 2 copies of $\widetilde{\mathbb{P}}^2$, and 6 copies of $\mathbb{P}^1 \times \mathbb{P}^1$, corresponding to the hexagons and squares respectively in this figure.

Figure 5: The threefold $\overline{T_{F_{ij}^{\varepsilon *}}}$ in $\widetilde{P} \setminus T$.

where $\widetilde{P}_{A_n}$ is the toric variety corresponding to the root lattice A_n, and $\widetilde{\mathbb{P}}^2$ is $\mathbb{P}^2$ blown up in 3 points. These are sketched in Figures 4 and 5.

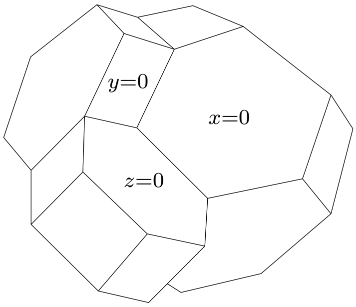
In terms of local coordinates x, y, z, w for $\mathbb{A}^4_{\mathrm{id}}$, we have

$$\begin{aligned}
T_{F_4^{-*}} &= T_{\frac{1}{5}(1,1,1,1,-4)} &&= \{x = 0 \neq yzw\},\\
T_{F_{45}^{-*}} &= T_{\frac{1}{5}(2,2,2,-3,-3)} &&= \{y = 0 \neq xzw\},\\
T_{F_{12}^{+*}} &= T_{\frac{1}{5}(3,3,-2,-2,-2)} &&= \{z = 0 \neq xyw\},\\
T_{F_1^{+*}} &= T_{\frac{1}{5}(4,-1,-1,-1,-1)} &&= \{w = 0 \neq xyz\}.
\end{aligned}$$

The intersections of the closures these hypersurfaces is sketched in Figure 6.

§3. The Calabi-Yau varieties

Following Batyrev, we define a family of hypersurfaces in $\widetilde{P}$, given by elements $X \in |-K_{\widetilde{P}}|$. The general member of the family is given by



The closures of $\{x = 0\}$, $\{y = 0\}$ and $\{z = 0\}$ intersect as indicated by the intersections of the corresponding polyhedra. A polyhedron corresponding to $\{w = 0\}$ meets these polyhedra in the labeled faces. Where two polyhedra meet in a hexagon the corresponding threefolds have intersection $\widetilde{\mathbb{P}}^2$, and where they meet in a square the corresponding threefolds intersect in $\mathbb{P}^1 \times \mathbb{P}^1$.

Figure 6: How $\overline{\{x = 0\}}, \overline{\{y = 0\}}$ and $\overline{\{z = 0\}} \subset \widetilde{P} \setminus T$ meet.

an equation containing exactly the monomials corresponding to the lattice points of Δ.

In our case, Δ has 21 lattice points, and so gives a 20 dimensional family. The general member, when restricted to the open torus T, has an equation

$$(9)\quad X_u : \sum_{1\leq i,j\leq 5,\, i\neq j} u_{ij} X_i X_j^{-1} = t \quad \text{for } u = (u_{12} : u_{13} : \cdots : u_{45} : t) \in \mathbb{P}^{20}$$

in terms of the homogenous coordinates for $T \subset \mathbb{P}^4$.

Given the above analysis of Δ and Δ^*, we can now prove the following.

Proposition 3.1. *For every smooth member X_u of the family of Calabi-Yau threefolds* (9), *we have*

(i) *The Euler number $e(X_u) = 20$.*

(ii) *The Hodge numbers of X_u are given by $h^{00} = h^{33} = h^{30} = h^{03} = 1$, $h^{10} = h^{01} = h^{20} = h^{02} = 0$, $h^{11} = 26$ and $h^{12} = h^{21} = 16$.*

Proof. Since X_u is smooth and Calabi-Yau the only Hodge numbers to be computed are h^{11} and h^{21}. By [Ba, p. 521] we have

$$h^{11}(X_u) = l(\Delta^*) - 5 - \sum_{\operatorname{codim}\Theta^*=1} l^*(\Theta^*) + \sum_{\operatorname{codim}\Theta^*=2} l^*(\Theta^*) l^*(\Theta),$$

$$h^{21}(X_u) = l(\Delta) - 5 - \sum_{\operatorname{codim}\Theta=1} l^*(\Theta) + \sum_{\operatorname{codim}\Theta=2} l^*(\Theta) l^*(\Theta^*),$$

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where $l(\Theta)$ denotes the number of lattice points in Θ and $l^*(\Theta)$ denotes the number of interior lattice points of Θ, for any face Θ of Δ.

Lemmas 2.2, 2.3 and 2.6 imply that $l(\Delta) = 21$, $l(\Delta^*) = 31$, $l^*(\Delta^*) = 1$ and $l^*(\Theta^*) = 0$ for all proper faces Θ of Δ^*. Hence $h^{11}(X_u) = 26$, $h^{21}(X_u) = 16$. The Euler characteristic is then given by $e(X_u) = 2h^{11} - 2h^{12} = 20$. □

3.1. Resolution of singular Calabi-Yau threefolds

We consider elements $X \in |-K_{\widetilde{P}}|$ which have s nodes, but no other singularities. We denote the big resolution of X, obtained by blowing up the nodes, by $\widetilde{X}$. We also have 2^s small resolutions of X, where each node is replaced by a $\mathbb{P}^1$. It is not clear whether there are any small projective resolutions as these could all contain null homologous lines. By $\widehat{X}$ we denote a small *projective* resolution, when one exists.

Let X_u be a smooth member of the family (9), and let $X_{\boldsymbol{a}}$ be an element of the family with s nodes, but no other singularities. Then we have

PROPOSITION 3.2. *Let $\tilde{h}^{pq} = h^{pq}(\widetilde{X}_{\boldsymbol{a}})$, resp. $\hat{h}^{pq} = h^{pq}(\widehat{X}_{\boldsymbol{a}})$ be the Hodge numbers of the big resolution $\widetilde{X}_{\boldsymbol{a}}$ of $X_{\boldsymbol{a}}$, resp. a small projective resolution of $X_{\boldsymbol{a}}$. Then the following holds*

(i) $e(X_{\boldsymbol{a}}) = e(X_u) + s, \quad e(\widehat{X}_{\boldsymbol{a}}) = e(X_u) + 2s, \quad e(\widetilde{X}_{\boldsymbol{a}}) = e(X_u) + 4s$

(ii) $\tilde{h}^{30} = \tilde{h}^{03} = \hat{h}^{30} = \hat{h}^{03} = 1$

(iii) $\tilde{h}^{10} = \tilde{h}^{01} = \hat{h}^{10} = \hat{h}^{01} = 0, \quad \tilde{h}^{20} = \tilde{h}^{02} = \hat{h}^{20} = \hat{h}^{02} = 0$

(iv) $\tilde{h}^{11} - \tilde{h}^{12} = \frac{1}{2}e(\widetilde{X}_{\boldsymbol{a}}), \quad \hat{h}^{11} - \hat{h}^{12} = \frac{1}{2}e(\widehat{X}_{\boldsymbol{a}})$.

Proof. (i) These formulae are well known, cf. [C, Section 1] or [We, Kapitel II].

(ii) Since $h^{p0} = h^{0p}$ are birational invariants, it is enough to prove the assertion for $\tilde{h}^{30}$. Let $Q_1, \ldots, Q_s$ be the exceptional quadrics in $\widetilde{X}_{\boldsymbol{a}}$ and note that their normal bundle is $(-1,-1)$. Since $X_{\boldsymbol{a}}$ is a Calabi-Yau variety with s nodes it follows that $\omega_{\widetilde{X}_{\boldsymbol{a}}} = \mathcal{O}_{\widetilde{X}_{\boldsymbol{a}}}\left(\sum_{i=1}^{s} Q_i\right)$ and $\tilde{h}^{30} = h^0(\omega_{\widetilde{X}_{\boldsymbol{a}}}) = 1$.

(iii) We consider the sequence

$$0 \longrightarrow \mathcal{O}_{\widetilde{P}}(-K_{\widetilde{P}}) \longrightarrow \mathcal{O}_{\widetilde{P}} \longrightarrow \mathcal{O}_{X_{\boldsymbol{a}}} \longrightarrow 0.$$

Since $h^1(\mathcal{O}_{\widetilde{P}}) = 0$ and $h^2(\mathcal{O}_{\widetilde{P}}(-K_{\widetilde{P}})) = h^2(\mathcal{O}_{\widetilde{P}}) = 0$, we have $h^1(\mathcal{O}_{X_{\boldsymbol{a}}}) = 0$. The resolution $\pi : \widetilde{X}_{\boldsymbol{a}} \to X_{\boldsymbol{a}}$ is the blow up of double points, i.e., of rational singularities, and hence $R^1\pi_*\mathcal{O}_{\widetilde{X}_{\boldsymbol{a}}} = 0$. By the Leray spectral sequence for the resolution this implies that $h^1(\mathcal{O}_{\widetilde{X}_{\boldsymbol{a}}}) = h^1(\mathcal{O}_{X_{\boldsymbol{a}}}) = 0$, and hence, since these numbers are birational invariants, $\tilde{h}^{10} = \tilde{h}^{01} = \hat{h}^{10} = \hat{h}^{01} = 0$.

It remains to prove that $\tilde{h}^{02} = h^2(\mathcal{O}_{\widetilde{X}_{\boldsymbol{a}}}) = h^1(\omega_{\widetilde{X}_{\boldsymbol{a}}}) = 0$. The latter can be deduced from the exact sequence

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}_{\boldsymbol{a}}} \longrightarrow \omega_{\widetilde{X}_{\boldsymbol{a}}} \longrightarrow \bigoplus_{i=1}^{s} \mathcal{O}_{Q_i}(-1,-1) \longrightarrow 0$$

together with the fact that $H^1(\mathcal{O}_{\widetilde{X}_{\boldsymbol{a}}}) = 0$ which we have already seen.

(iv) This follows immediately from (ii) and (iii). ∎

3.2. A five dimensional subfamily $X_{\boldsymbol{a}}$ with 30 nodes

We now turn our attention to a certain subfamily of (9) of Calabi-Yau threefolds of the form

$$(10) \quad X_{\boldsymbol{a}} \cap T: \ (X_1 + \cdots + X_5)\left(\frac{a_1}{X_1} + \cdots + \frac{a_5}{X_5}\right) = a_6 = t \quad \text{for } \boldsymbol{a} = (a_1 : a_2 : a_3 : a_4 : a_5 : t) \in \mathbb{P}^5.$$

The variety $X_{\boldsymbol{a}}$ is the closure of $X_{\boldsymbol{a}} \cap T$ in the toric variety $\widetilde{P}$. For $t \in \mathbb{C}$, we will also use the notation $X_t := X_{(1:1:1:1:1:t)}$.

In terms of the local coordinates $x = X_1/X_2$, $y = X_2/X_3$, $z = X_3/X_4$, $w = X_4/X_5$ for $\mathbb{A}^4_{\text{id}}$, given in Section 2.2, the equation (10) for $X_{\boldsymbol{a}}$ becomes

$$(11) \quad (a_1+a_2x+a_3xy+a_4xyz+a_5xyzw)(1+w+wz+wzy+wzyx) = a_6xyzw.$$

Remark 3.3. If $\boldsymbol{a} = (a_1 : \cdots : a_6) \in \mathbb{P}^5$ with $\prod_{i=1}^6 a_i = 0$, then $X_{\boldsymbol{a}}$ is not irreducible, and so in general we assume $\prod_{i=1}^6 a_i \neq 0$.

Other local equations are given by permuting the a_i appropriately. Although this equation is not symmetric in the a_i, we will now see that up to birational equivalence a_6 plays the same role as the other a_i.

Lemma 3.4. *The variety $X_{\boldsymbol{a}}$ defined by* (10) *is birational to a variety in $\mathbb{P}^5$ defined by two equations*:

$$(12) \quad \sum_{i=1}^{6} \frac{a_i}{X_i} = \sum_{i=1}^{6} X_i = 0.$$

Proof. This follows immediately from setting $X_6 = -\sum_{i=1}^5 X_i$. ∎

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Corollary 3.5. *For any permutation $\sigma \in S_6$, the varieties $X_{\boldsymbol{a}}$ and $X_{\sigma(\boldsymbol{a})}$ are birational.*

Remark 3.6. The Barth-Nieto quintic N_5 is the variety in $\mathbb{P}^5$ defined by (12) with all $a_i = 1$. A corollary of the above lemma is that

$$X_{(1:1:1:1:1:1)} = X_1 \sim_{\text{bir}} N_5.$$

3.2.1. The singularities of $X_{\boldsymbol{a}}$ on $X_{\boldsymbol{a}} \cap T$

Lemma 3.7. *For $\boldsymbol{a} = (a_1 : a_2 : a_3 : a_4 : a_5 : t) \in \mathbb{P}^5$, with $t \neq 0$, the variety $X_{\boldsymbol{a}}$ (over any field) has a singularity at $\boldsymbol{b} \in T$ if and only if $\boldsymbol{a} = \phi(\boldsymbol{b})$ for some $\boldsymbol{b} \in T \subset \mathbb{P}^4$, where ϕ is the map*

$$\begin{gathered} \phi : T \longrightarrow \mathbb{P}^5 \\ (a : b : c : d : e) \longmapsto (a^2 : b^2 : c^2 : d^2 : e^2 : (a+b+c+d+e)^2). \end{gathered} \tag{13}$$

Proof. Writing (10) as $f \cdot g = t$, and differentiating with respect to X_i gives

$$g - \frac{a_i}{X_i^2} f = 0, \quad i = 1, \ldots, 5, \tag{14}$$

which implies that a singular point has the form $P = (\pm\sqrt{a_1} : \cdots : \pm\sqrt{a_5}\,)$. Substituting into (10), we obtain

$$(\sqrt{a_1} \pm \sqrt{a_2} \pm \sqrt{a_3} \pm \sqrt{a_4} \pm \sqrt{a_5}\,)^2 = t,$$

and so $\boldsymbol{a}$ has the claimed form. ∎

Proposition 3.8. *The singularity $\boldsymbol{b} = (b_1 : b_2 : b_3 : b_4 : b_5) \in X_{\phi(\boldsymbol{b})} \cap T \subset \mathbb{P}^4$ in Lemma 3.7 is an A_1 singularity.*

Proof. Consider the affine cover of T with coordinates $y_i = X_i/X_5 - b_i/b_5$ for $1 \leq i \leq 4$. In terms of these coordinates, the singularity is at $(0,0,0,0)$, and the equation for $X_{\phi(\boldsymbol{b})}$ is given by

$$\begin{gathered} (b_1+b_2+b_3+b_4+b_5)\left(\frac{y_1^2}{b_1} + \frac{y_2^2}{b_2} + \frac{y_3^2}{b_3} + \frac{y_4^2}{b_4}\right) \\ - (y_1+y_2+y_3+y_4)^2 + \text{higher order terms} = 0. \end{gathered} \tag{15}$$

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The matrix of the quadratic form given by the degree two part of (15) is

$$M := \begin{pmatrix} s/b_1 - 1 & -1 & -1 & -1 \\ -1 & s/b_2 - 1 & -1 & -1 \\ -1 & -1 & s/b_3 - 1 & -1 \\ -1 & -1 & -1 & s/b_4 - 1 \end{pmatrix}, \tag{16}$$

where $s = b_1 + b_2 + b_3 + b_4 + b_5$. We have

$$\det(M) = \frac{b_5(b_1 + b_2 + b_3 + b_4 + b_5)^3}{b_1 b_2 b_3 b_4},$$

which is non zero since $\boldsymbol{b} \in T$. Thus the singularity is as claimed. □

3.2.2. The singularities of $X_{\boldsymbol{a}}$ on $X_{\boldsymbol{a}} \setminus T$

LEMMA 3.9. *Let $\boldsymbol{a} = (a_1 : \cdots : a_6) \in \mathbb{P}$ with all $a_i \neq 0$, and $\varsigma \in S_5$. Then over any field, $X_{\boldsymbol{a}}$ has singularities in $(X_{\boldsymbol{a}} \setminus T) \cap \mathbb{A}^4_\varsigma$ only at the point*

$$(x_\varsigma, y_\varsigma, z_\varsigma, w_\varsigma) = (-a_{\varsigma(1)}/a_{\varsigma(2)}, 0, 0, -1),$$

where x_ς, y_ς, z_ς, w_ς are as in Section 2.2. This singularity is a node.

Proof. This follows easily by computing the partial derivatives of (11), and setting them to zero. □

LEMMA 3.10. *For general $\boldsymbol{a} \in \mathbb{P}^5$ there are exactly 30 singularities on $X_{\boldsymbol{a}}$.*

Proof. By Lemma 3.7, in general, $X_{\boldsymbol{a}} \cap T$ is smooth. By Lemma 3.9, there is only one singularity in the affine piece $\mathbb{A}^4_\varsigma$, contained in $(y_\varsigma = z_\varsigma = 0 \neq x_\varsigma w_\varsigma) = T_{\varsigma(F^+_{12} \cap F^-_{45})^*}$. Since Δ has $\binom{5}{2}\binom{3}{2} = 30$ faces $\sigma(F^+_{12} \cap F^-_{45})$ for $\sigma \in S_5$, this implies the result. □

3.2.3. Classification of singular subfamilies

By Lemma 3.7, to find the number of singularities on $X_{\boldsymbol{a}} \cap T$, we need to determine the number of $\boldsymbol{b}$ with $\boldsymbol{a} = \phi(\boldsymbol{b})$. We obtain the following result.

PROPOSITION 3.11. *For $\boldsymbol{b} = (b_1 : \cdots : b_5) \in T$, the number of nodes on $X_\phi(\boldsymbol{b})$ is given by*

$$30 + \#\left\{J : J \subset \{1,2,3,4,5\},\ \sum_{i \in J} b_i = 0\right\}. \tag{17}$$

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Proof. Lemma 3.10 gives 30 nodes on $X_{\boldsymbol{a}} \setminus T$. By Lemma 3.7, the only nodes on $X_{\boldsymbol{a}} \cap T$ are at $\boldsymbol{c}$ with $\phi(\boldsymbol{c}) = \phi(\boldsymbol{b})$. Let $\boldsymbol{c} = (c_1 : \cdots : c_5) \in \mathbb{P}^4$. After scaling, $b_i = \varepsilon_i c_i$, with $\varepsilon_i = \pm 1$ and $\sum b_i = \sum c_i$. From this we obtain

$$\sum (1 - \varepsilon_i) b_i = 0.$$

Thus the correspondence between nodes in T and subsets of $\{1, 2, 3, 4, 5\}$ is given by

$$\boldsymbol{c} \longleftrightarrow \{i : c_i \neq b_i\}, \tag{18}$$

with the empty set corresponding to $\boldsymbol{b}$. ∎

Remark 3.12. By considering how subsets of the b_i can intersect, this lemma allows us to classify all subfamilies of $X_{\boldsymbol{a}}$ with more than 30 nodes. These are given in Table 1. In this table, sets $\mathcal{F}_i$ are given in a shortened form, e.g., $\mathcal{F}_7 = \phi\{(a : a : a : -a : b)\}$ should be read as

$$\mathcal{F}_7 := \phi\{(a : a : a : -a : b) \in \mathbb{P}^4 \mid a, b \in k \setminus \{0\}\}.$$

To use this table, one must take the smallest set $\mathcal{F}_i$ containing a given $\boldsymbol{a}$, up to permutation of coordinates; e.g.; $(1 : 1 : 1 : 1 : 1 : 1) \in \mathcal{F}_{15} \subset \mathcal{F}_{11} \subset \mathcal{F}_4 \subset \mathcal{F}_2 \subset \mathcal{F}_0$; the data for X_1 is given by the last line of the table.

As a corollary of Proposition 3.11, we have the following result.

PROPOSITION 3.13. *The Euler numbers of* $\widetilde{X}_{\boldsymbol{a}}$, $\widetilde{\widetilde{X}}_{\boldsymbol{a}}$, *and of the small resolution* $\widehat{X}_{\boldsymbol{a}}$, *if it exists, are given in Table* 1.

Proof. This follows from Proposition 3.1(i) and Proposition 3.2(i). ∎

We will discuss whether $\widehat{X}_{\boldsymbol{a}}$ exists in Sections 3.2.4, 3.2.7 and 4.1.

Remark 3.14. In Proposition 4.8 we will see that h^{12} of the general member of any of the families in Table 1 is equal to the dimension of the family. In particular, we will see that for $\boldsymbol{a}$ in a zero dimensional family, $X_{\boldsymbol{a}}$ is rigid.

Smallest family $\mathcal{F}_i$ containing $\boldsymbol{a}$	dimension of $\mathcal{F}_i$	number of nodes on $X_{\boldsymbol{a}}$	$e(\widetilde{X}_{\boldsymbol{a}})$	$e(\widetilde{\overline{X}}_{\boldsymbol{a}})$	$e(\widehat{X}_{\boldsymbol{a}})$
$\mathcal{F}_0 = \{\boldsymbol{a} \in \mathbb{P}^6,\ a_i \neq 0\}$	5	30	140	80	80
$\mathcal{F}_1 = \phi\{(a : b : c : d : e)\}$	4	$30 + 1$	144	84	–
$\mathcal{F}_2 = \phi\{(a : -a : b : c : d)\}$	3	$30 + 2$	148	88	–
$\mathcal{F}_3 = \phi\{(a : b : -a - b : c : d)\}$	3	$30 + 2$	148	88	–
$\mathcal{F}_4 = \phi\{(a : -a : a : b : c)\}$	2	$30 + 3$	152	92	–
$\mathcal{F}_5 = \phi\{(a : -a : b : a - b : c)\}$	2	$30 + 3$	152	92	–
$\mathcal{F}_6 = \phi\{(a : -a : b : -b : c)\}$	2	$30 + 4$	156	96	88
$\mathcal{F}_7 = \phi\{(a : a : a : -a : b)\}$	1	$30 + 4$	156	96	–
$\mathcal{F}_8 = \phi\{(a : a : a : -2a : b)\}$	1	$30 + 4$	156	96	–
$\mathcal{F}_9 = \phi\{(a : a : b : -b : b - a)\}$	1	$30 + 4$	156	96	–
$\mathcal{F}_{10} = \phi\{(a : a : b : b : -a - b)\}$	1	$30 + 5$	160	100	–
$\mathcal{F}_{11} = \phi\{(a : a : -a : -a : b)\}$	1	$30 + 6$	164	104	92
$\mathcal{F}_{12} = \phi\{(1 : 1 : 1 : 1 : -1)\}$	0	$30 + 5$	160	100	–
$\mathcal{F}_{13} = \phi\{(1 : 1 : 1 : 2 : -2)\}$	0	$30 + 5$	160	100	–
$\mathcal{F}_{14} = \phi\{(1 : 1 : 1 : 1 : -2)\}$	0	$30 + 7$	168	108	–
$\mathcal{F}_{15} = \phi\{(1 : 1 : 1 : -1 : -1)\}$	0	$30 + 10$	180	120	100

Table 1: Number of nodes on $X_{\boldsymbol{a}}$ for subfamilies of $\boldsymbol{a}$, up to permutation of coordinates, as images of subfamilies of $\boldsymbol{b} \in T$, under ϕ, given by (13); see Remark 3.12. The last 4 sets are $\{(1 : 1 : 1 : 1 : 1 : 9)\}$, $\{(1 : 1 : 1 : 4 : 4 : 9)\}$, $\{(1 : 1 : 1 : 1 : 4 : 4)\}$ and $\{(1 : 1 : 1 : 1 : 1 : 1)\}$ respectively.

3.2.4. Weil divisors and small resolutions

We shall now discuss whether $X_{\boldsymbol{a}}$ has a small projective resolution. If this is the case then $X_{\boldsymbol{a}}$ possesses a smooth projective Calabi-Yau model. We shall see later (Proposition 4.4) that these two statements are in fact equivalent.

First we recall a fact about small resolutions. Let X be a projective variety, and $P \in X$ a double point. Near P, X is locally analytically a cone over a quadric surface Q. The local divisor class group is generated by the cones Λ_1, Λ_2 over the rulings of Q. Blowing up X in P defines a (big) resolution $\pi : \widetilde{X} \to X$, with $\pi^{-1}(P) \cong Q$. Locally analytically we can also blow up Λ_1 and Λ_2, obtaining small resolutions $p_i : X_i \to X$, $i = 1, 2$, with $p_i^{-1}(P) \cong \mathbb{P}^1$. However, this is an analytic construction, and it is not clear

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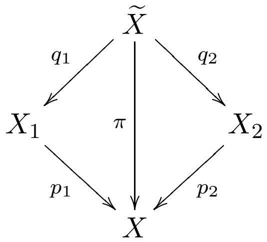
whether the X_i are projective. We have a commutative diagram



where the q_i each blow down one family of rulings. This is Atiyah's flop. If W is a global Weil divisor on X through P, blowing up along W gives a projective variety. If W is not Cartier near P then it is analytically locally equivalent to $a\Lambda_1$ or $a\Lambda_2$ for some $a > 0$. By the universal property of the blow up, blowing up X along W is the same as performing a small resolution. For a discussion of this material see [EH, Example IV-27].

3.2.5. Small resolution $\overline{X}_{\boldsymbol{a}}$ of the 30 singularities on $X_{\boldsymbol{a}} \setminus T$

We now show that we can find a projective variety which is a small projective resolution of each of the 30 nodes on $X_{\boldsymbol{a}} \setminus T$. First we will define surfaces which define the Weil divisors we use for the blow up, as described above.

Definition 3.15. Let $1 \leq i < j \leq 5$, $i \neq j$ and $\varepsilon = \pm$. For some $\varsigma \in S_5$, we have $\overline{T_{F_{ij}^{\varepsilon}{}^*}} = \overline{(y_\varsigma = 0)}$ in the affine piece $\mathbb{A}^4_\varsigma$. We define the surface $S_{\boldsymbol{a}}^{\varepsilon ij}$ to be one of the two components of $\overline{(y_\varsigma = 0)} \cap X_{\boldsymbol{a}}$, given in terms of the coordinates for $\mathbb{A}^4_\varsigma$ by

$$S_{\boldsymbol{a}}^{\varepsilon ij} : \overline{(a_{\varsigma(1)} + a_{\varsigma(2)} x_\varsigma = y_\varsigma = 0)}. \tag{19}$$

This is independent of which of the 6 possible ς is chosen. The surface $S_{\boldsymbol{a}}^{-45}$ is indicated in Figure 7.

Lemma 3.16. *The* 10 *surfaces* $S_{\boldsymbol{a}}^{-ij}$ *are smooth, disjoint, and each contains* 3 *nodes of* $X_{\boldsymbol{a}}$.

Proof. Smoothness is clear. Set $T_{ij} := T_{F_{ij}^{-*}}$. For $\{i,j\} \neq \{k,l\}$ the vertices F_{ij}^{-*} and F_{kl}^{-*} do not lie on a common edge of Δ^* (see Figure 3), so $\overline{T_{ij}} \cap \overline{T_{kl}} = \varnothing$, from which disjointness follows. From the defining equation, $S_{\boldsymbol{a}}^{-ij}$ contains the node $(-a_{\varsigma(1)}/a_{\varsigma(2)}, 0, 0, -1)$ in $(X_{\boldsymbol{a}} \setminus T) \cap \mathbb{A}^4_\varsigma$. As in Lemma 3.10, each node lies on a surface $T_{(F_{ij}^+ \cap F_{kl}^-)^*}$. Since T_{ij} contains 3 such surfaces, $S_{\boldsymbol{a}}^{-ij}$ contains 3 nodes. This can be seen in Figure 7. □

Remark 3.17. Since $S_{\boldsymbol{a}}^{\varepsilon ij} \cap T \subset \overline{T_{F_{ij}^{\varepsilon}*}} \cap T = \varnothing$, the surfaces $S_{\boldsymbol{a}}^{\varepsilon ij}$ do not pass through any singularities $\boldsymbol{b} \in X_{\boldsymbol{a}} \cap T$, and may be ignored in considering resolutions of such singularities.

DEFINITION 3.18. We define $\overline{X_{\boldsymbol{a}}}$ to be the blow up of $X_{\boldsymbol{a}}$ along all 10 surfaces $S_{\boldsymbol{a}}^{-ij}$. This is a projective small resolution of all 30 nodes in $X_{\boldsymbol{a}} \backslash T$.

PROPOSITION 3.19. *Let $\boldsymbol{a} \neq \phi(\boldsymbol{b})$ for any $\boldsymbol{b} \in \mathbb{P}^4$. Then the nodal Calabi-Yau variety $X_{\boldsymbol{a}}$ has a small projective resolution $\overline{X}_{\boldsymbol{a}}$.*

Proof. This is because $X_{\boldsymbol{a}} \cap T$ is smooth in this case, by Lemma 3.7. ∎

We have now constructed a five dimensional family of Calabi-Yau threefolds, $\overline{X}_{\boldsymbol{a}}$. In Proposition 4.8 we will see that h^{12} of the general member is 5. By Proposition 3.2 the Euler number of $X_{\boldsymbol{a}}$ for general $\boldsymbol{a}$ is 80, and so the Hodge diamond is as follows:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 45 & & 0 & \\ 1 & & 5 & & 5 & & 1 \\ & 0 & & 45 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

3.2.6. Big resolutions of singularities on $X_{\boldsymbol{a}} \cap T$

As discussed in Section 3.2.4, the big resolution of a singularity $\boldsymbol{b} \in X_{\boldsymbol{a}} \cap T$ replaces $\boldsymbol{b}$ by a quadric, which in this case has equation given by the quadratic part of (15). By transforming (16) to a diagonal matrix, one can see that the rulings of this quadric are defined over the field

$$\mathbb{Q}\Big[\sqrt{-b_0b_1b_2(b_0+b_1+b_2)}, \sqrt{-b_3b_4b_5(b_3+b_4+b_5)}\Big],$$

and other similarly defined fields. For later use, we now consider the number of points on these quadrics over finite fields. First we need the following:

LEMMA 3.20. *Let f be an irreducible quadric in $\mathbb{P}^3(\mathbb{F}_p)$, for $p \neq 2$, with corresponding matrix $M \in M_4(\mathbb{F}_p)$. Then we have*

$$\#(f=0)(\mathbb{F}_p) = \begin{cases} (p+1)^2 & \textit{if } \det M \textit{ is a square in } \mathbb{F}_p \\ p^2+1 & \textit{if } \det M \textit{ is not a square in } \mathbb{F}_p. \end{cases}$$

Proof. This follows from [Se1, Proposition 5, IV §1.7]. □

Together with Proposition 3.8, this immediately implies the following:

COROLLARY 3.21. *For the quadric $Q_{\boldsymbol{b}}$ introduced in resolving the singularity at $\boldsymbol{b} := (b_1 : b_2 : b_3 : b_4 : b_5)$ on $X_{\phi(\boldsymbol{b})}$, we have*

$$\#Q_{\boldsymbol{b}}(\mathbb{F}_p) = \begin{cases} (p+1)^2 & \text{if } \sqrt{b_0 b_1 b_2 b_3 b_4 b_5} \in \mathbb{F}_p \\ p^2+1 & \text{if } \sqrt{b_0 b_1 b_2 b_3 b_4 b_5} \notin \mathbb{F}_p, \end{cases}$$

where $b_0 = b_1 + b_2 + b_3 + b_4 + b_5$.

3.2.7. Small resolutions of singularities on $X_{\boldsymbol{a}} \cap T$

PROPOSITION 3.22. *For $\boldsymbol{b} = (a : b : -b : c : -c)$ (up to permutation of coordinates), there is a small projective resolution of $\boldsymbol{b} \in X_{\phi(\boldsymbol{b})}$.*

Proof. In this case the surface $(X_2 + X_3 = X_4 + X_5 = 0)$ lies in $X_{\phi(\boldsymbol{b})}$, contains the point $\boldsymbol{b}$, and its closure is a smooth Weil divisor in $X_{\phi(\boldsymbol{b})}$. Thus a small resolution is achieved by blowing up this surface. □

Remark 3.23. Note that for $\boldsymbol{b}$ as in the proposition, $X_{\phi(\boldsymbol{b})}$ has at least four singularities, at $(a : b : -b : c : -c)$, $(a : -b : b : c : -c)$, $(a : -b : b : -c : c)$ and $(a : b : -b : -c : c)$.

COROLLARY 3.24. *There is a small projective resolution of all the singularities of X_1.*

Proof. This follows from Proposition 3.22, since if $\boldsymbol{a} = (1 : 1 : 1 : 1 : 1 : 1) = \phi(\boldsymbol{b})$ then $\boldsymbol{b} = (1 : 1 : 1 : -1 : -1)$, up to permutation of coordinates. □

Remark 3.25. The only Calabi-Yau threefolds in Table 1 for which all singularities can be resolved by the small resolution of Proposition 3.22 are $X_{\phi(\boldsymbol{b})}$ for $\boldsymbol{b} = (1 : -1 : b : -b : c)$ up to permutation, with $1 \pm b \pm c \neq 0$ for all sign choices; this can easily be seen by listing all possible singularities, by using (17) and (18). In Corollary 4.5 we will see that other $X_{\boldsymbol{a}}$ have no small projective resolution.

§4. $X_{\boldsymbol{a}}$ as a fibre product

We now show that $X_{\boldsymbol{a}}$ is birational to the fibre product of families of elliptic curves. This enables us to apply Schoen's results [Sc] on such threefolds.

First we define the families of elliptic curves.

Definition 4.1. For a, b, $c \in k \setminus \{0\}$ define an elliptic surface $\mathcal{E}_{a,b,c}$ to be the resolution of the surface $\mathcal{E}'_{a,b,c} \subset \mathbb{P}^2 \times \mathbb{P}^1$ given by

$$\mathcal{E}'_{a,b,c}: \ (x+y+z)(axy+byz+czx)t_0 = t_1 xyz, \tag{20}$$

where $(x:y:z) \in \mathbb{P}^2$, $(t_0 : t_1) \in \mathbb{P}^1$. There is a projection $p : \mathcal{E}_{a,b,c} \to \mathbb{P}^1$, with fibre $\mathcal{E}_{a,b,c,t} := p^{-1}(1:t)$. We write $\mathcal{E}_a := \mathcal{E}_{1,1,a}$ and $\mathcal{E}_{a,t} := \mathcal{E}_{1,1,a,t}$.

The only singular points of $\mathcal{E}'_{a,b,c}$ are the three singular points of the fibre at infinity. When these are resolved the fibre at infinity becomes an I_6 fibre.

When $\mathcal{E}_{a,b,c,t}$ is smooth, taking the zero to be $(0:1:-1)$, the elliptic curve $\mathcal{E}_{a,b,c,t}$ is isomorphic to the cubic curve with equation

$$y^2 = x\left(\left(x+\frac{2(t^2+a^2+b^2+c^2)-(t+a+b+c)^2}{8a^2}\right)^2 - \frac{A(a,b,c,t)}{64a^4}\right), \tag{21}$$

where

$$A(a,b,c,t) := \prod_{(\nu,\mu)\in\{-1,1\}^2} \left(t - (\sqrt{a}+\nu\sqrt{b}+\mu\sqrt{c}\,)^2\right).$$

From this we find that the j-invariant of $\mathcal{E}_{a,b,c,t}$ is given by

$$j(\mathcal{E}_{a,b,c,t}) = \frac{(A(a,b,c,t)+16abct)^3}{(abct)^2 A(a,b,c,t)}. \tag{22}$$

This implies that in general $\mathcal{E}_{a,b,c}$ has 6 singular fibres, corresponding to the values of t for which $j(\mathcal{E}_{a,b,c,t}) = \infty$. The singular fibres together with their fibre type, in the general and all special cases are given in Table 2. As examples, we also tabulate the data for $\mathcal{E}_1$, $\mathcal{E}_9$, $\mathcal{E}_{25}$ and $\mathcal{E}_{1,4,4}$.

Lemma 4.2. *$X_{(a_1:a_2:a_3:a_4:a_5:a_6)}$ and $\mathcal{E}_{a_1,a_2,a_3} \times_{\mathbb{P}^1} \mathcal{E}_{a_4,a_5,a_6}$ are birational.*

Proof. We can rewrite (12) as

$$X_1 + X_2 + X_3 = -(X_4 + X_5 + X_6),$$
$$\frac{a_1}{X_1} + \frac{a_2}{X_2} + \frac{a_3}{X_3} = -\left(\frac{a_4}{X_4} + \frac{a_5}{X_5} + \frac{a_6}{X_6}\right),$$

from which, introducing a new parameter $(\lambda_0 : \lambda_1) \in \mathbb{P}^1$, we obtain

$$(X_1 + X_2 + X_3)\left(\frac{a_1}{X_1} + \frac{a_2}{X_2} + \frac{a_3}{X_3}\right)\lambda_0 = \lambda_1, \tag{23}$$

$$(X_4 + X_5 + X_6)\left(\frac{a_4}{X_4} + \frac{a_5}{X_5} + \frac{a_6}{X_6}\right)\lambda_0 = \lambda_1, \tag{24}$$

But these are equations for $\mathcal{E}_{a_1,a_2,a_3}$ and $\mathcal{E}_{a_4,a_5,a_6}$, so we have a map

$$\varphi_T : X_{\boldsymbol{a}} \cap T \longrightarrow (\mathcal{E}_{a_1,a_2,a_3} \cap (\lambda_0\lambda_1 \neq 0)) \times_{\mathbb{P}^1} (\mathcal{E}_{a_4,a_5,a_6} \cap (\lambda_0\lambda_1 \neq 0)),$$
$$(X_1 : X_2 : X_3 : X_4 : X_5 : X_6) \longmapsto (X_1 : X_2 : X_3) \times (X_4 : X_5 : X_6).$$

If $P := (X_1 : X_2 : X_3) \in \mathcal{E}_{a_1,a_2,a_3} \cap (\lambda_0\lambda_1 \neq 0)$ and $Q := (X_4 : X_5 : X_6) \in \mathcal{E}_{a_4,a_5,a_6} \cap (\lambda_0\lambda_1 \neq 0)$, then $(X_1+X_2+X_3)+\mu(X_4+X_5+X_6) = 0$ for some unique $\mu \neq 0$, and then $R := (X_1 : X_2 : X_3 : \mu X_4 : \mu X_5 : \mu X_6) \in X_{\boldsymbol{a}}$ and $\varphi(R) = (P, Q)$. Thus φ_T defines a birational map. ∎

For example, this result implies that X_1, X_9, $X_{(1:1:1:1:4:4)}$ and $X_{(1:1:1:4:4:9)}$ are birational to $\mathcal{E}_1 \times_{\mathbb{P}^1} \mathcal{E}_1$, $\mathcal{E}_1 \times_{\mathbb{P}^1} \mathcal{E}_9$, $\mathcal{E}_1 \times_{\mathbb{P}^1} \mathcal{E}_{1,4,4}$ and $\mathcal{E}_9 \times_{\mathbb{P}^1} \mathcal{E}_{1,4,4}$ respectively.

4.1. Existence of smooth projective Calabi-Yau models

Now we can apply results of Schoen [Sc] to determine whether $X_{\boldsymbol{a}}$ has a small projective resolution or not.

LEMMA 4.3. *If* $a, b, c, d, e, f \in k \setminus \{0\}$ *with* $a+b+c+d+e+f = 0$ *satisfy* (i) $0 \notin \{\pm a \pm b \pm c, \pm d \pm e \pm f\}$, (ii) $(a+b+c)^2 \notin \{(a+b-c)^2, (a-b+c)^2, (a-b-c)^2\}$, *and* (iii) $\{a^2, b^2, c^2\} \neq \{d^2, e^2, f^2\}$, *then* $\mathcal{E}_{a^2,b^2,c^2} \times_{\mathbb{P}^1} \mathcal{E}_{d^2,e^2,f^2}$ *has no small projective resolution.*

Proof. This follows from [Sc, Lemma 3.1 (iii)], as in this case $\mathcal{E}_{a,b,c}$ and $\mathcal{E}_{d,e,f}$ are not isogenous, and from Table 2, both have singular fibres at $(a+b+c)^2$, with that for $\mathcal{E}_{a,b,c}$ being I_1 type. ∎

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$\mathcal{E}_{a^2,b^2,c^2}$ $a+b+c=t\neq 0$, $\#\{2a,2b,2c,t,0\}=5$	∞	0	t^2	$(t-2a)^2$	$(t-2b)^2$	$(t-2c)^2$
	I_6	I_2	I_1	I_1	I_1	I_1
$\mathcal{E}_{a^2,a^2,b^2}$ $\#\{b,a,2a,0\}=4$	∞	0	b^2	$(2a-b)^2$	$(2a+b)^2$	
	I_6	I_2	I_2	I_1	I_1	
$\mathcal{E}_{a^2,a^2,a^2}$ $a\neq 0$	∞	0	a^2	$9a^2$		
	I_6	I_2	I_3	I_1		
$\mathcal{E}_{a^2,a^2,4a^2}$ $a\neq 0$	∞	0	$4a^2$	$16a^2$		
	I_6	III	I_2	I_1		
$\mathcal{E}_{a^2,b^2,c^2}$ $a+b+c=0$, $\#\{a,b,c,0\}=4$	∞	0	$4a^2$	$4b^2$	$4c^2$	
	I_6	III	I_1	I_1	I_1	

$\mathcal{E}_1$	∞	0	1	9	
	I_6	I_2	I_3	I_1	
$\mathcal{E}_9$	∞	0	1	9	25
	I_6	I_2	I_1	I_2	I_1

$\mathcal{E}_{1,4,4}$	∞	0	1	9	25
	I_6	I_2	I_2	I_1	I_1
$\mathcal{E}_{25}$	∞	0	9	25	49
	I_6	I_2	I_1	I_2	I_1

Table 2: The fibre types of the singular fibres of $\mathcal{E}_{\boldsymbol{a}}$.

In Corollary 3.24 we saw that X_1 has a smooth projective Calabi-Yau model. However, as we shall now see, this is not true for $X_{\phi(\boldsymbol{b})}$ in general. The following was pointed out to us by J. Kollár.

Proposition 4.4. *Suppose Y is a nodal threefold with trivial canonical bundle and that there is no small resolution of all singularities. Then Y does not posess a smooth projective Calabi-Yau model.*

Proof. Assume that Z is a smooth projective model of Y which is Calabi-Yau. We can successively blow up Weil divisors of Y until we obtain a projective variety Y' with all Weil divisors of Y' Cartier at all nodes. In other words Y' is factorial. By assumption Y' cannot be smooth. We have a birational map $f : Y' \dashrightarrow Z$. Since both Y' and Z have trivial (and hence nef) canonical bundle the map f factors as a finite sequence of flops [Ko, Theorem 4.9]. Since flops in dimension 3 do not change the type of singularity [Ko, Theorem 2.4] this gives a contradiction. ∎

COROLLARY 4.5. *For* $\boldsymbol{b} \in T$*, the nodal variety* $X_{\phi(\boldsymbol{b})}$ *does not posess smooth projective Calabi-Yau model unless* $\phi(\boldsymbol{b}) = \phi((1 : -1 : b : -b : c))$*, up to permutations, with* $1 \pm b \pm c$ *for some sign choice.*

Proof. Let $\phi(\boldsymbol{b}) = (b_1^2 : b_2^2 : b_3^2 : b_4^2 : b_5^2 : b_6^2)$ with $b_1 + b_2 + b_3 + b_4 + b_5 + b_6 = 0$. If for some permutation of these variables the condition of Lemma 4.3 is satisfied, $\mathcal{E}_{b_1^2,b_2^2,b_3^2} \times_{\mathbb{P}^1} \mathcal{E}_{b_4^2,b_5^2,b_6^2}$ has no smooth projective Calabi-Yau model. Then Proposition 4.4 and Lemma 4.2 imply that $X_{\phi(\boldsymbol{b})}$ has no smooth projective Calabi-Yau model.

Suppose that conditions (i), (ii) and (iii) of Lemma 4.3 are not satisfied for any permutations, or sign changes preserving $\sum b_i = 0$, of the b_i.

First note that $(a+b+c)^2 \notin \{(a+b-c)^2, (a-b+c)^2, (a-b-c)^2\} \iff abc(a+b)(b+c)(c+a) \neq 0$, so if $b_i + b_j \neq 0$ for $1 \leq i, j \leq 6$, condition (ii) can not fail. If we had $b_i + b_j + b_k = 0$ for all triples $1 \leq i, j, k \leq 6$, then all b_i are equal, but this contradicts $\sum b_i = 0$. If (iii) fails for all permutations, then all b_i^2 are equal, and if $b_i + b_j \neq 0$, this again gives the contradiction that all b_i are equal. Hence we can assume $b_1 + b_2 = 0$. Now suppose that $b_i + b_j \neq 0$ for $3 \leq i, j \leq 6$. Then (ii) holds for b_3, b_4, b_5. Now $b_3 + b_4 + b_5 + b_6 = -(b_1 + b_2) = 0$, so (i) also holds, since $a_6 \neq 0$. Similarly, (i) and (ii) hold for all other triples from b_3, b_4, b_5, b_6. If (iii) fails for all of these, then again, all b_i^2 are equal, but this is not possible with $\sum b_i = 0$. Thus we can assume $b_3 + b_4 = 0$, and since $\sum b_i = 0$ we also have $b_5 + b_6 = 0$.

Thus, if (i), (ii) or (iii) always fail, this implies that up to permutations, $\boldsymbol{b} = (1 : -1 : b : -b : c)$. If $1 + \pm b + \pm c = 0$, then after a sign change, we have $1 + b + c = 0$, and then $\phi(\boldsymbol{b}) = (1 : 1 : b : b : -c)$, and taking b_i in this order satisfies (i), (ii) and (iii) of Lemma 4.3.

Conversely, if $\boldsymbol{b} = (1 : -1 : b : -b : c)$, then (i), (ii) or (iii) always fail, unless $\phi(\boldsymbol{b}) \neq \phi(\boldsymbol{c})$ for some $\boldsymbol{c}$ not of this form. As in Proposition 3.11, we obtain $\boldsymbol{c}$ from $\boldsymbol{b}$ by changing the signs of the b_i in a nontrivial subset of the b_i which sum to 0. For $\boldsymbol{c}$ to have a different form is equivalent to $1 \pm b \pm c = 0$ for some choice of signs.

In case $\phi(\boldsymbol{b}) = \phi((1 : -1 : b : -b : c))$, up to permutations, with $1 \pm b \pm c \neq 0$ for all sign choices, by Proposition 3.22 and Remark 3.25 there is a small resolution of $X_{\phi(\boldsymbol{b})}$, by explicit construction. ∎

In particular, this means that X_9, X_{25}, $X_{(1:1:1:1:4:4)}$ and $X_{(1:1:1:4:4:9)}$ have no small projective resolution. Later we will see that X_1 and X_9 have the same L-series. The Tate conjecture then says that there should be

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a correspondence between them. In this case such a correspondence can not be a birational map, since Corollaries 4.5 and 3.24 imply the following result.

Corollary 4.6. *X_1 and X_9 are not birational.*

4.2. Computation of h^{12}

By Proposition 3.2, we know all the Hodge numbers of the varieties $\widetilde{X}_{\boldsymbol{a}}$ and $\widehat{X}_{\boldsymbol{a}}$, except for h^{11} and h^{12}, for which we only know the difference. Note that $\tilde{h}^{12} = \hat{h}^{12}$, so it is enough to compute $\tilde{h}^{12}$. We will use the fact that in our situation h^{12} is a birational invariant. This can either be deduced from [Ko, Corollary (4.12)] or, as Batyrev has informed us, by using motivic integration.

Then from Lemma 4.2 it is enough to compute h^{12} for the corresponding elliptic families fibre product. First we need to know that the elliptic families in question are semistable, for which we need the following lemma.

Lemma 4.7. *Suppose that $b_1, \dots, b_6 \neq 1, 1, 1, 1, 1, 2$ or $1, 1, 1, 1, 2, 3$ (up to scaling, permutations and sign changes). Then, after possible permutation, we can assume that $b_1 \pm b_2 \pm b_3 \neq 0$ and $b_4 \pm b_5 \pm b_6 \neq 0$, for all sign choices.*

Proof. Suppose that this is not the case. Then for each partition $\{i_1, i_2, i_3\}$, $\{i_4, i_5, i_6\}$ of $1, \dots, 6$ into 2 sets of size 3, we have either $(b_{i_1} \pm b_{i_2} \pm b_{i_3}) = 0$ or $(b_{i_4} \pm b_{i_5} \pm b_{i_6}) = 0$, (or both) (for some choice of signs).

First we will show that some of the b_i must be equal, up to sign. There are 10 partitions, so we have ten distinct sets $\{i_1, i_2, i_3\}$ of size 3, with $(b_{i_1} \pm b_{i_2} \pm b_{i_3}) = 0$. These sets have $30 = 3 \cdot 10$ entries, and so each of $1, \dots, 6$ must occur at least $5 = 30/6$ times, i.e., in at least 5 of the 10 sets.

Take 5 of the sets containing 1. Suppose we always have $b_i \neq \pm b_j$ for $i \neq j$. Then for each other element occurring in these 5 sets, it must occur exactly twice, with opposite signs; or else we would have, e.g., $b_1 + \varepsilon b_2 \pm b_4 = 0$ and $b_1 + \varepsilon b_2 \pm b_5 = 0$ where $\varepsilon = \pm 1$, so $b_4 = \pm b_5$. But then when we take the sum of all 5 equalities, we would get $5b_1 = 0$, a contradiction. Hence we have $b_i = \pm b_j$ for some i, $j \in \{2, 3, 4, 5, 6\}$. This is similarly the case for every other subset of 5 elements. This implies that there are two (not necessarily disjoint) pairs of equal elements amongst the b_i. Thus the b_i have the form a, a, a, b, c, d or a, a, b, b, c, d, up to sign. In the first case we must have $b \pm c \pm d = 0$. From the other possible partitions, we get (possibly after replacing b with $-b$ or c with $-c$),

$$\begin{aligned} & b = \pm 2a \quad \text{or} \quad a + c + d = 0, \\ \text{and} \quad & c = \pm 2a \quad \text{or} \quad a + b \pm d = 0, \\ \text{and} \quad & d = \pm 2a \quad \text{or} \quad a \pm c \pm b = 0. \end{aligned}$$

We can not have all of the first column true, since this would imply $a = 0$ by taking the sum (up to sign), since $b \pm c \pm d = 0$. If two of the first column hold, e.g., $b = \pm 2a$, and $c = \pm 2a$, then from the last line we get $a = \pm 4a$, a contradiction. If from the first column we just have $d = \pm 2a$, then from the second, we get $b, c \in \{a, -3a\}$; since $b \pm c \pm d = 0$, we must have $b = c = a$ or $\{b, c\} = \{a, -3a\}$. The first possibility gives that the b_i are (after scaling) given, up to sign, by $1, 1, 1, 1, 2, 3$, and the second gives $1, 1, 1, 1, 1, 2$.

If the whole of the last column is true, then substituting $b = \pm c \pm d$ and eliminating a, we get $c + d = \pm c \pm 2d$ or $\pm c$ and $c + d = \pm d \pm 2c$ or $\pm d$. Since $c, d \neq 0$, this gives $2c = \pm d$ or $\pm 3d$ and $2d = \pm c$ or $\pm 3c$. But all possibilities lead to $c = d = 0$, a contradiction.

Now suppose the b_i have the form a, a, b, b, c, d. This gives us

$$\begin{aligned} & c = \pm 2a \quad \text{or} \quad d = \pm 2b, \\ \text{and} \quad & d = \pm 2a \quad \text{or} \quad c = \pm 2b. \end{aligned}$$

If we have one true from each column, we get $b = a$, and this gives the previous case. Otherwise, we can assume we have $c = \pm d = \pm 2a$. The b_i now must have the form $a, a, b, b, 2a, 2a$. Thus $b = a$ or $b = 2a$. This gives, up to sign change, scaling and permutation, $1, 1, 1, 1, 2, 2$ or $1, 1, 2, 2, 4, 4$. But in the first case we have $(1 \pm 1 \pm 1)(1 \pm 2 \pm 2) \neq 0$, and in the second $(1 \pm 2 \pm 4)(1 \pm 2 \pm 4) \neq 0$, so neither of these is possible.

Hence the only possibility for the b_i such that there is no permutation with $b_1 \pm b_2 \pm b_3 \neq 0$ and $b_4 \pm b_5 \pm b_6 \neq 0$, are those given. □

PROPOSITION 4.8. *If $\mathcal{F}_i$ is the smallest set in Table 1 containing $\boldsymbol{a}$, then $h^{12}(\widetilde{X}_{\boldsymbol{a}}) = \dim \mathcal{F}_i$. In particular, if $\boldsymbol{a} \in \mathbb{P}^5 \setminus \phi(\mathbb{P}^4)$ with all coordinates nonzero, then $h^{12}(\overline{X}_{\boldsymbol{a}}) = 5$.*

Proof. For some $b_i \neq 0$ we have $\boldsymbol{a} = (b_1^2 : \cdots : b_6^2) \in \mathbb{P}^5$. We set $\mathcal{E}^1 = \mathcal{E}_{b_1^2, b_2^2, b_3^2}$ and $\mathcal{E}^2 = \mathcal{E}_{b_4^2, b_5^2, b_6^2}$. By Lemma 4.2, $X_{\boldsymbol{a}}$ is birational to $\mathcal{E}^1 \times_{\mathbb{P}^1} \mathcal{E}^2$, and since h^{12} is a birational invariant, it is enough to compute $h^{12}(\mathcal{E}^1 \times_{\mathbb{P}^1} \mathcal{E}^2)$.

Assuming b_i are not $1,1,1,1,1,2$ or $1,1,1,1,2,3$, by Lemma 4.7, we can assume that $b_1 \pm b_2 \pm b_3$, $b_4 \pm b_5 \pm b_6 \neq 0$ for all sign choices, so $\mathcal{E}^1$ and $\mathcal{E}^2$ are semistable (see Table 2). Let $c_i(s)$ be the number of components of the fibre $\mathcal{E}^i_s$, $d = 1$ if $\mathcal{E}^1$ and $\mathcal{E}^2$ are isogenous, and 0 otherwise, $S_i := \{s \in \mathbb{P}^1 \mid \mathcal{E}^i_s \text{ is singular}\}$, and $S' := S_1 \cap S_2 \setminus \{0, \infty\}$. We have $S_1 = \{(b_1 \pm b_2 \pm b_3)^2\} \cup \{0, \infty\}$, $c_1(s) = \#\{(\varepsilon_2, \varepsilon_3) \in \{\pm 1\}^2 \mid (b_1 + \varepsilon_2 b_2 + \varepsilon_3 b_3)^2 = s\}$, for $s \neq 0, \infty$, and S_2 and c_2 are given similarly in terms of b_4, b_5, b_6. Then by [Sc, (7.4)], for the smooth resolution $\mathcal{E}^1 \widetilde{\times_{\mathbb{P}^1}} \mathcal{E}^2$ of $\mathcal{E}^1 \times_{\mathbb{P}^1} \mathcal{E}^2$ we have

$$
\begin{aligned}
h^{12}(\mathcal{E}^1 \widetilde{\times_{\mathbb{P}^1}} \mathcal{E}^2) &= \#(S_1 \cup S_2) + d - 5 + \sum_{s \in \mathbb{P}^1 \setminus (S_1 \cap S_2)} (c_1(s) c_2(s) - 1) \\
&= 5 + d - \#S' - \sum_{s \in S'} (c_1(s) + c_2(s) - 2) \\
&= 5 + d - \sum_{s \in S'} (c_1(s) + c_2(s) - 1) \\
&= 5 + d - \sum_{s \in S'} c_1(s) c_2(s) - (c_1(s) - 1)(c_2(s) - 1).
\end{aligned}
$$

If $(b_1^2 : \cdots : b_6^2) \in \mathcal{F}_0$, i.e., if $\widetilde{\sum \pm b_i} \neq 0$ for all sign choices, then $d = 0$ and $S' = \varnothing$, so $h^{12}(\overline{X}_{\boldsymbol{a}}) = h^{12}(\mathcal{E}^1 \times \mathcal{E}^2) = 5$, as claimed.

Now suppose $\sum b_i = 0$. Note we can take $d = 0$, unless all the $|b_i|$ are all equal, since $d = 1$ only if $|b_i|$ have the form a, b, c, a, b, c, but these can be rearranged as a, a, b, c, c, b with $b \neq c, a$, unless all are equal. Set $A := \sum_{s \in S'} (c_1(s) - 1)(c_2(s) - 1)$. This is 0 unless the $|b_i|$ have the form a, a, b, c, c, b, with $b \neq a, c$ or a, a, a, a, b, b, with $b \neq a$, or a, a, a, a, a, a, in which case $A = 1, 2$ or 4 respectively. Now $-h^{12} + 5 + d + A$ is equal to

$$
\begin{aligned}
&\#\{(\varepsilon_2, \varepsilon_3, \varepsilon_5, \varepsilon_6) \in \{\pm 1\}^4 \mid (b_1 + \varepsilon_2 b_2 + \varepsilon_3 b_3)^2 = (b_4 + \varepsilon_5 b_5 + \varepsilon_6 b_6)^2\} \\
&\quad = \#\{(\varepsilon_2, \ldots, \varepsilon_6) \in \{\pm 1\}^5 \mid b_1 + \varepsilon_2 b_2 + \varepsilon_3 b_3 + \varepsilon_4 b_4 + \varepsilon_5 b_5 + \varepsilon_6 b_6 = 0\} \\
&\quad = \text{number of singularities on } \overline{X}_{\boldsymbol{a}}.
\end{aligned}
$$

The first equality follows since if $u = b_1 + \varepsilon_2 b_2 + \varepsilon_3 b_3$, $v = b_4 + \varepsilon_5 b_5 + \varepsilon_6 b_6$, and $u^2 = v^2$, then exactly one of $u = \pm v$ holds, since we assume $b_1 \pm b_2 \pm b_3$,

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$b_4 \pm b_5 \pm b_6 \neq 0$. Proposition 3.11 gives the second equality. Now we have

$$h^{12} = 5 - \text{number of singularities of } \overline{X}_{\boldsymbol{a}} + d + A$$
$$= 5 - \#(\text{sings of } \overline{X}_{\boldsymbol{a}}) + \begin{cases} 0+1 & \text{if } \boldsymbol{a} = (1,1,b,c,c,b),\ b \neq 1, c, \\ 0+2 & \text{if } \boldsymbol{a} = (1,1,1,1,b,b),\ b \neq 1, \\ 1+4 & \text{if } \boldsymbol{a} = (1,1,1,1,1,1), \\ 0 & \text{otherwise.} \end{cases}$$

The four cases correspond to (1) $\mathcal{F}_6$ and $\mathcal{F}_{10}$, (2) $\mathcal{F}_{11}$ and $\mathcal{F}_{14}$, (3) $\mathcal{F}_{15}$, and (4) everything else, respectively. Comparing the number of singularities with the dimension of $\mathcal{F}_i$ in Table 1 gives the result.

If the b_i are $1,1,1,1,1,2$ or $1,1,1,1,2,3$, then instead of the above, we can use van Geemen's point counting method [GW] to show that the values of h^{12} in these cases are also given as stated. ∎

We immediately have the following, which also can be quickly deduced from [Sc, Proposition 7.1].

Corollary 4.9. *The varieties X_1, X_9, $X_{(1:1:1:1:4:4)}$ and $X_{(1:1:1:4:4:9)}$ are rigid, i.e., have $h^{12} = 0$.*

§5. Elliptic surfaces in $X_{\boldsymbol{a}}$

Definition 5.1. Let $\boldsymbol{a} = (a_1 : \cdots : a_6) \in \mathbb{P}^5$, $1 \leq i < j \leq 5$, $i \neq j$, and $k < l < m$, with $\{i,j,k,l,m\} = \{1,2,3,4,5\}$. Suppose that

$$\prod_{n=1}^{6} a_n \neq 0,\ a_i = a_j, \quad \text{and} \quad \sqrt{a}_k \pm \sqrt{a}_l \pm \sqrt{a}_m \pm \sqrt{a}_6 \neq 0. \tag{25}$$

Let H_{ij} denote the hyperplane given in T by $X_i + X_j = 0$. We define

$$E^{ij}_{\boldsymbol{a}} := \overline{(X_{\boldsymbol{a}} \cap H_{ij} \cap T)} \subset X_{\boldsymbol{a}},$$

and let $\widetilde{E}^{ij}_{\boldsymbol{a}}$ be the strict transform of E^{ij} in $\widetilde{\overline{X}}_{\boldsymbol{a}}$.

Substituting $X_i = -X_j$ in Equation (10) for $X_{\boldsymbol{a}} \cap T$, gives

$$(X_l + X_m + X_n)\left(\frac{a_l}{X_l} + \frac{a_m}{X_m} + \frac{a_n}{X_n}\right) = a_6.$$

This is the equation for $\mathcal{E}_{a_l,a_m,a_n,a_6}$ (see (20)), and so $E^{ij}_{\boldsymbol{a}}$ is birational to $\mathcal{E}_{a_l,a_m,a_n,a_6} \times \mathbb{P}^1$.

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Remark 5.2. For $\boldsymbol{a}$ satisfying condition (25), in terms of the fibre product structure, $X_{\boldsymbol{a}} \cong \mathcal{E}_{a_i,a_i,a_6} \times_{\mathbb{P}^1} \mathcal{E}_{a_k,a_l,a_m}$, $E^{ij}_{\boldsymbol{a}}$ corresponds to the component $L_{a_6} \times \mathcal{E}_{a_k,a_l,a_m,a_6}$ of the fibre over a_6, where L_{a_6} is a component of the I_2 fibre $\mathcal{E}_{a_i,a_i,a_6,a_6}$ of the family $\mathcal{E}_{a_i,a_i,a_6}$.

Remark 5.3. With $\boldsymbol{a}$ as above, $\widetilde{E}^{ij}_{\boldsymbol{a}}$ is isomorphic to $\mathcal{E}_{a_k,a_l,a_m,a_6} \times \mathbb{P}^1$ blown up in the 6 points

$$\begin{array}{lll} (1:0:0)\times(1:0), & (0:1:0)\times(1:0), & (0:0:1)\times(1:0), \\ (1:-1:0)\times(0:1), & (0:1:-1)\times(0:1), & (1:0:-1)\times(0:1). \end{array}$$

LEMMA 5.4. *Let a_i, a_j, a_k, a_l, a_m be as in Definition* 5.1. *Then $E^{ij}_{\boldsymbol{a}}$ is smooth and contains no singularities of $X_{\boldsymbol{a}} \cap T$.*

Proof. Smoothness follows from considering all possible local equations, and the fact that $\sqrt{a}_k \pm \sqrt{a}_l \pm \sqrt{a}_m \pm \sqrt{a}_6 \neq 0$ for all possible sign choices, so $\mathcal{E}_{a_k,a_l,a_m,a_6}$ is smooth.

By Lemma 3.7, a singularity on $X_{\boldsymbol{a}}$ has the form $\boldsymbol{b} = (b_1 : \cdots : b_5)$ with $b_i^2 = a_i$, and $(\sum b_i)^2 = a_6$. But if $\boldsymbol{b} \in E^{ij}_{\boldsymbol{a}}$, then $b_i + b_j = 0$, which implies $b_k + b_l + b_m \pm \sqrt{a_6} = 0$. However this contradicts condition (25). ∎

DEFINITION 5.5. With $\boldsymbol{a}, i, j, k, l, m$ as in Definition 5.1, and $\mathcal{E}_{a_k,a_l,a_m,a_6}$ as in Definition 4.1, let ϕ_{ij} be the birational map defined where all coordinates are nonzero by

$$\begin{aligned} \phi_{ij} : \mathcal{E}_{a_k,a_l,a_m,a_6} \times \mathbb{P}^1 &\dashrightarrow X_{\boldsymbol{a}}, \\ (x:y:z)\times(r:s) &\longmapsto (X_1 : X_2 : X_3 : X_4 : X_5), \\ \text{with } (X_i : X_j : X_l : X_m : X_n) &= (rz : -rz : sx : sy : sz). \end{aligned}$$

We want to consider the subspace of $H_3(\widetilde{\widetilde{X}}_{\boldsymbol{a}}, \mathbb{Z})$, spanned by the images of the induced maps

$$\phi^*_{ij} : H_1(\mathcal{E}_{a_k,a_l,a_m,a_6}, \mathbb{Z}) \times H_2(\mathbb{P}^1, \mathbb{Z}) \longrightarrow H_3(\widetilde{\widetilde{X}}_{\boldsymbol{a}}, \mathbb{Z}).$$

DEFINITION 5.6. For $\boldsymbol{c} = (c_1, c_2, c_3, c_4)$, $c_i \neq 0$, $\sum \pm c_i \neq 0$, fix $\alpha_{\boldsymbol{c}}$, $\beta_{\boldsymbol{c}}$ to be 1-cycles, with classes $[\alpha_{\boldsymbol{c}}]$, $[\beta_{\boldsymbol{c}}]$ spanning $H_1(\mathcal{E}_{\boldsymbol{c}}, \mathbb{Z})$, with $\alpha_{\boldsymbol{c}}.\beta_{\boldsymbol{c}} = 1$, and with

(26)
$$(1:0:0), (0:1:0), (0:0:1), (1:-1:0), (0:1:-1), (1:0:-1) \notin \alpha_{\boldsymbol{c}}, \beta_{\boldsymbol{c}}.$$

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Definition 5.7. For $\boldsymbol{a}$ satisfying condition (25), and $\boldsymbol{c} = (a_k, a_l, a_m, a_6)$, define 3-cycles on $X_{\boldsymbol{a}}$ by

$$\alpha^{ij} := \phi_{ij}(\alpha_{\boldsymbol{c}} \times \mathbb{P}^1), \quad \beta^{ij} := \phi_{ij}(\beta_{\boldsymbol{c}} \times \mathbb{P}^1).$$

Lemma 5.8. *For $\boldsymbol{a}$ satisfying condition* (25), *we have*

$$A := \begin{pmatrix} (\alpha^{ij})^2 & \alpha^{ij}.\beta^{ij} \\ \beta^{ij}.\alpha^{ij} & (\beta^{ij})^2 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}.$$

Proof. We apply a general result in intersection theory (see [F, 19.2.2]). Namely, for a closed embedding $i : Y \hookrightarrow X$ of compact oriented manifolds, and $\alpha \in H_r(Y)$, $\beta \in H_s(Y)$, we have

$$i_*(\alpha) \cap i_*(\beta) = i_*(Y|_Y \cap \alpha \cap \beta).$$

We take $Y = \widetilde{E}^{ij}_{\boldsymbol{a}} \sim_{\mathrm{bir}} \mathcal{E} \times \mathbb{P}^1$, where $\mathcal{E} = \mathcal{E}_{a_k, a_l, a_m, a_n}$ for appropriate indices, and $X = \widetilde{\widetilde{X}}_{\boldsymbol{a}}$. We have

$$Y|_Y = (K_X + Y)|_Y = K_Y = -2(\mathcal{E} \times \{\text{point}\}) + \sum E_j, \tag{27}$$

where E_j are the exceptional divisors in the blow up $\widetilde{E}^{ij}_{\boldsymbol{a}} \to \mathcal{E} \times \mathbb{P}^1$. The first equality of (27) follows from the fact that $K_X = \sum Q_i$, where the Q_i are quadrics coming from blowing up nodes in $X_{\boldsymbol{a}} \cap T$; by Lemma 5.4 their restriction to $\widetilde{E}^{ij}_{\boldsymbol{a}}$ is trivial. The second equality comes from the adjunction formula. The third equality comes from the fact that the canonical bundle of the ruled surface $\mathcal{E} \times \mathbb{P}^1$ has degree -2 on the general fibre. Since α^{ij}, β^{ij} are chosen to avoid the E_j (Remark 5.3 and Definition 5.6), we have

$$\alpha^{ij}.\beta^{ij} = \alpha^{ij} \cap \beta^{ij} \cap Y|_Y = -2(\{\text{point}\} \times \mathbb{P}^1) \cap (\mathcal{E} \times \{\text{point}\}) = -2.$$

The claim $(\alpha^{ij})^2 = (\beta^{ij})^2 = 0$ can be seen directly from the geometry. ∎

Lemma 5.9. *For $\boldsymbol{a} = (a_1 : \cdots : a_6)$ with $a_i = a_j$, $a_k = a_l$, and $\pm a_m \pm a_6 \notin \{0, 2\sqrt{a}_i, 2\sqrt{a}_k\}$, where $\{i,j,k,l,m\} = \{1,2,3,4,5\}$, we have*

$$\begin{pmatrix} \alpha^{ij}.\alpha^{kl} & \alpha^{ij}.\beta^{kl} \\ \beta^{ij}.\alpha^{kl} & \beta^{ij}.\beta^{kl} \end{pmatrix} = 0.$$

Proof. First note that substituting $X_i + X_j = X_k + X_l = 0$ in (10) gives $a_m = a_6$, but we have assumed $\sqrt{a}_m \pm \sqrt{a}_6 \neq 0$, so $H_{ij} \cap H_{lm} \cap X_{\boldsymbol{a}} \cap T = \varnothing$. Local considerations show that having picked $\alpha_{\boldsymbol{c}}$ and $\beta_{\boldsymbol{c}}$ to avoid certain points means that these cycles can also not intersect in $X_{\boldsymbol{a}} \setminus T$. ∎

Lemma 5.10. *For* $\boldsymbol{a} = (a_1 : \cdots : a_6)$ *with* $a_i = a_j = a_k$ *and* $\pm\sqrt{a_l} \pm \sqrt{a_m} \pm \sqrt{a_6} \notin \{\sqrt{a}_i, 3\sqrt{a}_i\}$, $i < j < k$ *and* $\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}$, *we have*

$$\begin{pmatrix} \alpha^{ij}.\alpha^{ik} & \alpha^{ij}.\beta^{ik} \\ \beta^{ij}.\alpha^{ik} & \beta^{ij}.\beta^{ik} \end{pmatrix} = \mathrm{sg}(\phi_{ij}^{-1} \circ \phi_{ik})B, \quad \textit{where } B := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and where sg *is the sign of* $\phi_{ij}^{-1} \circ \phi_{ik}$ *as a permutation of coordinates of* $\mathbb{P}^2$.

Proof. By local considerations, one can show that the cycles do not meet in $X_{\boldsymbol{a}} \setminus T$. In T, the surfaces $E_{\boldsymbol{a}}^{ij}$ and $E_{\boldsymbol{a}}^{ik}$ meet in an elliptic curve, with α^{ij} and β^{ij} restricting to the images of α and β on this curve, so up to sign the intersection matrix is B as above.

The sign is determined by whether or not the map $\phi_{km}^{-1} \circ \phi_{ij}$ preserves the orientation of the chosen cycles. Suppose $i, j, k = 1, 2, 4$. We have maps

$$\phi_{12} : (x : y : z) \times (r : s) \longmapsto (rz : -rz : sx : sy : sz),$$
$$\phi_{14} : (x : y : z) \times (r : s) \longmapsto (rz : sx : sy : -rz : sz).$$

Since $E_{\boldsymbol{a}}^{12} \cap E_{\boldsymbol{a}}^{14}$ is given by setting $X_2 = X_4$, the map $\phi_{12}^{-1} \circ \phi_{14}$ is given by

$$(x : y : z) \longmapsto (y : x : z).$$

This is an odd permutation of the coordinates, and so $\alpha_{14} \cap E_{12}$ and $\beta_{14} \cap E_{12}$ have the opposite orientation to $\alpha_{12} \cap E_{14}$ and $\beta_{12} \cap E_{14}$, and so the intersection matrix is $-B$. Similar considerations hold in general. ∎

Corollary 5.11. (i) *If* $\boldsymbol{a} = (1 : 1 : 1 : 1 : 1 : t)$, *for* $t \neq 0, 1, 9$, *then the intersection matrix of* $\alpha_{\boldsymbol{a}}^{ij}$, $\beta_{\boldsymbol{a}}^{ij}$ *for* $1 \leq i < j \leq 5$ *is given by the block matrix*

	E^{12}	E^{13}	E^{14}	E^{15}	E^{23}	E^{24}	E^{25}	E^{34}	E^{35}	E^{45}
E^{12}	A	$+B$	$-B$	$+B$	$+B$	$-B$	$+B$	0	0	0
E^{13}	$+B$	A	$+B$	$-B$	$+B$	0	0	$-B$	$+B$	0
E^{14}	$-B$	$+B$	A	$+B$	0	$+B$	0	$-B$	0	$+B$
E^{15}	$+B$	$-B$	$+B$	A	0	0	$+B$	0	$-B$	$+B$
E^{23}	$+B$	$+B$	0	0	A	$+B$	$-B$	$+B$	$-B$	0
E^{24}	$-B$	0	$+B$	0	$+B$	A	$+B$	$+B$	0	$-B$
E^{25}	$+B$	0	0	$+B$	$-B$	$+B$	A	0	$+B$	$-B$
E^{34}	0	$-B$	$-B$	0	$+B$	$+B$	0	A	$+B$	$+B$
E^{35}	0	$+B$	0	$-B$	$-B$	0	$+B$	$+B$	A	$+B$
E^{45}	0	0	$+B$	$+B$	0	$-B$	$-B$	$+B$	$+B$	A

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This matrix has rank 8.

(ii) *If* $\boldsymbol{a} = (1:1:1:t:t:t)$, *for* $t \neq 0, 1$, *then the intersection matrix of the* 6 *three cycles* $\alpha_{\boldsymbol{a}}^{ij}$, $\beta_{\boldsymbol{a}}^{ij}$ *for* $1 \leq i < j \leq 3$ *is given by*

$$\begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix}.$$

This matrix has rank 4.

Proof. This follows from Lemmas 5.8, 5.9 and 5.10. The conditions on t in (i) and (ii) ensure that $1 \pm 1 \pm 1 \pm \sqrt{t}$ and $1 \pm \sqrt{t} \pm \sqrt{t} \pm \sqrt{t} \neq 0$, so that the elliptic curves $E_{\boldsymbol{a}}^{ij}$ are nonsingular. ∎

We now look at how much of $H^3(\widetilde{\overline{X}}_{\boldsymbol{a}})$ comes from the elliptic surfaces $E_{\boldsymbol{a}}^{ij}$. The maps ϕ_{ij}, defined when $a_i = a_j$, and when the roots of the remaining coefficients can not sum to zero, gives us a homomorphism

$$H^3_{\text{ét}}(\widetilde{\overline{X}}_{\boldsymbol{a}}, \mathbb{Q}_\ell) \longrightarrow \bigoplus_{\substack{i,\, j \text{ with } a_i = a_j \text{ and} \\ \sum_{n \neq i,j} \pm a_n \neq 0}} H^1(E_{\boldsymbol{a}}^{ij}, \mathbb{Q}_\ell) \times H^2(\mathbb{P}^1, \mathbb{Q}_\ell).$$

Let $W_{\boldsymbol{a}}$ be the image of this map. The dimension of $W_{\boldsymbol{a}}$ can be determined by computing the dimension of the corresponding intersection matrix, as in the examples in Corollary 5.11. Let $V_{\boldsymbol{a}}$ be the kernel of this map. We have a sequence

$$0 \longrightarrow V_{\boldsymbol{a}} \longrightarrow H^3_{\text{ét}}(\widetilde{\overline{X}}_{\boldsymbol{a}}, \mathbb{Q}_\ell) \longrightarrow W_{\boldsymbol{a}} \longrightarrow 0 \tag{28}$$

of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ representations. (Strictly speaking the $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ representation lives on the dual spaces but by abuse of language we shall still refer to the cohomology groups as $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules.) By basic linear algebra

$$\operatorname{trace} \text{Frob}_p | H^3_{\text{ét}}(\widetilde{\overline{X}}_{\boldsymbol{a}}, \mathbb{Q}_\ell) = \operatorname{trace} \text{Frob}_p | V_{\boldsymbol{a}} + \operatorname{trace} \text{Frob}_p | W_{\boldsymbol{a}}. \tag{29}$$

Since all elliptic curves over $\mathbb{Q}$ are modular, if $\boldsymbol{a} \in \mathbb{P}^5(\mathbb{Q})$, the Galois representation on $W_{\boldsymbol{a}}$ is given in terms of a weight 2 modular form, with level given by the conductor of the curve. This modular form can be determined by counting points. The values $\operatorname{trace} \text{Frob}_p | H^3_{\text{ét}}(\widetilde{\overline{X}}_{\boldsymbol{a}}, \mathbb{Q}_\ell)$ can also be determined by counting points, and so by subtraction we obtain the traces of the representation $V_{\boldsymbol{a}}$. We will be most interested in cases where $V_{\boldsymbol{a}}$ is 2-dimensional. These are the rigid examples of Corollary 4.9, and the following nonrigid cases.

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Corollary 5.12. *For $\boldsymbol{a}$ given by one of the following,*

$\boldsymbol{a}$	$\dim \mathcal{F}_i$
$(1:1:1:1:1:25) \in \mathcal{F}_1$	4
$(1:1:1:9:9:9) \in \mathcal{F}_4$	2
$(1:1:4:4:4:16) \in \mathcal{F}_8$	1

The semi-simplification of the Galois representation on $H^3_{ét}(\widetilde{\overline{X}}_{\boldsymbol{a}}, \mathbb{Q}_\ell)$ splits into a sum of Galois representations corresponding to elliptic curves $W_{\boldsymbol{a}}$, and a 2-dimensional Galois-representation, $V_{\boldsymbol{a}}$.

Proof. The dimension of the pieces coming from elliptic surfaces $E^{ij}_{\boldsymbol{a}}$ for $1 \leq i \leq j \leq M$, with $M = 5$, 3 and 2 respectively, is given by Corollary 5.11, parts (i), (ii), and Lemma 5.8 respectively. These values of $\boldsymbol{a}$ are in the indicated families $\mathcal{F}_i$, defined in Table 1, and in no smaller families. In each case we have that $\dim W_{\boldsymbol{a}} = h^{12}(\widetilde{\overline{X}}_{\boldsymbol{a}}) = \dim \mathcal{F}_i$, (see Proposition 4.8) and so Galois representation $V_{\boldsymbol{a}}$, which is the kernel, must have dimension 2. ∎

Remark 5.13. In the next section we will compute the Galois representations of the $V_{\boldsymbol{a}}$ in the above corollary, and show that they correspond to weight 4 modular forms.

§6. Computing the L-series

We will use the Lefschetz fixed point theorem to compute coefficients of the L-series of $H^3_{\text{ét}}(\widetilde{\overline{X}}_{\boldsymbol{a}})$. This says that for a variety Z defined over $\mathbb{Q}$, the number of rational points of Z over $\mathbb{F}_p$ is given by

$$\#Z(\mathbb{F}_p) = \sum (-1)^j \left(\mathrm{Frob}^*_p|_{H^j_{\text{ét}}(Z)}\right). \tag{30}$$

The spaces $H^0_{\text{ét}}(\widetilde{\overline{X}}_{\boldsymbol{a}})$ and $H^6_{\text{ét}}(\widetilde{\overline{X}}_{\boldsymbol{a}})$ are 1 dimensional, and Frob_p acts trivially and by multiplication by p^3 respectively. The following result gives some information about the Galois action on $H^2_{\text{ét}}(\widetilde{\overline{X}}_{\boldsymbol{a}})$, and (by duality) on $H^4_{\text{ét}}(\widetilde{\overline{X}}_{\boldsymbol{a}})$.

Proposition 6.1. *For a prime p of good reduction for $\widetilde{\overline{X}}_{\boldsymbol{a}}$, all eigenvalues of the Frobenius action of* Frob_p *on $H^2_{ét}(\widetilde{X}_{\boldsymbol{a}})$ are equal to p, provided the rulings of the quadrics Q_i which are obtained by blowing up the 30 singularities are defined over the field $\mathbb{F}_p$.*

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Proof. We claim that $H^2_{\text{ét}}(\widetilde{P}) \cong H^2_{\text{ét}}(X_{\boldsymbol{a}})$, where $\widetilde{P}$ is the ambient toric variety. This suffices since $\widetilde{P}$ is a smooth toric variety and hence $H^2(\widetilde{P}, \mathbb{Z})$ is spanned by divisors defined over $\mathbb{Z}$. In order to prove that the restriction $H^2(\widetilde{P}, \mathbb{Q}) \to H^2(X_{\boldsymbol{a}}, \mathbb{Q})$ is an isomorphism we proceed as follows. We first observe that as in the proof of [C, (1.28)] one has an isomorphism $H^2(X_{\boldsymbol{a}}, \mathbb{Q}) \cong H^2(X_u, \mathbb{Q})$ where X_u is a general (smooth) Calabi-Yau in $|-K_{\widetilde{P}}|$. Using this isomorphism it is enough to show that $H^2(\widetilde{P}, \mathbb{Q}) \to H^2(X_u, \mathbb{Q})$ is an isomorphism. Both vector spaces have dimension 26. In the case of X_u this was shown in Proposition 3.1, and for $\widetilde{P}$ this is standard toric geometry. (Viewing $\widetilde{P}$ as a repeated blow up of $\mathbb{P}^4$ one sees that the Picard group of $\widetilde{P}$ is spanned by the pullback of the hyperplane sections and the $5 + 10 + 10 = 25$ exceptional divisors. Note also that there are 5 T-invariant hyperplanes in $\widetilde{P}$ which are, of course, linearly equivalent). It is shown in the proof of [Ba, Theorem 4.42] that the Picard group of X_u is spanned by components of divisors of the form $Y = H \cap X_u$ where H is a T-invariant divisor on $\widetilde{P}$. In our case there are no proper faces of Δ which have interior points. Again from the proof of [Ba, Theorem 4.42, p. 520] one can conclude that Y is always irreducible and hence $H^2(\widetilde{P}, \mathbb{Q}) \to H^2(X_u, \mathbb{Q})$ is an epimorphism. Since both vector spaces have the same dimension this is indeed an isomorphism.

In order to go from $X_{\boldsymbol{a}}$ to $\widetilde{X}_{\boldsymbol{a}}$ we consider the Leray spectral sequence

$$0 \longrightarrow H^2_{\text{ét}}(X_{\boldsymbol{a}}) \longrightarrow H^2_{\text{ét}}(\widetilde{X}_{\boldsymbol{a}}) \longrightarrow \bigoplus_{i=1}^{s} H^2_{\text{ét}}(Q_i).$$

Hence it is enough to check that the rulings of the quadrics Q_i are defined over $\mathbb{F}_p$. ∎

We shall see later that to compute values of Frob_p acting on $H^3_{\text{ét}}(\widetilde{\overline{X}}_{\boldsymbol{a}})$, it will be enough to count points on $X_{\boldsymbol{a}}$ over finite fields.

6.1. Counting points on $\widetilde{X}_{\boldsymbol{a}}$

In this section we give a formula for counting points on $\widetilde{\overline{X}}_{\boldsymbol{a}}$ over finite fields. First we determine the primes of bad reduction for $X_{\boldsymbol{a}}$, since we will only count points on $X_{\boldsymbol{a}}$ at primes of good reduction.

LEMMA 6.2. *Let* $\boldsymbol{a} = (a_1 : \cdots : a_6) \in \mathbb{P}^5(\mathbb{Z})$, *with* $\prod_{i=1}^5 a_i \neq 0$, *and let* $F(\boldsymbol{a})$ *be the degree* 16 *polynomial in* $\mathbb{Z}[a_1, \ldots, a_6]$ *given by*

$$F(\boldsymbol{a}) = \prod_{(\epsilon_1, \ldots, \epsilon_5) \in \{\pm 1\}^5} \left(\sum_{i=1}^{5} \epsilon_i \sqrt{a}_i + \sqrt{a}_6 \right). \tag{31}$$

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If $\boldsymbol{a} \notin \phi(\mathbb{P}^4)$, then $\overline{X}_{\boldsymbol{a}} \otimes \mathbb{F}_p$ is smooth over $\overline{\mathbb{F}}_p$ for $p \nmid a_1a_2a_3a_4a_5a_6F(\boldsymbol{a})$. Furthermore,

$$\begin{array}{ll} \widetilde{X}_1 \otimes \mathbb{F}_p,\ \widetilde{X}_9 \otimes \mathbb{F}_p \text{ and } \widetilde{X}_{(1:1:1:9:9:9)} \otimes \mathbb{F}_p & \text{are smooth if } p \neq 2,3, \\ \widetilde{X}_{25} \otimes \mathbb{F}_p,\ \widetilde{X}_{(1:1:1:1:4:4)} \otimes \mathbb{F}_p \text{ and } \widetilde{X}_{(1:1:4:4:4:16)} \otimes \mathbb{F}_p & \\ & \text{are smooth if } p \neq 2,3,5, \\ \widetilde{X}_{(1:1:1:4:4:9)} \otimes \mathbb{F}_p & \text{is smooth if } p \neq 2,3,5,7. \end{array}$$

Proof. Lemmas 3.10 and 3.7 describe the singularities of $X_{\boldsymbol{a}}$ over any field $\mathbb{F}_p$ with $p \nmid a_i$. The resolutions we have described over $\overline{\mathbb{Q}}$ remain resolutions over $\overline{\mathbb{F}}_p$. Thus if $\boldsymbol{a} \notin \phi(\mathbb{P}^4)$, $\overline{X}_{\boldsymbol{a}} \otimes \mathbb{F}_p$ is smooth over $\overline{\mathbb{F}}_p$ unless $\boldsymbol{a} \equiv \phi(\boldsymbol{b}) \mod p$ for some $\boldsymbol{b}$. This is the case only if $F(\boldsymbol{a}) \equiv 0 \mod p$.

If $\boldsymbol{a} = \phi(\boldsymbol{b})$, the primes of bad reduction are the prime factors of a_i, and of the nonzero factors of $F(\boldsymbol{a})$. E.g., if all $a_i = 1$, then $|\sum \pm\sqrt{a_i}| = 0$, 2, 4 or 6, so the primes of bad reduction are 2, 3. Other examples are computed similarly. ∎

We now count points on $X_{\boldsymbol{a}}$ by considering points on $X_{\boldsymbol{a}} \cap T$ and on $X_{\boldsymbol{a}} \setminus T$, and points added in the resolution of singularities, separately.

Lemma 6.3. *If $\boldsymbol{a} = (a_1 : a_2 : a_3 : a_4 : a_5 : a_6) \in \mathbb{P}^5(\mathbb{F}_p^\times)$, then*

$$\begin{aligned} \#(X_{\boldsymbol{a}} \cap T)(\mathbb{F}_p) &= \sum_{y,z,w=1}^{p-1} \left(\left(\frac{\left((1+x+y+z)(\frac{a_2}{x}+\frac{a_3}{y}+\frac{a_4}{z}+a_5)-a_1-a_6\right)^2-4a_1a_6}{p} \right) + 1 \right) \\ &\quad - 2(p^2-3p+3) + \rho(a_1,a_6)\left(\#\mathcal{E}_{a_2,a_3,a_5,a_4}(\mathbb{F}_p) - 6\right), \end{aligned} \tag{32}$$

where $\rho(a_1,a_6) = p$ if $a_1 \equiv a_6 \mod p$, and 0 otherwise, $\left(\frac{x}{p}\right)$ is the Kronecker symbol, and $\mathcal{E}_{a_2,a_3,a_5,a_4}(\mathbb{F}_p)$ is the elliptic curve given by (20).

Proof. We must compute the number of solutions to (10) with all $X_i \neq 0$. Setting $X_5 = 1$, $A = X_2 + X_3 + X_4 + 1$ and $B = a_2/X_2 + a_3/X_3 + a_4/X_4 + a_5$, and multiplying through by X_1, (10) becomes

$$BX_1^2 + (AB + a_1 - a_6)X_1 + a_1A = 0. \tag{33}$$

For fixed X_2, X_3, X_4, this has $\left(\frac{d}{p}\right) + 1$ solutions, where $d = (AB - a_1 - a_6) - 4a_1a_6$ is the discriminant of (33). This gives the term which is the sum in (32). However,

(i) this sum counts solutions to (33) where $X_1 = 0$,

(ii) if $A = B = 0$, $a_1 = a_6$, there are $p-1$ solutions, but the sum counts 1.

(iii) If $B = 0$, $a_1 = a_6$, $A \neq 0$, there are 0 solutions, but the sum counts 1.

If $a_1 = a_6$, then (i) occurs exactly when $A = 0$ and $B \neq 0$. We must add

$$- \#\{(X_2, X_3, X_4) \in (\mathbb{F}_p^{\times})^3 \mid A = 0\} \tag{34}$$

$$+ \#\{(X_2, X_3, X_4) \in (\mathbb{F}_p^{\times})^3 \mid A = B = 0\}p \tag{35}$$

$$- \#\{(X_2, X_3, X_4) \in (\mathbb{F}_p^{\times})^3 \mid B = 0\} \tag{36}$$

to the sum in (32). Since $A = 0$, $X_2X_3X_4 \neq 0$ is a plane with 3 lines removed, the set in line (34) has $p^2 - 3(p-1)$ points. The set in line (36) similarly has $p^2 - 3(p-1)$ points. The equations $A = B = 0$ for the set in line (35) can be rearranged to give the the equation for $\mathcal{E}_{a_2,a_3,a_5,a_4}$, with all coordinates nonzero. Thus in the case $a_1 = a_6$ we obtain (32).

The case $a_1 \neq a_6$ is similar. ∎

Lemma 6.4. *We have*

$$\#\big(X_{\boldsymbol{a}} \setminus T\big)(\mathbb{F}_p) = 50p^2 + 10p + 20. \tag{37}$$

Proof. The components of the decomposition $X_{\boldsymbol{a}} = \bigsqcup_{\sigma \in \widetilde{\Sigma}} (X_{\boldsymbol{a}} \cap T_\sigma)$ are listed in Table 3, and illustrated in Figure 7. If $\dim T_\sigma < 4$, then $\widetilde{X}_{\boldsymbol{a}} \cap T_\sigma$ is rational, and so the number of points $\#(\widetilde{X}_{\boldsymbol{a}} \cap T_\sigma)(\mathbb{F}_p)$, can easily be computed, and is given in Table 4. From Tables 3 and 4 we have

$$\begin{aligned}\#\big(X_{\boldsymbol{a}} \setminus T\big)(\mathbb{F}_p) &= 10(p^2 - 3p + 3) + 20(2p^2 - 6p + 5) \\ &\quad + 40(p-2) + 60(p-1) + 30(2p-3) + 120,\end{aligned}$$

which sums to give the required result. ∎

Lemma 6.5. *For $\boldsymbol{a} = (a_1 : \cdots : a_6) \in \mathbb{P}^5(\mathbb{Q})$, if the big resolution $\widetilde{\overline{X}}_{\boldsymbol{a}}$ of $\overline{X}_{\boldsymbol{a}}$ has smooth reduction mod p, then*

$$\begin{aligned}\#\widetilde{\overline{X}}_{\boldsymbol{a}}(\mathbb{F}_p) &= 48p^2 + 46p + 14 \\ &\quad + \sum_{x,y,z=1}^{p-1} \left(\left(\frac{\big((1+x+y+z)(\frac{a_2}{x}+\frac{a_3}{y}+\frac{a_4}{z}+a_5)-a_1-a_6\big)^2 - 4a_1a_6}{p} \right) + 1 \right) \\ &\quad + \sum_{\boldsymbol{b}=(b_1:\cdots:b_5)\in\mathbb{P}^4,\ \phi(\boldsymbol{b})=\boldsymbol{a}} p\left(p + 1 + \left(\frac{b_1b_2b_3b_4b_5(\sum b_i)}{p} \right) \right) \\ &\quad + \rho(a_1, a_6)\big(\#\mathcal{E}_{a_2,a_3,a_5,a_4}(\mathbb{F}_p) - 6\big),\end{aligned}$$

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$$
\begin{array}{rl}
\tilde{\tilde{X}}_{\boldsymbol{a}} = & \text{The open threefold } \tilde{X}_{\boldsymbol{a}} \cap T, \\
+ & 10 \text{ translates of the surface } \tilde{X}_{\boldsymbol{a}} \cap (x=0), \\
+ & 20 \text{ translates of the surface } \tilde{X}_{\boldsymbol{a}} \cap (y=0), \\
+ & 40 \text{ translates of the curve } \tilde{X}_{\boldsymbol{a}} \cap (x=y=0), \\
+ & 60 \text{ translates of the curve } \tilde{X}_{\boldsymbol{a}} \cap (x=z=0), \\
+ & 30 \text{ translates of the curve } \tilde{X}_{\boldsymbol{a}} \cap (y=z=0), \\
+ & 120 \text{ translates of the point } (x,y,z,w) = (0,0,0,-1), \\
+ & 30\ \mathbb{P}^1\text{s obtained in resolving the singularities in } X_{\boldsymbol{a}} \setminus T, \\
+ & \text{the } Q_{\boldsymbol{b}} \text{ obtained in the resolution of singularities in } X_{\boldsymbol{a}} \cap T.
\end{array}
$$

Table 3: Decomposition of $\tilde{\tilde{X}}_{\boldsymbol{a}}$. Translates mean images under the S_5 action.

defining equations of $X_{\boldsymbol{a}} \cap T_\sigma$			$\#(X_{\boldsymbol{a}} \cap T_\sigma)(\mathbb{F}_p)$
$x=0$	$yzw \neq 0$	$a_1(1+w+wz+wzy)=0$	p^2-3p+3
$y=0$	$xzw \neq 0$	$(a_1+a_2x)(1+w+wz)=0$	$2p^2-6p+5$
$x=y=0$	$zw \neq 0$	$a_1(1+w+wz)=0$	$p-2$
$x=z=0$	$yw \neq 0$	$a_1(1+w)=0$	$p-1$
$y=z=0$	$xw \neq 0$	$(a_1+a_2x)(1+w)=0$	$2p-3$

Table 4: Number of points on $X_{\boldsymbol{a}} \cap T_\sigma$.

where $\rho(a_1,a_6) = p$ *if* $a_1 \equiv a_6 \mod p$, *and* 0 *otherwise*, $\left(\frac{x}{p}\right)$ *is the Kronecker symbol, and* $\mathcal{E}_{a_2,a_3,a_5,a_4}(\mathbb{F}_p)$ *is the elliptic curve given by* (20).

Proof. Table 3 lists the components of $\tilde{\tilde{X}}_{\boldsymbol{a}}$. By Proposition 3.10, if $\boldsymbol{a} \in \mathbb{P}^5(\mathbb{Z})$, all 30 nodes on $X_{\boldsymbol{a}} \setminus T$ are defined over $\mathbb{Q}$, and the $\mathbb{P}^1$s added in the small resolution contribute $30p$ to the sum. If $\boldsymbol{a} = \phi(\boldsymbol{b})$, Corollary 3.21 gives the number of points on the quadric $Q_{\boldsymbol{b}}$ introduced when $\boldsymbol{b}$ is blown up. Thus the number of points added in resolving the singularities of $X_{\boldsymbol{a}}$ is

$$
30p + \sum_{\boldsymbol{b}=(b_1:\cdots:b_5)\in\mathbb{P}^4,\ \phi(\boldsymbol{b})=\boldsymbol{a}} \left(p+1+\left(\frac{b_1b_2b_3b_4b_5\sum_{i=1}^{5} b_i}{p}\right)\right)p. \tag{38}
$$

Adding up (32), (37) and (38) gives the result. ∎

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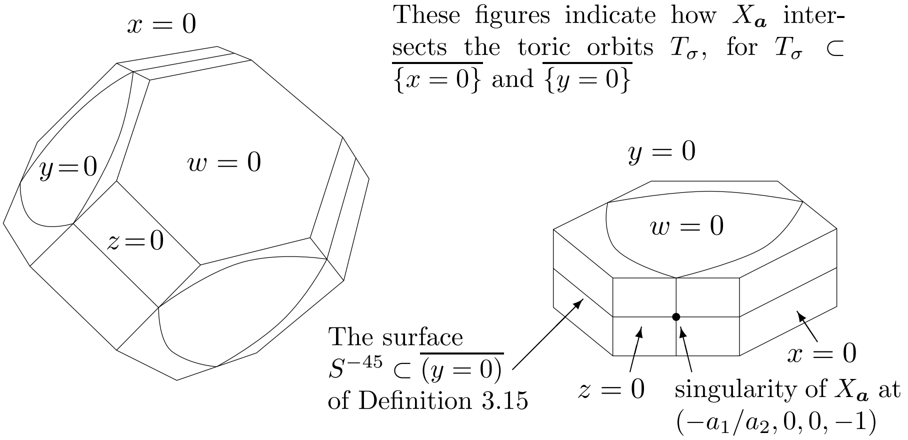

These figures indicate how $\widetilde{X}_{\boldsymbol{a}}$ intersects the toric orbits T_σ, for $T_\sigma \subset \overline{\{x = 0\}}$ and $\overline{\{y = 0\}}$

Figure 7: How $\widetilde{X}_{\boldsymbol{a}}$ intersects $P \setminus T$.

6.2. Applying Livné's method

We now want to prove that the 2 dimensional Galois representations $V_{\boldsymbol{a}}$, for $\boldsymbol{a} = (1:1:1:1:1:1)$, $(1:1:1:1:1:9)$, $(1:1:1:1:1:25)$, $(1:1:1:1:4:4)$, $(1:1:1:4:4:9)$, $(1:1:4:4:4:16)$ and $(1:1:1:9:9:9)$ (see Corollaries 4.9, and 5.12) are modular, by comparing explicit computation and comparison of the coefficients of these L-series with coefficients of certain modular forms. In order to do this, we apply Falting's method, as given by Serre and Livné [Li, Theorem 4.3]. A simplified form is as follows:

Theorem 6.6. (Faltings-Serre-Livné) *Let ρ_1 and ρ_2 be two 2-adic 2-dimensional Galois representations, unramified outside a set of primes S. Let K_S be the smallest field containing all quadratic extensions of $\mathbb{Q}$ ramified at primes in S, and let T be a set of primes disjoint from S. Then if*

(L.1) $\operatorname{trace}\rho_1 \equiv \operatorname{trace}\rho_2 \equiv 0$ *and* $\det\rho_1 \equiv \det\rho_2$,

(L.2) $\{\mathrm{Frob}_p|_{K_s} : p \in T\}$ *is "non-cubic" in* $\mathrm{Gal}(K_s/K)$; *in particular, it is sufficient for these sets to be equal,*

(L.3) *for all* $p \in T$,

$$(a)\ \operatorname{trace}\rho_1 \mathrm{Frob}_p = \operatorname{trace}\rho_2 \mathrm{Frob}_p,$$
$$(b)\ \det\rho_1 \mathrm{Frob}_p = \det\rho_2 \mathrm{Frob}_p,$$

then ρ_1 and ρ_2 have isomorphic semi-simplifications.

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We want to apply this result to the Galois representation on $H^3_{\text{ét}}(\widetilde{\overline{X}}_{\boldsymbol{a}} \otimes \bar{\mathbb{F}}_p, \mathbb{Q}_\ell)$ and the modular Galois representation corresponding to a cuspidal Hecke eigenform, constructed by Serre and Deligne. Thus we need to do the following:

(L.1a) Check that $\operatorname{trace}(\operatorname{Frob}_p | H^3_{\text{ét}}(\widetilde{\overline{X}}_{\boldsymbol{a}} \otimes \bar{\mathbb{F}}_p, \mathbb{Q}_\ell))$ is always even for primes p of good reduction.

(L.1b) Check that the coefficients a_p of the modular forms in question are all even for primes p not dividing the level.

(L.1c) and (L.3b) Remark that the determinants of both representations are given by χ^3, where χ is the cyclotomic character.

(L.2) Determine a suitable set of primes T_s.

(L.3a) Compute $\operatorname{trace}(\operatorname{Frob}_p | H^3_{\text{ét}}(\widetilde{\overline{X}}_{\boldsymbol{a}} \otimes \bar{\mathbb{F}}_p))$ for all $p \in T_s$, and verify these are equal to the corresponding coefficients of the modular forms.

We now treat each of these points in turn.

(L.1a) We first remark that the trace of Frobenius on $H^2_{\text{ét}}(\widetilde{X}_{\boldsymbol{a}})$ is an integer multiple of p (and hence by duality the trace of Frobenius on $H^4_{\text{ét}}(\widetilde{X}_{\boldsymbol{a}})$ is an integer multiple of p^2). This can be deduced either from the proof of Proposition 6.1 or directly from the analogue of the Riemann hypothesis. Thus the coefficients of the L-series are given by

$$\operatorname{trace}(\operatorname{Frob}_p | H^3_{\text{ét}}(\widetilde{\overline{X}}_{\boldsymbol{a}} \otimes \bar{\mathbb{F}}_p)) = p^3 + p(p+1)h + 1 - \#\widetilde{\overline{X}}_{\boldsymbol{a}}(\mathbb{F}_p) \tag{39}$$

for some integer h.

Lemma 6.7. *The numbers* $\#\widetilde{\overline{X}}_{\boldsymbol{a}}(\mathbb{F}_p)$ *and* $\operatorname{trace}(\operatorname{Frob}_p | H^3_{\acute{e}t}(\widetilde{\overline{X}}_{\boldsymbol{a}} \otimes \bar{\mathbb{F}}_p))$ *are even for primes p of good reduction.*

Proof. Equation (37) in Lemma 6.4 implies that $\#(\widetilde{\overline{X}}_{\boldsymbol{a}} \setminus T)(\mathbb{F}_p)$ is even. On $X_{\boldsymbol{a}} \cap T$ there is an involution $X_i \mapsto a_i/X_i$. The fixed points are exactly the singularities of $X_{\boldsymbol{a}}$ in T. After blowing up, these are replaced by a quadratic Q, with $\#Q(\mathbb{F}_p) = (p+1)^2$ or p^2+1 by Corollary 3.21. In either case this is an even number and so $\widetilde{\overline{X}}_{\boldsymbol{a}}(\mathbb{F}_p)$ is even. Now (39) implies that $\operatorname{trace}(\operatorname{Frob}_p | H^3_{\text{ét}}(\widetilde{\overline{X}}_{\boldsymbol{a}} \otimes \bar{\mathbb{F}}_p))$ is also even. ∎

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(L.1b) The cuspidal Hecke eigen forms we are interested in are f_6, f_{12}, g_{30}, f_{30}, f'_{30}, f_{60}, and f_{90}, with q expansions starting as in (1), (2), (6), (5), (7), (3) and (8). Arbitrarily many coefficients of the q-expansion of these forms may be computed by Stein's MAGMA package [BCP], [St].

LEMMA 6.8. *If f is a cuspidal Hecke eigen form of level N coprime to 2, 3, 5, 7, with q-expansion $f = \sum_{n\geq 1} a_n q^n$, and if a_p is even for primes p with $11 \leq p \leq 37$, then a_p is even for all primes $p \geq 11$.*

Proof. This is proved by the same method as [Li, Proposition 4.10]. Tables of [J], list all C_3 and S_3 extensions of $\mathbb{Q}$ unramified outside 2, 3, 5, 7, and one can easily compute to see that the Frobenius at p acting on any of these extensions has order 3 for at least one prime p with $11 \leq p \leq 37$. □

From this and the computation of enough terms of the q expansions, we immediately obtain the following result.

COROLLARY 6.9. *For a prime $p \neq 2, 3, 5, 7$, the coefficient of q^p in the q-expansion of the modular forms given by f_6, f_{12}, g_{30}, f_{30}, f'_{30}, f_{60}, and f_{90} are all even.*

(L.1c) and (L.3b). This is a well known consequence of Poincaré duality. Let V be a two dimensional piece of H^3 invariant under the Frobenius homomorphism. There is a map $\bigwedge^2 V \to H^6$, which is nonzero by Poincaré duality. Since both spaces are 1-dimensional, this is an isomorphism. It follows from the fact that Frob_p acts by multiplication by p^3 on H^6 that the determinant of the action on H^3 is p^3.

(L.2) By [Li, Proposition 4.11 b], when $\widetilde{\overline{X}}_{\boldsymbol{a}}$ has smooth reduction for $p \neq 2, 3, 5$, we can take

$$T_{\{2,3,5\}} = \{7, 11, 13, 17, 19, 23, 29, 31, 41, 43, 53, 61, 71, 73\}. \tag{40}$$

Note that we can replace 7, 11, 13 here by 103, 59, 37 respectively. This will become important later on when we shall determine the trace of Frobenius on $H^3_{\text{ét}}(\widetilde{\overline{X}}_{\boldsymbol{a}})$. If $\widetilde{\overline{X}}_{\boldsymbol{a}}$ has bad reduction at 2, 3, 5, 7, then we use the following easy lemma.

LEMMA 6.10. *The elements of* $\mathrm{Gal}(\mathbb{Q}[\sqrt{-1}, \sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}]/\mathbb{Q})$ *are given by the identity, together with* Frob_p *for p in the set*

$$T_{\{2,3,5,7\}} := \{11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 71, 73, 79, 83, \\ 101, 103, 107, 109, 113, 127, 173, 193, 211, 241, 281, 283, 311\}.$$

p	$\#X_1$	$\#X_9$	$\#X_{4,4}$	$\#X_{4,4,9}$	$\#X_{25}$	$\#X_{9,9,9}$	$\#X_{4,4,4,16}$
7	3720	3160	3360	3172	3000	3092	3120
11	9240	7920	8424	7956	7464	7680	7848
13	13080	11260	12036	11368	10500	10940	11088
17	23400	20340	21420	20112	18540	19464	19920
19	29640	25840	27480	25840	24720	25352	25416
23	45120	39600	41904	39840	37560	38796	39144
29	76560	67860	71604	67584	65100	66408	66984
31	89400	79480	83376	79528	74664	76760	77880
41	172200	154980	161820	155172	148884	151632	153744
43	193080	174160	181224	174400	167640	170636	172656
53	320400	291780	303012	292392	281580	287112	289512
61	454440	416620	430788	416500	403884	408836	412608
71	663840	612720	634320	613032	592944	603720	609168
73	712920	658900	680700	658684	636180	647660	654048
103	1735320	1628200	1671168	1627156	1586040	1608716	1616976
59	418440	383040	397224	383124	367560	375720	378600
37	134760	120700	126804	121168	114900	118028	119808

Table 5: Number of points on $\widetilde{\overline{X}}_{\boldsymbol{a}}(\mathbb{F}_p)$, where for a sequence $\boldsymbol{b} = b_1, \dots, b_i$ of length $i < 6$, $X_{\boldsymbol{b}}$ means $\widetilde{\overline{X}}_{\boldsymbol{a}}$, where $\boldsymbol{a} = (b_1 : \cdots : b_i : 1 : \cdots : 1)$.

Note that in $T_{\{2,3,5,7\}}$ we may replace 11 and 13 by 179 and 157 respectively.

(L.3a) Using the formula of Lemma 6.5, we obtain the data in Table 5.

From these values we shall be able to compute trace $\mathrm{Frob}_p|_{V_{\boldsymbol{a}}}$, where $V_{\boldsymbol{a}}$ is the two dimensional Galois representation given by $H^3_{\text{ét}}(\widetilde{\overline{X}}_{\boldsymbol{a}} \otimes \bar{\mathbb{F}}_p)$ in the rigid case, or by the subrepresentation given in Corollary 5.12 for the three nonrigid cases.

In the nonrigid cases, the elliptic curves $\mathcal{E}_{1,1,1,25}$, $\mathcal{E}_{1,9,9,9}$ and $\mathcal{E}_{4,4,4,16}$ have j-invariants $11^3 1259^3 2^{-1} 3^{-3} 5^{-4}$, $11^3 13^3 23^3 2^{-1} 3^{-12} 5^{-1}$ and $71^3 2^{-4} \cdot 3^{-3} 5^{-1}$ respectively, but they all have conductor 30. They are isogenous to each other, since there is only one weight 2 level 30 Hecke eigen newform, $g_{30} = \sum b_n q^n$, with b_p for primes $p \in T_{2,3,5}$ (see (40)) as in the following

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p	V_1	V_9	$V_{4,4}$	$V_{4,4,9}$	V_{25}	$V_{9,9,9}$	$V_{4,4,4,16}$
17	-126	-126	18	102	42	-66	-114
19	20	20	-100	20	-76	-100	140
23	168	168	72	-72	0	-132	72
29	30	30	-234	306	6	90	210
31	-88	-88	-16	-136	-232	152	272
41	42	42	90	-150	234	438	-198
43	-52	-52	452	-292	-412	32	-268
53	198	198	414	-414	222	-222	-78
61	-538	-538	422	-418	-490	902	302
71	792	792	-360	480	120	-432	-768
73	218	218	26	434	746	362	-478
103	128	128	8	1172	-560	-1812	640
59	-660	-660	-684	-744	660	-420	240
37	254	254	-226	-214	430	114	-260
level	6	6	12	60	30	90	30

Table 6: Values of $\mathrm{trace}(\mathrm{Frob}_p|_{V_{\boldsymbol{a}}})$.

table.

p	7, 103	11, 59	13, 37	17	19	23	29	31	41	43	53	61	71	73
b_p	-4	0	2	6	-4	0	-6	8	-6	-4	-6	-10	0	2

Now from (29) and (39), we have

$$\mathrm{trace}(\mathrm{Frob}_p|_{V_{\boldsymbol{a}}}) = p^3 + 1 + p(1+p)h - \#\widetilde{\widetilde{X}}_{\boldsymbol{a}}(\mathbb{F}_p) - h^{12}(\widetilde{\widetilde{X}}_{\boldsymbol{a}})pb_p.$$

From this we see that using the data in Tables 1 and 5 we can compute the values of $\mathrm{trace}(\mathrm{Frob}_p|_{V_{\boldsymbol{a}}})$ provided we know the integer h. At this point we make use of a trick which to our knowledge goes back to van Geemen and Werner. Recall that by the analogue of the Riemann hypothesis the absolute value of the trace of Frobenius on an invariant 2-dimensional piece of $H^3_{\text{ét}}(\widetilde{\widetilde{X}}_{\boldsymbol{a}})$ is bounded by $2p^{3/2}$. If $p \geq 17$ then the value of $p(1+p)$ exceeds $4p^{3/2}$ and hence h and thus also $\mathrm{trace}(\mathrm{Frob}_p|_{V_{\boldsymbol{a}}})$ can be determined by the above formula. Using this observation we obtain values as in Table 6, where we use the same indexing convention as in Table 5.

Note that in all of these cases $h = h^{11}(\widetilde{\widetilde{X}}_{\boldsymbol{a}})$. We conjecture that this is always the case. To prove this, it would be enough to prove that the action

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p	37	47	53	59	61	71	79	83	101	103
$\#\widetilde{\widetilde{X}}_{\boldsymbol{a}}(\mathbb{F}_p)$	121168	216696	292392	383124	416500	613032	807688	921000	1546944	1627156
trace	−214	−72	−414	−744	−418	480	1352	−612	−1542	1172

p	107	109	113	127	173	193	211	241	281
$\#\widetilde{\widetilde{X}}_{\boldsymbol{a}}(\mathbb{F}_p)$	1800888	1896388	2086824	2863252	6680856	9063196	11627272	16915444	26156796
trace	1956	−1858	174	−2068	1962	−2038	3260	−1822	−6654

p	283	311	179	157
$\#\widetilde{\widetilde{X}}_{\boldsymbol{a}}(\mathbb{F}_p)$	26685544	34931928	7345764	5110360
trace	−1756	−96	576	−166

Table 7: Values of $\#\widetilde{\widetilde{X}}_{\boldsymbol{a}}(\mathbb{F}_p)$ and $\mathrm{trace}(\mathrm{Frob}_p|_{V_{\boldsymbol{a}}})$ for $\boldsymbol{a} = (1:1:4:4:9)$.

of the Frobenius on H^2 and H^4 is multiplication by p and p^2 respectively. The above numbers can be verified to be coefficients of weight 4 cuspidal Hecke eigen modular forms of levels 6, 6, 12, 60, 30, 90 and 30 respectively, as indicated in the last column. In Lemma 6.2 we saw that $\widetilde{X}_{(1:1:4:4:9)}$ also has bad reduction at 7. Thus in this case we have to compute the number of point over $\mathbb{F}_p$ for additional primes as given in Lemma 6.10, and from this, as before, we can compute the values of the traces on $V_{\boldsymbol{a}}$. This data is given in Table 7.

Thus, applying [Li, Theorem 4.3], we have the following.

THEOREM 6.11. *For* $\boldsymbol{a}$ *as in Corollaries* 4.9 *and* 5.12 *the two dimensional Galois representations* $V_{\boldsymbol{a}}$ *given by* (28) *are modular, corresponding to the weight* 4 *modular forms, with coefficients and level indicated in Table* 6.

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