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RATIONALITY AND SYLOW 2-SUBGROUPS

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Abstract Let G be a finite group. If G has a cyclic Sylow 2-subgroup, then G has the same number of irreducible rational-valued characters as of rational conjugacy classes. These numbers need not be the same even if G has Klein Sylow 2-subgroups and a normal 2-complement.

Keywords: rational characters; rational classes; characters

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1. Introduction

Suppose that G is a finite group. An irreducible complex character $\chi \in \mathrm{Irr}(G)$ of G is *rational* if $\chi(g) \in \mathbb{Q}$ for all $g \in G$, while a conjugacy class $C = \mathrm{cl}_G(g)$ of G is *rational* if $\chi(g) \in \mathbb{Q}$ for all $\chi \in \mathrm{Irr}(G)$. That is, the rational characters correspond to the rational rows in the character table, while the rational classes correspond to the rational columns. Let us write $\mathrm{Irr}_{\mathbb{Q}}(G)$ and $\mathrm{Cl}_{\mathbb{Q}}(G)$ for the sets of irreducible rational characters and rational conjugacy classes of G, respectively.

Often we have that $|\mathrm{Irr}_{\mathbb{Q}}(G)| = |\mathrm{Cl}_{\mathbb{Q}}(G)|$, but not always: the smallest counter-example is a group of order 2^5. It is convenient to mention now that it is not true in general that the natural actions of the Galois group $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ on $\mathrm{Irr}(G)$ and on the set $\mathrm{Cl}(G)$ of conjugacy classes of G are permutation isomorphic, where n is the exponent of G and $\mathbb{Q}_n$ is the cyclotomic field. Whenever this is the case, of course, we have that $|\mathrm{Irr}_{\mathbb{Q}}(G)| = |\mathrm{Cl}_{\mathbb{Q}}(G)|$. In particular, it follows from the main result of [**1**] that $|\mathrm{Irr}_{\mathbb{Q}}(G)| = |\mathrm{Cl}_{\mathbb{Q}}(G)|$ if all Sylow subgroups are abelian.

Rationality questions in a finite group G are believed to be somehow related to the Sylow 2-subgroups P of G, the first reason being, perhaps, that groups of odd order do not possess non-trivial rational characters or classes. (It continues to be an open question as to whether G being rational, i.e. if all $\chi \in \mathrm{Irr}(G)$ are rational, implies that P is rational.) But there are no general results relating Sylow 2-subgroups and rationality. In fact, there are very few results guaranteeing that $|\mathrm{Irr}_{\mathbb{Q}}(G)| = |\mathrm{Cl}_{\mathbb{Q}}(G)|$ (see [**9**] for another result of this type).

In this paper we count rational characters and rational classes of certain groups. In particular, we can prove the following.

Theorem 1.1. *Suppose that $P \in \mathrm{Syl}_2(G)$ is cyclic. Then* $|\mathrm{Irr}_{\mathbb{Q}}(G)| = |\mathrm{Cl}_{\mathbb{Q}}(G)|$.

Even the case where $|P| = 2$ in Theorem 1.1 is non-trivial and relies on the recent paper [**8**] on quadratic characters of groups of odd order. (See also [**2**].)

Unfortunately, Theorem 1.1 does not seem to allow a variety of results of similar type: if we change the word 'cyclic' to $C_2 \times C_2$, Theorem 1.1 is already false (even assuming that G has a normal 2-complement). The same happens if P is a quaternion group or is dihedral of order 8. We will provide examples in § 5.

Our methods here are in fact slightly more general and apply, for instance, in analysing when groups with a normal Sylow p-subgroup G have equal numbers of rational classes and rational characters.

2. Characters

If n is any positive integer, then we denote by

$$\mathcal{G}_n = \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$$

the Galois group of the nth cyclotomic extension $\mathbb{Q}_n$ over the rationals. In general, we use the notation of [**5**].

If G is a finite group and $C = \mathrm{cl}_G(x) \in \mathrm{cl}(G)$ is the conjugacy class of $x \in G$, then $\mathbb{Q}(C) = \mathbb{Q}(\chi(x) \mid \chi \in \mathrm{Irr}(G))$. If $\chi \in \mathrm{Irr}(G)$, then $\mathbb{Q}(\chi) = \mathbb{Q}(\chi(x) \mid x \in G)$. Notice that if $a \in \mathrm{Aut}\, G$, then $\mathbb{Q}(\chi) = \mathbb{Q}(\chi^a)$ and $\mathbb{Q}(C) = \mathbb{Q}(C^a)$. (In this case, χ^a is the unique irreducible character of G such that $\chi^a(g^a) = \chi(g)$ for $g \in G$ and $C^a = \mathrm{cl}_G(x^a)$ if $C = \mathrm{cl}_G(x)$.)

Now, we have that $\mathcal{G}_n$ acts on the conjugacy classes of G consisting of elements $x \in G$ of order dividing n via

$$\mathrm{cl}_G(x)^{\sigma_t} = \mathrm{cl}_G(x^t),$$

where $\sigma_t \in \mathcal{G}_n$, t is an integer and $\sigma_t(\xi) = \xi^t$ for nth roots of unity ξ. Of course, t is uniquely determined modulo n, and is coprime with n. If $\chi \in \mathrm{Irr}(G)$, then $\chi(x) \in \mathbb{Q}_n$ and

$$\chi(x)^{\sigma_t} = \chi(x^t).$$

Now we have that $\sigma_t \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}(C))$ if and only if $\chi(x)^{\sigma_t} = \chi(x)$ for all $\chi \in \mathrm{Irr}(G)$. This happens if and only if x^t is G-conjugate to x: in other words, if $C^{\sigma_t} = C$. Hence, we see that the stabilizer of the conjugacy class $C = \mathrm{cl}_G(x)$ in $\mathcal{G}_n$ is $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}(C))$. Using this fact, we can prove that the groups $\mathrm{Gal}(\mathbb{Q}_{o(x)}/\mathbb{Q}(C))$ and $\boldsymbol{N}_G(\langle x\rangle)/\boldsymbol{C}_G(x)$ are naturally isomorphic.

Let us record the following elementary fact.

Lemma 2.1. *Suppose that $F/\mathbb{Q}$ is a finite Galois abelian complex extension of $\mathbb{Q}$ and let $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$. Suppose that χ is a character of G such that $\mathbb{Q}(\chi) \subseteq F$. Then χ^σ is also a character of G, where*

$$\chi^\sigma(x) = \chi(x)^\sigma.$$

Also $\chi \in \mathrm{Irr}(G)$ if and only if $\chi^\sigma \in \mathrm{Irr}(G)$.

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Proof. Let $n = |G|$. Then we know that $\mathbb{Q}_n$ is a splitting field for G and therefore $\mathcal{G}_n = \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ acts naturally on $\mathrm{Irr}(G)$. Also, $\chi^\tau \in \mathrm{Irr}(G)$ if and only if $\chi \in \mathrm{Irr}(G)$, for $\tau \in \mathcal{G}_n$. Now $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\chi)$. Since $F/\mathbb{Q}$ is Galois and abelian, we have that $\psi = \sigma_{\mathbb{Q}(\chi)}$ is a Galois automorphism of $F \cap \mathbb{Q}_n$. Since $\mathbb{Q}_n/\mathbb{Q}$ is Galois, ψ extends to some $\tau \in \mathcal{G}_n$. Now, $\chi^\sigma = \chi^\tau$, and we are done. □

From Lemma 2.1, we deduce that $\mathcal{G}_n$ acts on the irreducible characters $\chi \in \mathrm{Irr}(G)$ with $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_n$, where

$$\chi^{\sigma_t}(x) = \chi(x)^{\sigma_t} = \chi(x^t)$$

for all $x \in G$. Notice too that the $\mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$-orbit of χ has size $|\mathbb{Q}(\chi) : \mathbb{Q}|$.

It is easy to check that all the Galois actions defined so far commute with the natural action of $\mathrm{Aut}\,G$ on $\mathrm{Irr}(G)$ and $\mathrm{Cl}(G)$. In particular, we have that $\mathcal{G}_{|G|} \times \mathrm{Aut}(G)$ naturally acts on $\mathrm{Irr}(G)$ and $\mathrm{Cl}(G)$.

When dealing with the field of values of characters and normal subgroups, it is convenient to use the so-called *semi-inertia* subgroup. Suppose that $N \lhd G$ and let $\theta \in \mathrm{Irr}(N)$. If $g \in G$, then we have that $\mathbb{Q}(\theta) = \mathbb{Q}(\theta^g)$. If $T = I_G(\theta)$ is the stabilizer of θ in G, then we define $T^* = \{g \in G \mid \theta^g = \theta^\sigma$ for some $\sigma \in \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})\}$. Of course, T^* is a subgroup of G containing T.

We use the notation $\mathrm{Irr}(G \mid \theta)$ for the irreducible constituents of θ^G.

Lemma 2.2. *Let $N \lhd G$, $\theta \in \mathrm{Irr}(N)$, T and T^* be as before.*

(a) *Suppose that $\chi \in \mathrm{Irr}(G \mid \theta)$ and assume that $\mathbb{Q}(\chi), \mathbb{Q}(\theta) \subseteq F$, where $F/\mathbb{Q}$ is a finite abelian Galois extension. If $\sigma \in \mathrm{Gal}(F/\mathbb{Q}(\chi))$, then there exists $g \in T^*$ such that $\theta^g = \theta^\sigma$.*

(b) *If $\psi \in \mathrm{Irr}(T \mid \theta)$, then $\mathbb{Q}(\psi^{T^*}) = \mathbb{Q}(\psi^G)$.*

Proof. By Lemma 2.1, we have that $\theta^\sigma \in \mathrm{Irr}(N)$. Now, since $[(\chi^\sigma)_N, \theta^\sigma] = [\chi_N, \theta]$, we see that θ^σ lies under $\chi^\sigma = \chi$. Hence, $\theta^\sigma = \theta^g$ for some $g \in G$ by Clifford's Theorem. Then we easily see that $g \in T^*$. This proves part (a).

To prove part (b), write $\eta = \psi^{T^*} \in \mathrm{Irr}(T^* \mid \theta)$ and $\chi = \psi^G \in \mathrm{Irr}(G \mid \theta)$. Since $\eta^G = \chi$, we have that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\eta) \subseteq \mathbb{Q}(\psi)$ by the induction formula. It suffices to show that if $\sigma \in \mathrm{Gal}(\mathbb{Q}(\psi)/\mathbb{Q}(\chi))$, then $\eta^\sigma = \eta$. Since $\psi_N = e\theta$, we have that $\mathbb{Q}(\theta) \subseteq \mathbb{Q}(\psi)$. By part (a) (with $F = \mathbb{Q}(\psi)$), we have that $\theta^\sigma = \theta^g$ for some $g \in T^*$. Now $(\psi^\sigma)^{g^{-1}}$ lies over θ and induces χ, so $(\psi^\sigma)^{g^{-1}} = \psi$ by the uniqueness in the Clifford correspondence. Thus, $\psi^\sigma = \psi^g$ and

$$\eta^\sigma = (\psi^\sigma)^{T^*} = (\psi^g)^{T^*} = \psi^{T^*} = \eta,$$

as desired. □

We shall use the following elementary lemma several times.

Lemma 2.3. *Let $N \lhd G$, $\theta \in \mathrm{Irr}(N)$, T and T^* be as before. Then the map $T^* \to \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})$ given by $g \mapsto \sigma$ and defined by the equation*

$$\theta^g = \theta^\sigma$$

is a well-defined group homomorphism with kernel T and image

$$\mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q}(\theta^G)).$$

In particular, we have that θ^G is rational valued if and only if $|T^*/T| = |\,\mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})|$. This happens, for instance, if there exists some $\chi \in \mathrm{Irr}(G \mid \theta)$ that is rational valued.

Proof. The first part is [**6**, Lemma 2.2]. Suppose that there exists $\chi \in \mathrm{Irr}(G \mid \theta)$ that is rational valued. Since θ^G is zero off N and $(\theta^G)_N$ is a rational multiple of χ_N, we see that $\mathbb{Q}(\theta^G) = \mathbb{Q}(\chi_N)$ and the last part follows. □

Hence, we see that if θ^G is rational valued, then there exists a natural isomorphism

$$\rho_\theta = \rho : \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q}) \to T^*/T.$$

If $\sigma \in \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})$, then $\rho(\sigma)$ is the unique (modulo T) element in T^* such that

$$\theta^\sigma = \theta^{\rho(\sigma)}.$$

Theorem 2.4. *Let $N \lhd G$, let $\theta \in \mathrm{Irr}(N)$, let T and T^* be as before, and suppose that θ^G is rational valued.*

(a) *Let $\psi \in \mathrm{Irr}(T \mid \theta)$ and $\chi = \psi^G$. Then χ is rational valued if and only if $\mathbb{Q}(\psi) = \mathbb{Q}(\theta)$ and*

$$\psi^\tau = \psi^{\rho(\tau)}$$

for all $\tau \in \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})$.

(b) *Let $\psi \in \mathrm{Irr}(T \mid \theta)$ be such that ψ^G is rational valued, and assume that $\psi_N = \theta$. Let $\epsilon \in \mathrm{Irr}(T/N)$. Then $(\epsilon\psi)^G$ is rational if and only if $\mathbb{Q}(\epsilon) \subseteq \mathbb{Q}(\theta)$ and $\epsilon^\tau = \epsilon^{\rho(\tau)}$ for $\tau \in \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})$.*

Proof. By hypothesis, we have a natural isomorphism $\rho : \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q}) \to T^*/T$.

(a) Suppose first that $\mathbb{Q}(\psi) = \mathbb{Q}(\theta)$ and $\psi^\tau = \psi^{\rho(\tau)}$ for all $\tau \in \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})$. We have that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\psi) = \mathbb{Q}(\theta)$. Take $\tau \in \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})$ and write $\rho(\tau) = gT \in T^*/T$. Then $\chi^\tau = (\psi^G)^\tau = (\psi^\tau)^G = (\psi^g)^G = \chi$ and we conclude that χ is rational valued.

Conversely, suppose that χ is rational valued. Since $\psi_N = e\theta$, we have that $\mathbb{Q}(\theta) \subseteq \mathbb{Q}(\psi)$. Now, take $\sigma \in \mathrm{Gal}(\mathbb{Q}(\psi)/\mathbb{Q}(\theta))$ and note that $\psi^\sigma \in \mathrm{Irr}(T \mid \theta)$. Since χ is rational, $(\psi^\sigma)^G = (\psi^G)^\sigma = \chi^\sigma = \chi = \psi^G$, and by the uniqueness in the Clifford correspondence we deduce that $\psi^\sigma = \psi$. Hence, σ is the identity in $\mathrm{Gal}(\mathbb{Q}(\psi)/\mathbb{Q}(\theta))$, and we conclude that $\mathbb{Q}(\theta) = \mathbb{Q}(\psi)$.

Now take $\tau \in \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})$ and suppose that $\rho(\tau) = gT$, so $(\theta^\tau)^{g^{-1}} = \theta$. Since $\mathbb{Q}(\theta) = \mathbb{Q}(\psi)$, we have that $(\psi^\tau)^{g^{-1}} \in \mathrm{Irr}(T)$, and it is clear that this character lies over θ. Also, $(\psi^\tau)^{g^{-1}}$ induces χ, and we conclude that $\psi^\tau = \psi^g$, again by the uniqueness in the Clifford correspondence. This proves (a).

(b) By hypothesis and part (a), we have that $\mathbb{Q}(\theta) = \mathbb{Q}(\psi)$ and $\psi^\tau = \psi^{\rho(\tau)}$ for all $\tau \in \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})$. Suppose first that $(\epsilon\psi)^G$ is rational valued. Let F be the smallest extension containing $\mathbb{Q}(\epsilon)$ and $\mathbb{Q}(\theta)$. Then $F/\mathbb{Q}$ is Galois and abelian, and we can use Lemma 2.1. Let $\tau \in \mathrm{Gal}(F/\mathbb{Q}(\theta))$. Since τ fixes θ, τ fixes ψ, because $\mathbb{Q}(\theta) = \mathbb{Q}(\psi)$. Now $(\epsilon\psi)^\tau$ induces $(\epsilon\psi)^G$ and lies over θ. By the uniqueness in the Clifford correspondence, we deduce that $(\epsilon\psi)^\tau = \epsilon\psi$. Now $\epsilon\psi = (\epsilon\psi)^\tau = \epsilon^\tau\psi^\tau = \epsilon^\tau\psi$, and we deduce that $\epsilon^\tau = \epsilon$ by the uniqueness in Gallagher's Theorem [**5**, Corollary 6.17]. Hence, $\mathbb{Q}(\epsilon) \subseteq \mathbb{Q}(\theta)$. Now, by part (a), we have that $(\epsilon\psi)^\tau = \epsilon^{\rho(\tau)}\psi^{\rho(\tau)}$, and we deduce that $\epsilon^\tau = \epsilon^{\rho(\tau)}$, again by the uniqueness in Gallagher's Theorem.

Conversely, if $\mathbb{Q}(\epsilon) \subseteq \mathbb{Q}(\theta)$, then $\mathbb{Q}(\theta) \subseteq \mathbb{Q}(\epsilon\psi) \subseteq \mathbb{Q}(\theta)$ (the first containment because $(\epsilon\psi)_N = \epsilon(1)\theta$ and the second because $\mathbb{Q}(\psi), \mathbb{Q}(\epsilon) \subseteq \mathbb{Q}(\theta)$). It is easy to see that $\mathbb{Q}(\epsilon\psi) = \mathbb{Q}(\theta)$, and the result easily follows from part (a). □

As we have seen, if θ^G is rational valued, then we have that $\theta^\tau = \theta^{\rho(\tau)}$ for $\tau \in \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})$. Now, if there is a *canonical* choice of some $\psi \in \mathrm{Irr}(T \mid \theta)$, then we might easily check that $\psi^\tau = \psi^{\rho(\tau)}$ too, and then we are ready to use Theorem 2.4. This happens, for instance when $(|N|, |G : N|) = 1$, if N is abelian and complemented in G or in certain unique extensions which respect fields of values [**7**].

Corollary 2.5. *Let $N \triangleleft G$, let $\theta \in \mathrm{Irr}(N)$, let T and T^* be as before, and suppose that θ^G is rational valued. If $(|N|, |G/N|) = 1$, then the number of $\varphi \in \mathrm{Irr}(G \mid \theta)$ which are rational valued is the number of rational $\epsilon \in \mathrm{Irr}(T/N)$ which are T^*-invariant. In particular, this number is $|\mathrm{Irr}_{\mathbb{Q}}(T/N)|$ if T^*/N is abelian.*

Proof. Since $(|N|, |T/N|) = 1$, by [**5**, Corollary 8.16], there exists a unique extension $\psi \in \mathrm{Irr}(T)$ of θ such that $o(\theta) = o(\psi)$ (where $o(\theta)$ is the determinantal order of θ). By uniqueness of the extension, $\mathbb{Q}(\psi) = \mathbb{Q}(\theta)$ and $\psi^\tau = \psi^g$ whenever $\theta^\tau = \theta^g$, for all $\tau \in \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})$ and $g \in G$. Now, by Theorem 2.4 (a) we see that ψ^G is rational valued. Now, note that $\epsilon \in \mathrm{Irr}(T/N)$ has values in $\mathbb{Q}(\theta) \subseteq \mathbb{Q}_{|N|}$ if and only if ϵ is rational valued by coprimeness. The result now follows from Theorem 2.4 (b). □

Notice that if θ^G is not rational, then there are no rational characters over θ by the last part of Lemma 2.3.

3. Classes

If $C = \mathrm{cl}_G(x)$ is a conjugacy class of a finite group G, we have defined $\mathbb{Q}(C) = \mathbb{Q}(\chi(x) \mid \chi \in \mathrm{Irr}(G))$. Now, if $\sigma \in \mathrm{Gal}(\mathbb{Q}(C)/\mathbb{Q})$, then σ uniquely defines a conjugacy class C^σ of G as follows. Extend σ to some $\sigma_t \in \mathcal{G}_{|G|}$ and define $C^\sigma = C^{\sigma_t} = \mathrm{cl}_G(x^t)$. If σ_s also extends σ, then we have that $\chi(x)^{\sigma_t} = \chi(x)^{\sigma_s}$ for all $\chi \in \mathrm{Irr}(G)$. Hence, x^t and x^s are G-conjugate. So we see that C^σ is well defined. Notice too that $C^\sigma = C$ if and only if $\sigma = 1$.

If $H \subseteq G$ and $C = \mathrm{cl}_H(h)$, then we define $C^G = \mathrm{cl}_G(h)$. Notice that $\mathbb{Q}(C^G) \subseteq \mathbb{Q}(C)$.

Suppose now that $C = \mathrm{cl}_N(n)$, where $N \triangleleft G$. We define

$$\boldsymbol{N}_G(C)^* = \{g \in G \mid C^g = C^\sigma \text{ for some } \sigma \in \mathrm{Gal}(\mathbb{Q}(C)/\mathbb{Q})\}.$$

Lemma 3.1. *The map*

$$\rho_C : \boldsymbol{N}_G(C)^* \to \mathrm{Gal}(\mathbb{Q}(C)/\mathbb{Q}(C^G))$$

given by $g \mapsto \sigma$, defined by the equation $C^g = C^\sigma$, is an onto group homomorphism with kernel $\boldsymbol{N}_G(C) = \{g \in G \mid C^g = C\}$.

Proof. Suppose that $C^g = C^\sigma = \mathrm{cl}_N(n^t)$. If $\chi \in \mathrm{Irr}(G)$, then

$$\chi(n)^\sigma = \chi(n^t) = \chi(n^g) = \chi(n),$$

so we see that $\sigma \in \mathrm{Gal}(\mathbb{Q}(C)/\mathbb{Q}(C^G))$. Conversely, if $\sigma \in \mathrm{Gal}(\mathbb{Q}(C)/\mathbb{Q}(C^G))$ and $\sigma = \sigma_t$ on $\mathbb{Q}(C)$, then we have that $\chi(n^t) = \chi(n)^\sigma = \chi(n)$ for all $\chi \in \mathrm{Irr}(G)$. Hence, n^t and n are G-conjugate. So there is $g \in \boldsymbol{N}_G(C)^*$ such that $C^g = C^\sigma$. □

Next, we give an appropriate set of representatives of conjugacy classes in a coprime action, which is fairly well known. We shall frequently use the fact that if A acts coprimely on G and C is an A-invariant conjugacy class of G, then there exists $c \in C \cap \boldsymbol{C}_G(A)$ [**5**, Corollary 13.10].

Lemma 3.2. *Suppose that $N \lhd G$ with $(|G : N|, |N|) = 1$ and let H be a complement of N in G.*

(a) *Let C be a conjugacy class of N and let $\boldsymbol{N}_G(C) = \{x \in G \mid C^x = C\}$. Write $C = \mathrm{cl}_N(n)$, where $[n, \boldsymbol{N}_H(C)] = 1$. Let $h_1, h_2 \in \boldsymbol{N}_H(C)$. Then nh_1 is G-conjugate to nh_2 if and only if h_1 and h_2 are $\boldsymbol{N}_H(C)$-conjugate.*

(b) *If $\{C_i \mid 1 \leqslant i \leqslant k\}$ is a complete set of representatives of the action of G on $\mathrm{Cl}(N)$ and $\{h_{ij} \mid 1 \leqslant j \leqslant k_i\}$ is a complete set of representatives of conjugacy classes of $H_i = \boldsymbol{N}_H(C_i)$, then $n_i h_{ij}$ is a complete set of representatives of the conjugacy classes of G, where $C_i = \mathrm{cl}_N(n_i)$ and $[n_i, H_i] = 1$.*

Proof. (a) Suppose that $nh_2 = (nh_1)^g$ for some $g \in G$. Since $(o(n), o(h_i)) = 1$ and n commutes with $h_i \in \boldsymbol{N}_H(C)$, where $i = 1, 2$, we have that $n^g = n$ and $h_1^g = h_2$. If $g = mh$ for $m \in N$ and $h \in H$, then $(h_1)^m = (h_2)^{h^{-1}} \in H \cap H^m = \boldsymbol{C}_H(m)$. Hence, $h_1^h = h_2$. Now, $(n^m)^h = n$ implies that $h \in \boldsymbol{N}_H(C)$. Since $[n, \boldsymbol{N}_H(C)] = 1$, the converse easily follows.

(b) Suppose that $x \in G$, and we prove that x is G-conjugate to some $n_i h_{ij}$, following the notation in the statement of the lemma. We have that $x = nu$ for some $n \in N$ and $[u, n] = 1$, where $(o(u), o(n)) = 1$. By conjugating by an appropriate element, we may assume that $n = n_i$ for some i. Now, $H_i \subseteq \boldsymbol{C}_G(n_i)$ and $u \in \boldsymbol{C}_G(n_i)$. Since H_i is a Hall π'-complement of $\boldsymbol{C}_G(n_i) \subseteq \boldsymbol{N}_G(C_i)$ and u is a π'-element (where π is the set of primes dividing $|N|$), it follows by the Schur–Zassenhaus Theorem that $u^v \in H_i$ for some $v \in \boldsymbol{C}_G(n_i)$. Now $u^{vw} = h_{ij}$ for some $w \in H_i$. Now,

$$x^{vw} = n_i^{vw} h_{ij} = n_i h_{ij}.$$

Finally, if $n_i h_{ij}$ is G-conjugate to $n_r h_{rs}$, it is clear that $n_i = n_r$, and the result now follows from part (a). □

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Recall that a conjugacy class $C = \mathrm{cl}_G(x)$ is rational if whenever $\langle x \rangle = \langle x^t \rangle$, where t is an integer, then x and x^t are G-conjugate.

Lemma 3.3. *Suppose that $N \triangleleft G$ with $(|G:N|, |N|) = 1$ and let H be a complement of N in G. Let C be a conjugacy class of N, and write $C = \mathrm{cl}_N(n)$ where $[n, \boldsymbol{N}_H(C)] = 1$. Let $h \in \boldsymbol{N}_H(C)$. Let also $g \in \boldsymbol{N}_G(C)^*$ with $g = ym$, where $y \in \boldsymbol{N}_H(C)^* = \boldsymbol{N}_G(C)^* \cap H$ and $m \in N$. If $n^g \in \boldsymbol{C}_N(h)$, then there exists $z \in \boldsymbol{C}_N(h)$ such that $n^g = n^{yz}$.*

Proof. Since $\boldsymbol{N}_H(C) \triangleleft \boldsymbol{N}_H(C)^*$, it is clear that $h^{y^{-1}} \in \boldsymbol{N}_H(C)$. Then $[n, h^{y^{-1}}] = 1$ or, equivalently, $[n^y, h] = 1$. Thus, $n^y \in \boldsymbol{C}_N(h)$, and since n^g and n^y are N-conjugate, we deduce by Glauberman's Lemma [**5**, Corollary 13.10] on coprime action that n^g and n^y are indeed $\boldsymbol{C}_N(h)$-conjugate. Thus, there exists $z \in \boldsymbol{C}_N(h)$ such that $n^{yz} = n^g$, as desired. □

Theorem 3.4. *Suppose that $N \triangleleft G$ with $(|G:N|, |N|) = 1$ and let H be a complement of N in G. Let C be a conjugacy class of N, and write $C = \mathrm{cl}_N(n)$ where $[n, \boldsymbol{N}_H(C)] = 1$. Suppose that C^G is rational. If $h \in \boldsymbol{N}_H(C)$, then $\mathrm{cl}_G(nh)$ is rational if and only if $L = \mathrm{cl}_{\boldsymbol{N}_H(C)}(h)$ is rational and invariant in $\boldsymbol{N}_H(C)^* = \boldsymbol{N}_G(C)^* \cap H$.*

Proof. Suppose first that $\mathrm{cl}_G(nh)$ is rational. Suppose that $\langle h^k \rangle = \langle h \rangle$, and we want to see that h^k is $\boldsymbol{N}_H(C)$-conjugate to h. By using the Chinese Remainder Theorem, we may assume that $n^k = n$. Now, since nh is rational in G, we have that $(nh)^k = nh^k$ is G-conjugate to nh, and we deduce from Lemma 3.2 (a) that h and h^k are $\boldsymbol{N}_H(C)$-conjugate, as desired.

Next we prove that L is $\boldsymbol{N}_H(C)^*$-invariant. Let $y \in \boldsymbol{N}_H(C)^*$. Then $\mathrm{cl}_N(n^y) = \mathrm{cl}_N(n^t)$ for some integer t coprime to $o(n)$. Again using the Chinese Remainder Theorem, we may assume that $h^t = h$. So $n^t = n^{ym}$ for some $m \in N$, and we can suppose by Lemma 3.3 that $m \in \boldsymbol{C}_N(h)$. Now nh is rational in G, and therefore G-conjugate to

$$(nh)^t = n^t h = n^{ym} h = (nh^{y^{-1}})^{ym}.$$

We deduce that $\mathrm{cl}_G(nh) = \mathrm{cl}_G(nh^{y-1})$. Now it follows from Lemma 3.2 (a) that h is $\boldsymbol{N}_H(C)$-conjugate to $h^{y^{-1}}$. This implies that L is y-invariant, as desired.

Conversely, suppose now that L is rational and $\boldsymbol{N}_H(C)^*$-invariant. We want to prove that $\mathrm{cl}_G(nh)$ is rational. Suppose that $\langle nh \rangle = \langle (nh)^k \rangle$. Note that $\langle h^k \rangle = \langle h \rangle$, and then $h^k = h^v$ for some $v \in \boldsymbol{N}_H(C)$, since L is rational. Thus, we have $(nh)^k = n^k h^v$. Also, since n is rational in G, we know that there exists $g \in \boldsymbol{N}_G(C)^*$ such that $n^k = n^g$. Write $g = ym$, where $y \in \boldsymbol{N}_H(C)^*$ and $m \in N$. Since $n^g = n^k \in \boldsymbol{C}_N(h^v)$, we can assume by Lemma 3.3 that $m \in \boldsymbol{C}_N(h^v)$. Now

$$(nh)^k = n^g h^v = (nh^{vy^{-1}})^g,$$

and we deduce that $(nh)^k$ is G-conjugate to $nh^{vy^{-1}}$. Since L is $\boldsymbol{N}_H(C)^*$-invariant, it follows that $h^{vy^{-1}}$ is $\boldsymbol{N}_H(C)$-conjugate to h. This implies by Lemma 3.2 (a) that $nh^{vy^{-1}}$ is G-conjugate to nh, and we deduce that $(nh)^k$ and nh are G-conjugate, as desired. □

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4. Main results

The proof of the following key result is inspired by an idea due to Wilde [**10**].

Theorem 4.1. *Suppose that P is a group that acts as automorphisms on N with $(|P|,|N|)=1$. Then the actions of $P\times\langle\sigma\rangle$ on $\mathrm{Irr}(N)$ and $\mathrm{cl}(N)$ are permutation isomorphic.*

Actually, we shall need a more general version of Theorem 4.1, where only certain characters and classes of N are taken into account. In order to state that version of Theorem 4.1, we need to introduce Isaacs B_π-characters. In any π-separable group G, Isaacs [**4**] defined a canonical subset $B_\pi(G)$ of the complex irreducible characters $\mathrm{Irr}(G)$. (The main point was that $B_{p'}$-characters constitute a canonical lifting of the irreducible Brauer characters of a p-solvable group.) It easily follows from their definition that if G is a π-group, then $B_\pi(G)=\mathrm{Irr}(G)$. Also, $\chi\in B_\pi(G)$ if and only if $\chi^\sigma\in B_\pi(G)$, where $\sigma\in\mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$, and B_π-characters are closed under automorphisms. We have that $\mathbb{Q}(\chi)\subseteq\mathbb{Q}_{|G|_\pi}$ for $\chi\in B_\pi(G)$ [**4**, Corollary 12.1]. A deeper fact on B_π-characters is that $|B_\pi(G)|$ is the number of conjugacy classes consisting of π-elements, and that the square matrix $A=(\chi(x))$, where $\chi\in B_\pi(G)$ and x runs over representatives of the conjugacy classes of π-elements of G, is invertible. (This follows from [**4**, Corollary 10.2].)

In the proof of the next theorem we shall also need the existence of an extension of the Glauberman–Isaacs correspondence. Namely, if A acts coprimely on a π-separable group G, we shall use the fact that there is a canonical bijection between the A-invariant B_π-characters of G and the B_π-characters of $\boldsymbol{C}_G(A)$. This fact was proved by Wolf [**11**], and later generalized in [**3**]. This canonical correspondence commutes with Galois action and automorphisms, as can be easily checked.

Now we are ready to prove the version of Theorem 4.1 that we really need. We can recover Theorem 4.1 by taking π to be the set of all primes dividing $|N|$. We denote by $\mathrm{Cl}_\pi(N)$ the set of conjugacy classes of N consisting of π-elements.

Theorem 4.2. *Suppose that Q acts coprimely as automorphisms on a π-separable group N, and let $\sigma\in\mathrm{Gal}(\mathbb{Q}_{|N|}/\mathbb{Q})$. Then the natural actions of $Q\times\langle\sigma\rangle$ on $B_\pi(N)$ and on $\mathrm{Cl}_\pi(N)$ are permutation isomorphic.*

Proof. It is sufficient to show that if H is a subgroup of $Q\times\langle\sigma\rangle$, then H fixes the same number of $\theta\in B_\pi(N)$ as of $C\in\mathrm{cl}_\pi(N)$. Hence, it is sufficient to show that if $(z_i,\sigma^{k_i})\in Q\times\langle\sigma\rangle$, where $1\leqslant i\leqslant s$, then the sets $A=\{\theta\in B_\pi(N)\mid\theta^{z_i}=\theta^{\sigma^{k_i}}\text{ for all }i\}$ and $B=\{C\in\mathrm{Cl}_\pi(N)\mid C^{z_i}=C^{\sigma^{k_i}}\text{ for all }i\}$ have the same cardinality.

Suppose that $\theta^u=\theta^{\sigma^k}$ and $\theta^v=\theta^{\sigma^l}$, where $u,v\in Q$, and k, l are integers. Then notice that

$$\theta^{uv}=\theta^{\sigma^{k+l}}=\theta^{vu}.$$

In particular,

$$\theta^{u^m}=\theta^{\sigma^{mk}}$$

for every integer m. Write $n=o(\sigma)$. Also, write $d=\gcd(n,k)$. Then

$$\theta^{u^{n/d}}=\theta^{\sigma^{nk/d}}=\theta.$$

Since $\langle\sigma^k\rangle = \langle\sigma^d\rangle$, we have that $(\sigma^k)^s = \sigma^d$ and $\sigma^{dt} = \sigma^k$ for some integers s and t. These integers satisfy that $st \equiv 1 \bmod(n/d)$. From here, we easily conclude that

$$\theta^u = \theta^{\sigma^k} \iff \theta^{u^s} = \theta^{\sigma^d}.$$

Since the same argument applies to any class C such that $C^u = C^{\sigma^k}$, with the same s and t, we see that we may assume that k_i divides n for all i.

Now, let $e = \gcd(k_1, \dots, k_s)$. Hence, e divides k_i and there are integers a_i such that $e = \sum_{i=1}^{s} a_i k_i$. Also, we have that

$$\langle\sigma^{k_1}, \dots, \sigma^{k_s}\rangle = \langle\sigma^e\rangle.$$

Let $z = z_1^{a_1} \cdots z_s^{a_s}$. Notice that

$$\theta^z = \theta^{\sigma^e}$$

for all $\theta \in A$. Also

$$\theta^{z^{k_i/e}} = \theta^{\sigma^{k_i}} = \theta^{z_i}.$$

Hence, $v_i = z^{k_i/e} z_i^{-1}$ fixes θ. Since $z_i z_j = z_j z_i$ modulo the stabilizer of θ, we conclude that every $\theta \in A$ is X-invariant, where

$$X = \langle v_i, v_i^z, \dots, v_i^{z^{m-1}} \mid 1 \leqslant i \leqslant s\rangle,$$

and $m = o(z)$. Notice that $X^z = X$.

Let us define $\mathrm{cl}_X(N) = \{C \in \mathrm{Cl}_\pi(N) \mid C^x = C \text{ for all } x \in X\}$, $\mathrm{Cl}_{z,\sigma^e}(N) = \{C \in \mathrm{Cl}_\pi(N) \mid C^z = C^{\sigma^e}\}$, $\mathrm{Irr}_X(N) = \{\theta \in B_\pi(N) \mid \theta^x = \theta \text{ for all } x \in X\}$ and $\mathrm{Irr}_{z,\sigma^e}(N) = \{\theta \in B_\pi(N) \mid \theta^z = \theta^{\sigma^e}\}$.

We claim that $A = \mathrm{Irr}_X(N) \cap \mathrm{Irr}_{z,\sigma^e}(N)$ and $B = \mathrm{Cl}_X(N) \cap \mathrm{Cl}_{z,\sigma^e}(N)$. Let us prove this for A. It suffices to show that $\mathrm{Irr}_X(N) \cap \mathrm{Irr}_{z,\sigma^e}(N) \subseteq A$. But this is obvious: if $\theta^z = \theta^{\sigma^e}$ and θ is X-invariant, then

$$\theta^{z_i} = \theta^{v_i z_i} = \theta^{z^{k_i/e}} = \theta^{\sigma^{(k_i/e)e}} = \theta^{\sigma^{k_i}}.$$

Hence, in order to finish the proof of this theorem, it suffices to prove that

$$|\mathrm{Irr}_X(N) \cap \mathrm{Irr}_{z,\sigma^e}(N)| = |\mathrm{Cl}_X(N) \cap \mathrm{Cl}_{z,\sigma^e}(N)|.$$

Now, since the X-Glauberman–Isaacs B_π-correspondence commutes with z action and Galois action, we have that

$$|\mathrm{Irr}_X(N) \cap \mathrm{Irr}_{z,\sigma^e}(N)| = |\mathrm{Irr}_{z,\sigma^e}(\boldsymbol{C}_N(X))|.$$

Now, the map $C \mapsto C \cap \boldsymbol{C}_N(X)$ is a bijection $\mathrm{Cl}_X(N) \to \mathrm{Cl}(\boldsymbol{C}_N(X))$ which commutes with z action and Galois action, and we deduce that

$$|\mathrm{Cl}_X(N) \cap \mathrm{Cl}_{z,\sigma^e}(N)| = |\mathrm{Cl}_{z,\sigma^e}(\boldsymbol{C}_N(X))|.$$

Finally, we have that $\rho = (z, \sigma^e) \in Q \times \mathrm{Gal}(\mathbb{Q}_{|N|}/\mathbb{Q})$ naturally acts on the conjugacy classes of π-elements of $\boldsymbol{C}_N(X)$ and on the B_π-characters of $\boldsymbol{C}_N(X)$. By Brauer's Lemma on character tables, ρ fixes the same number of elements in both sets. This finishes the proof. □

The following key observation explains why we need to introduce B_π-characters.

Theorem 4.3. *Suppose that G has a cyclic Sylow 2-subgroup, and let N be the normal 2-complement of G. Let $\theta \in \mathrm{Irr}(N)$ and $C = \mathrm{cl}_N(n)$.*

(a) *If C^G is rational, then n is a p-element for some prime p.*

(b) *If θ^G is rational valued, then $\theta \in B_p(N)$ for a unique p.*

Proof. If C^G is rational, then $\mathrm{Gal}(\mathbb{Q}(C)/\mathbb{Q}) \cong \boldsymbol{N}_G(C)^*/\boldsymbol{N}_G(C)$ is cyclic by Lemma 3.1. Hence, n is a p-element by [**8**, Lemma 2.1]. If θ^G is rational, then $T^*/T \cong \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})$ is cyclic and therefore $\theta \in B_p(N)$ by [**8**, Theorem 2.2]. □

Lemma 4.4. *Suppose that N is a normal Hall subgroup of G. Suppose that p is an odd prime and let $\sigma \in \mathrm{Gal}(\mathbb{Q}_{p^a}/\mathbb{Q})$ be a generator.*

(a) *If $\theta \in \mathrm{Irr}(N)$ has values in $\mathbb{Q}_{p^a}$, then θ^G is rational if and only if $\theta^\sigma = \theta^y$ for some $y \in G$.*

(b) *If $n \in N$ is a p-element of order dividing p^a, then $\mathrm{cl}_G(n)$ is rational if and only if $\mathrm{cl}_N(n)^\sigma = \mathrm{cl}_N(n)^y$ for some $y \in G$.*

Proof. We know that the equation $\theta^g = \theta^\tau$ defines a map $T^* \to \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q}(\theta^G))$. In case (a), $\tau = \sigma_{\mathbb{Q}(\theta)} \in \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q}(\theta^G))$. Since τ generates $\mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})$, part (a) follows. Part (b) is proved similarly using Lemma 3.1. □

Now we are able to prove Theorem 1.1.

Theorem 4.5. *Suppose that G has a cyclic Sylow 2-subgroup Q. Then $|\mathrm{Irr}_{\mathbb{Q}}(G)| = |\mathrm{Cl}_{\mathbb{Q}}(G)|$.*

Proof. Let N be the normal 2-complement of G. If p is an odd prime, let $\mathrm{Irr}_{\mathbb{Q},p}(G)$ be the set of rational-valued $\chi \in \mathrm{Irr}(G)$ lying over some non-trivial B_p-character $\theta \in B_p(N)$. By Theorem 4.3 and Lemma 2.3, we have that $\mathrm{Irr}_{\mathbb{Q}}(G)$ is the disjoint union

$$\mathrm{Irr}_{\mathbb{Q}}(G) = \mathrm{Irr}_{\mathbb{Q}}(G/N) \cup \bigcup_{p \mid |N|} \mathrm{Irr}_{\mathbb{Q},p}(G).$$

Notice that $\mathrm{Irr}_{\mathbb{Q}}(G/N)$ has size 2. Also, let $\mathrm{Cl}_{\mathbb{Q},p}(G)$ be the set of rational classes $\mathrm{cl}_G(x)$ such that $1 \neq x_{2'}$ is a p-element of N. Suppose that $\mathrm{cl}_G(x)$ is rational, and write $x = nh$, where $[n,h] = 1$ and $h \in Q$. Since $n = x_{2'}$ is rational (see, for example, [**9**, Lemma 5.1 (d)]) we have that n is a p-element by Theorem 4.3. Now, by Lemma 3.2 (b), $n = 1$ if and only if $\mathrm{cl}_G(x)$ is either 1 or the unique class of involutions of G. Hence,

$$|\mathrm{Cl}_{\mathbb{Q}}(G)| = 2 + \sum_{p \mid |N|} |\mathrm{Cl}_{\mathbb{Q},p}(G)|.$$

It then suffices to show that

$$|\mathrm{Irr}_{\mathbb{Q},p}(G)| = |\mathrm{Cl}_{\mathbb{Q},p}(G)|.$$

Let $\langle\sigma\rangle = \mathrm{Gal}(\mathbb{Q}_{|G|_p}/\mathbb{Q})$. By Theorem 4.2, we have that the actions of $Q \times \langle\sigma\rangle$ on $B_p(N)$ and $\mathrm{Cl}_p(N)$ are permutation isomorphic. Hence, using Lemma 4.4, we may choose representatives $\{\theta_i\}_{1\leqslant i\leqslant s}$ of the Q-action on the $B_p(N)$-characters lying over rational characters and $\{C_i\}_{1\leqslant i\leqslant s}$ of the conjugacy classes of p-elements of N with C_i^G being rational, such that $\boldsymbol{N}_G(C_i) = I_G(\theta_i)$. Now the theorem follows from Corollary 2.5 and Theorem 3.4. □

5. Remarks and an example

As we mentioned in § 1, by working in similar fashion to that above it is possible to prove the following result.

Theorem 5.1. *Suppose that G has a normal Sylow p-subgroup P, where p is odd. Suppose that for every section $X \lhd Y \subseteq G/P$, where Y/X is abelian, the number of Y-invariant rational characters of X is the number of Y-invariant rational classes of X. Then $|\mathrm{Irr}_{\mathbb{Q}}(G)| = |\mathrm{Cl}_{\mathbb{Q}}(G)|$.*

For instance, if Q is an abelian, dihedral, generalized quaternion or semi-dihedral 2-group, then it is possible to prove that for every subgroup X of Q there is a natural bijection between $\mathrm{Irr}_{\mathbb{Q}}(X)$ and $\mathrm{Cl}_{\mathbb{Q}}(X)$. In particular, by Theorem 5.1, we have that groups of the form $G = PQ$, where $P \lhd G$ and Q are as before, satisfy that $|\mathrm{Irr}_{\mathbb{Q}}(G)| = |\mathrm{Cl}_{\mathbb{Q}}(G)|$.

Finally, our next example shows that we cannot greatly extend our hypotheses in Theorem 1.1.

Example 5.2. There is a finite group G of order $2^2 \cdot 3^4 \cdot 7$ with an elementary abelian Sylow 2-subgroup, having a normal 2-complement and such that the number of rational characters of G is not the number of rational classes of G.

Let $X = \langle u, v, w\rangle = C_9 \times C_3 \times C_7$, where $o(u) = 9$, $o(v) = 3$ and $o(w) = 7$. Consider the automorphism σ of X of order 3 with $u^\sigma = u$, $v^\sigma = u^3 v$, $w^\sigma = w^2$. Now consider the automorphisms τ, ρ of order 2 such that τ inverts the 3-elements of X and fixes the 7-elements, and ρ fixes the 3-elements but inverts the 7-elements. Then it can be checked in Groups, Algorithms and Programming (GAP) that the semidirect product

$$G = X\langle\sigma, \tau, \rho\rangle$$

has 15 rational classes and 18 rational characters. Now, since $Q_8/\boldsymbol{Z}(Q_8) = C_2 \times C_2 = D_8/\boldsymbol{Z}(S_8)$, we can modify this group to obtain similar examples with dihedral or quaternion Sylow 2-subgroups.

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