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Similarities and differences between specification and non-uniform specification

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Abstract. Pavlov [*Adv. Math.* **295** (2016), 250–270; *Nonlinearity* **32** (2019), 2441–2466] studied the measures of maximal entropy for dynamical systems with weak versions of specification property and found the existence of intrinsic ergodicity would be influenced by the assumptions of the gap functions. Inspired by these, in this article, we study the dynamical systems with non-uniform specification property. We give some basic properties these systems have and give an assumption for the gap functions to ensure the systems have the following five properties: CO-measures are dense in invariant measures; for every non-empty compact connected subset of invariant measures, its saturated set is dense in the total space; ergodic measures are residual in invariant measures; ergodic measures are connected; and entropy-dense. In addition, we will give examples to show the assumption is optimal.

Key words: specification, invariant measure, periodic orbit, generic point, subshift
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1. *Introduction*

Throughout this paper, suppose that (X, T) is a topological dynamical system, which means that (X, d) is a compact metric space and $T : X \to X$ is a continuous map. Let $\mathcal{M}(X)$, $\mathcal{M}_T(X)$, and $\mathcal{M}_T^{\mathrm{erg}}(X)$ denote the space of Borel probability measures, T-invariant Borel probability measures, and T-ergodic Borel probability measures, respectively. Let $\mathbb{Z}$, $\mathbb{N}_0$, and $\mathbb{N}$ denote integers, non-negative integers, and positive integers, respectively. Let $C(X)$ denote the space of real continuous functions on X with the norm $\|\varphi\| := \sup_{x\in X} |\varphi(x)|$ for any $\varphi \in C(X)$. A subset of a Baire space is said to be residual if it has a dense G_δ subset.

Let ρ be the first Wasserstein metric on $\mathcal{M}(X)$, a metrization of the weak* topology of $\mathcal{M}(X)$, see [33] for more information. By [33, p. 95, equation (6.3)], for any $\mu, \nu \in \mathcal{M}(X)$,

$$\rho(\mu, \nu) = \sup_{f \in \mathrm{Lip}^1(X)} \left| \int_X f \, d\mu - \int_X f \, d\nu \right|, \tag{1.1}$$

where $\mathrm{Lip}^1(X)$ is the space of all real Lipschitz continuous functions on X, whose Lipschitz constants are bounded by 1. Then we have $\rho(\delta_x, \delta_y) = d(x, y)$ for any $x, y \in X$.

Given $x \in X$, for any $n \in \mathbb{N}$, denote $\delta_x^n := (1/n) \sum_{i=0}^{n-1} \delta_{T^i x}$. Let $V_T(x)$ denote the set of accumulation points of $\{\delta_x^n : n \geq 1\}$, then $V_T(x)$ is a non-empty compact connected subset of $\mathcal{M}_T(X)$ [8, Proposition 3.8]. For any non-empty compact connected subset $K \subset \mathcal{M}_T(X)$, denote $G_K := \{x \in X : V_T(x) = K\}$ (called *saturated set*) and $G^K := \{x \in X : V_T(x) \supset K\}$. For convenience, when $K = \{\mu\}$ for some $\mu \in \mathcal{M}_T(X)$, we denote $G_\mu = G_{\{\mu\}}$, $G^\mu = G^{\{\mu\}}$, and every $x \in G_\mu$ is called a generic point of μ.

Specification-like properties, which were first considered by Bowen [3] in the study of Axiom A diffeomorphisms, play important roles in the study of uniqueness of equilibrium states, density of periodic measures, the existence of saturated sets, etc. The existence of saturated sets was proved by Sigmund [30] for systems with uniform hyperbolicity or specification property and generalized to systems with non-uniform hyperbolicity [19], g-almost product property [26], and asymptotic average shadowing property [9]. In addition, the notions of closability and linkability were introduced by Gelfert and Kwietniak in [12] to give a general method to show the density of periodic measures in the ergodic measures and the existence of saturated sets. We refer to [16] for a survey of many results for specification-like properties.

In this article, we mainly consider dynamical systems with the non-uniform specification property.

Definition 1.1. We say that a dynamical system (X, T) satisfies the *non-uniform specification property* with $M(n, \varepsilon)$ if the following holds: there exists a function $M(n, \varepsilon) : \mathbb{N} \times (0, \infty) \to \mathbb{N}$, such that:

- $M(n, \varepsilon)$ is non-decreasing with n and non-increasing with ε;
- for any integer $k \geq 2$, for any points $x_1, \ldots, x_k \in X$, for any non-negative integers $a_1, b_1, \ldots, a_k, b_k$ with

$$a_1 \leq b_1 < \cdots < a_k \leq b_k$$

and

$$a_{i+1} - b_i \geq M(b_i - a_i + 1, \varepsilon) \quad \text{for } 1 \leq i \leq k - 1,$$

there exists a point $z \in X$ such that

$$d(T^{n-a_i} x_i, T^n z) \leq \varepsilon \quad \text{for } a_i \leq n \leq b_i, 1 \leq i \leq k.$$

If further, for any integer p with $p \geq b_k - a_1 + M(b_k - a_k + 1, \varepsilon)$, z can be chosen as a periodic point with $T^p z = z$, then we say that (X, T) satisfies the *non-uniform periodic specification property* with $M(n, \varepsilon)$.

Remark 1.2

(1) This definition is modified from the definitions of weak specification property [16, Definition 14] and non-uniform specification for subshifts [23, Definition 2.14]. In fact, the weak specification property is equivalent to the non-uniform specification property with $\sup_{\varepsilon>0}\lim_{n\to\infty}(M(n,\varepsilon)/n)=0$. Compared with many other definitions of specification-like properties, in this definition, we do not give the assumptions of the asymptotic behavior of $M(n,\varepsilon)$. In particular, when $M(n,\varepsilon)=M(1,\varepsilon)=M(\varepsilon)$ for any $n\in\mathbb{N}$, the definition of non-uniform (periodic) specification property with $M(n,\varepsilon)$ is consistent with the definition of (periodic) specification property with $M(\varepsilon)$.

(2) The weak specification property was first used by Marcus [22] without a name to show the density of periodic measures in the invariant measures for ergodic toral automorphisms. It was consistent with the almost weak specification property introduced by Dateyama [7]. A remarkable result for dynamical systems with weak specification property is the universality, which was first shown by Quas and Soo [27] under additional conditions: asymptotic entropy expansiveness and the small boundary property; later, Burguet [4] proved that these additional conditions can be removed.

Let $|A|$ denote the cardinality of the set A. According to Lemma 2.11, if $|X|\geq 2$ and (X,T) satisfies the non-uniform specification property with $M(n,\varepsilon)$, then $\lim_{\varepsilon\to 0}M(1,\varepsilon)=\infty$.

Let $\mathrm{Per}_n(T):=\{x\in X:T^n x=x\}$ denote the set of periodic points of period n and $\mathrm{Per}(T):=\bigcup_{n\geq 1}\mathrm{Per}_n(T)$ denote the set of periodic points. Following [8, Definition 21.7], a measure supported on the orbit of a periodic point x is called a *CO-measure* of x and the set of CO-measures is denoted by

$$\mathcal{M}_T^{\mathrm{co}}(X):=\bigcup_{n\geq 1}\mathcal{M}_{T,n}^{\mathrm{co}}(X),$$

where

$$\mathcal{M}_{T,n}^{\mathrm{co}}(X):=\{\delta_x^n : x\in\mathrm{Per}_n(T), n\geq 1\}.$$

It is clear that we always have the following:

$$\mathcal{M}_T^{\mathrm{co}}(X)\subset\mathcal{M}_T^{\mathrm{erg}}(X)\subset\mathcal{M}_T(X).$$

The phenomenon that $\overline{\mathcal{M}_T^{\mathrm{co}}(X)}=\mathcal{M}_T(X)$ was shown by Sigmund [29] to occur for Axiom A diffeomorphisms. Afterward, a similar result was extended by Hirayama [13] to $C^{1+\alpha}$ diffeomorphism preserving mixing hyperbolic measures. More precisely, given a $C^{1+\alpha}$ diffeomorphism preserving a mixing hyperbolic measure μ, there exists Λ with $\mu(\Lambda)=1$ such that the set of CO-measures supported by hyperbolic periodic points is dense in the set of invariant measures supported by Λ. After that, it was shown by Liang *et al* [18] that the assumption of mixing can be removed. In the C^1 case, it was shown by Abdenur *et al* [1], for an isolated non-trivial transitive set Λ of a C^1-generic diffeomorphism f, $\overline{\mathcal{M}_T^{\mathrm{co}}(\Lambda)}=\mathcal{M}_T(\Lambda)$. Moreover, it was shown by Gelfert and

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Kwietniak [12] that $\overline{\mathcal{M}_T^{\mathrm{co}}(X)} = \mathcal{M}_T(X)$ provided that $\mathrm{Per}(T)$ is linkable and $\overline{\mathcal{M}_T^{\mathrm{co}}(X)} = \overline{\mathcal{M}_T^{\mathrm{erg}}(X)}$. Recently, it was shown by Hou *et al* [15] that for transitive dynamical systems satisfying the periodic shadowing property, the CO-measures with large supports are dense in invariant measures.

For any $\mu \in \mathcal{M}_T(X)$, let $S_\mu := \{x \in X : \mu(U) > 0$ for any neighborhood U of $x\}$ denote the *support* of μ. Let $C_T(X) := \overline{\bigcup_{\mu \in \mathcal{M}_T(X)} S_\mu}$ denote the *measure center* of (X, T). Let $h_{\mathrm{top}}(T)$ denote the *topological entropy* of (X, T). Given $\delta > 0$ and $n \in \mathbb{N}$, let

$$\mathcal{M}_{T,n,\delta}^{\mathrm{co}}(X) := \{\mu \in \mathcal{M}_{T,n}^{\mathrm{co}}(X) : d_H(S_\mu, X) < \delta\},$$

$$\mathcal{M}_{T,\delta}^{\mathrm{co}}(X) := \bigcup_{n \geq 1} \mathcal{M}_{T,n,\delta}^{\mathrm{co}}(X),$$

and

$$\mathrm{Per}_{n,\delta}(T) := \{x \in \mathrm{Per}_n(T) : \delta_x^n \in \mathcal{M}_{T,n,\delta}^{\mathrm{co}}(X)\},$$

where d_H is the Hausdorff distance.

A topological dynamical system (X, T) is said to be *positively expansive* if there exists $c > 0$, which is called a positively expansive constant, such that $x \neq y \in X$ implies $d(T^n x, T^n y) > c$ for some $n \in \mathbb{N}_0$. If further T is a homeomorphism, (X, T) is said to be *expansive* if there exists $c > 0$, which is called an expansive constant, such that $x \neq y \in X$ implies $d(T^n x, T^n y) > c$ for some $n \in \mathbb{Z}$. Inspired by these, we introduce the notion of pseudo-expansiveness. A topological dynamical system (X, T) is said to be *pseudo-expansive* if there exists $c > 0$, which is called a pseudo-expansive constant, such that for any $x, y \in X$, $\limsup_{i\to\infty} d(T^i x, T^i y) > 0$ implies $d(T^n x, T^n y) > c$ for some $n \in \mathbb{N}_0$. Directly from the definition, pseudo-expansiveness is weaker than positively expansiveness. In fact, pseudo-expansiveness is also weaker than expansiveness, see Lemma 2.5.

For any $f \in C(X)$, denote the *f-irregular set* by

$$I_f(T) := \left\{x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) \text{ diverges}\right\},$$

and

$$\mathcal{C}(T) := \{f \in C(X) : I_f(T) \neq \emptyset\}.$$

As for dynamical systems with specification-like properties, the f-irregular set has been studied a lot in the sense of residuality, topological entropy, topological pressure, metric mean dimension, and so on. For examples, the f-irregular set is either empty or residual in X for dynamical systems satisfying the specification property [17]; is either empty or carrying full topological entropy for dynamical systems satisfying the specification property [5] or almost specification property [32]; is either empty or carrying full topological pressure for dynamical systems satisfying the specification property [31] or gluing orbit property [20]; is either empty or carrying full metric mean dimension for dynamical systems satisfying the gluing orbit property [20] or almost weak specification property [21].

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According to Theorem 3.2, when (X, T) satisfies the non-uniform periodic specification property, we always have the following:

(1) $\overline{\mathrm{Per}(T)} = X$ and $C_T(X) = X$;

(2) (X, T) is topologically mixing;

(3) if (X, T) is pseudo-expansive, then

$$h_{\mathrm{top}}(T) \geq \limsup_{n\to\infty} \frac{1}{n} \log |\mathrm{Per}_n(T)|;$$

(4) for every compact subset $K \subset \mathcal{M}_T(X)$ with $G^K \neq \emptyset$, we have G^K is residual in X. If further, (X, T) is pseudo-expansive, then for every non-empty compact connected subset $K \subset \mathcal{M}_T(X)$ with $G_K \neq \emptyset$, G_K is dense in X.

If further, $|X| \geq 2$, then

(5) $h_{\mathrm{top}}(T) > 0$ and for any $\delta > 0$,

$$\limsup_{n\to\infty} \frac{1}{n} \log |\mathrm{Per}_n(T)| \geq \limsup_{n\to\infty} \frac{1}{n} \log |\mathrm{Per}_{n,\delta}(T)| > 0;$$

(6) (X, T) is DC2 chaotic;

(7) (X, T) is Devaney chaotic;

(8) $\mathcal{C}(T) \neq \emptyset$ and $\bigcap_{f\in\mathcal{C}(T)} I_f(T)$ is residual in X.

The definitions of DC2 chaos and Devaney chaos can be found in §2.7.

Given $\mu \in \mathcal{M}_T(X)$, let $h_\mu(T)$ denote the metric entropy of μ. Here, μ is said to be a measure of maximal entropy if $h_\mu(T) = h_{\mathrm{top}}(T)$. A dynamical system (X, T) is said to be intrinsically ergodic if there exists a unique measure of maximal entropy. The intrinsic ergodicity of subshifts having the non-uniform specification property with $\sup_{\varepsilon>0} \lim_{n\to\infty}(M(n, \varepsilon)/n) = 0$ was studied by Pavlov [23]. It was shown in [23] by giving examples, when $\inf_{\varepsilon>0} \liminf_{n\to\infty}(M(n, \varepsilon)/\log n) > 0$, we cannot guarantee the subshifts are intrinsically ergodic. After that, in [24], the controlled specification property with gap function $f(n)$ was introduced by Pavlov and $\liminf_{n\to\infty}(f(n)/n) = 0$ was proved to be the critical condition to guarantee the intrinsic ergodicity. Inspired by these, the goal of this article is to study that under which assumptions of $M(n, \varepsilon)$, a dynamical system (X, T) satisfying non-uniform periodic specification property with $M(n, \varepsilon)$ has the following ergodic properties:

(R1) $\mathcal{M}_T^{\mathrm{co}}(X)$ is dense in $\mathcal{M}_T(X)$;

(R2) for any non-empty compact connected subset $K \subset \mathcal{M}_T(X)$, G_K is dense in X;

(R3) $\mathcal{M}_T^{\mathrm{erg}}(X)$ is residual in $\mathcal{M}_T(X)$;

(R4) $\mathcal{M}_T^{\mathrm{erg}}(X)$ is connected;

(R5) (X, T) is entropy-dense.

Remark 1.3. Since $\mathcal{M}_T^{\mathrm{erg}}(X)$ is the set of extreme points of $\mathcal{M}_T(X)$ [8, Proposition 5.6] and is a G_δ subset of $\mathcal{M}_T(X)$ [8, Proposition 5.7], $\mathcal{M}_T(X)$ is a Choquet simplex [28, §2.2], when (X, T) is not uniquely ergodic, property (R3) is equal to that $\mathcal{M}_T(X)$ is a Poulsen simplex, that is, a non-trivial Choquet simplex whose extreme points are dense in it, see [16, §1.3] for more information.

When $\sup_{\varepsilon>0} \liminf_{n\to\infty}(M(n,\varepsilon)/n) = 0$, according to Theorems 3.4 and 3.6, properties (R1), (R2), (R3), (R4), and (R5) hold.

When $\sup_{\varepsilon>0} \liminf_{n\to\infty}(M(n,\varepsilon)/n) > 0$, we cannot guarantee that properties (R1), (R2), (R3), (R4), or (R5) hold. More precisely, we have the following theorems.

THEOREM A. *Let $M(n,\varepsilon) : \mathbb{N} \times (0,\infty) \to \mathbb{N}$ be a function such that $M(n,\varepsilon)$ is non-decreasing with n and non-increasing with ε. Suppose*

$$\lim_{\varepsilon\to 0} M(1,\varepsilon) = \infty$$

and

$$\sup_{\varepsilon>0} \liminf_{n\to\infty} \frac{M(n,\varepsilon)}{n} > 0.$$

Then there exist $c_0 > 0$ and a cluster of dynamical systems $\{(X_c, T_c)\}_{0<c\le c_0}$ satisfying the non-uniform periodic specification property with $M(n,\varepsilon)$, such that for any $0 < c \le c_0$:

(1)

$$\overline{\mathcal{M}^{\mathrm{co}}_{T_c}(X_c)} = \overline{\mathcal{M}^{\mathrm{erg}}_{T_c}(X_c)} \subsetneq \mathcal{M}_{T_c}(X_c)$$

and

$$\overline{\mathcal{M}^{\mathrm{co}}_{T_c,\delta}(X_c)} \subsetneq \overline{\mathcal{M}^{\mathrm{co}}_{T_c}(X_c)}$$

for $\delta > 0$ sufficiently small;

(2)

$$h_{\mathrm{top}}(T_c) = \lim_{n\to\infty} \frac{1}{n} \log |\mathrm{Per}_{n,\delta}(T_c)| > 0,$$

for any $\delta > 0$;

(3) *(X_c, T_c) does not satisfy properties (R4) and (R5).*

THEOREM B. *For the cluster of dynamical systems $\{(X_c, T_c)\}_{0<c\le c_0}$ from Theorem A and any $0 < c \le c_0$, there exists some $\mu \in \mathcal{M}_{T_c}(X_c)$ with $G_\mu = \emptyset$. Moreover, there exists a non-empty open subset $\mathcal{U} = \mathcal{M}_{T_c}(X_c) \setminus \overline{\mathcal{M}^{\mathrm{co}}_{T_c}(X_c)}$, such that:*

(1) *if $\mu \in \mathcal{U}$, then $G_\mu = \emptyset$;*
(2) *if $\mu \notin \mathcal{U}$, then G_μ is dense in X_c;*
(3) *there exists $\mu_0 \in \mathcal{U}$ such that G^{μ_0} is residual in X_c.*

1.1. *Organization of this paper.* In §2, we will introduce some preliminary results. In §3, we will study some basic properties of dynamical systems with non-uniform specification property. In §4, we will prove Theorem A by constructing a cluster of subshifts of $\{0,1\}^{\mathbb{N}_0}$. In §5, we will prove Theorem B by using the cluster of subshifts constructed in §4.

2. *Preliminaries*

2.1. *The first Wasserstein metric on $\mathcal{M}(X)$.* Let ρ be the first Wasserstein metric on $\mathcal{M}(X)$, then the following can be easily checked by using equation (1.1).

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PROPOSITION 2.1

(1) *Given* $\mu, \mu_1, \ldots, \mu_t \in \mathcal{M}(X)$ *and* $0 \le s_1, \ldots, s_t \le 1$ *with* $\sum_{i=1}^{t} s_i = 1$, *then*

$$\rho\bigg(\mu, \sum_{i=1}^{t} s_i \mu_i\bigg) \le \sum_{i=1}^{t} s_i \rho(\mu, \mu_i).$$

(2) *Given* $\mu_1, \ldots, \mu_t, \nu_1, \ldots, \nu_t \in \mathcal{M}(X)$ *and* $0 \le s_1, \ldots, s_t \le 1$ *with* $\sum_{i=1}^{t} s_i = 1$, *then*

$$\rho\bigg(\sum_{i=1}^{t} s_i \mu_i, \sum_{i=1}^{t} s_i \nu_i\bigg) \le \sum_{i=1}^{t} s_i \rho(\mu_i, \nu_i).$$

LEMMA 2.2. *For any* $0 \le \varepsilon, \delta \le 1$ *and any two sequences* $\{x_i\}_{i=0}^{n-1}$, $\{y_i\}_{i=0}^{n-1}$ *of* X, *let* $\Lambda = \{0 \le i \le n-1 : d(x_i, y_i) \le \varepsilon\}$. *If* $(n - |\Lambda|)/n \le \delta$, *then*

$$\rho\bigg(\frac{1}{n}\sum_{i=0}^{n-1} \delta_{x_i}, \frac{1}{n}\sum_{i=0}^{n-1} \delta_{y_i}\bigg) \le \varepsilon + \delta \mathrm{diam}(X),$$

where $\mathrm{diam}(X) := \sup_{x,y \in X} d(x, y)$.

Proof.

$$\begin{aligned}\rho\bigg(\frac{1}{n}\sum_{i=0}^{n-1} \delta_{x_i}, \frac{1}{n}\sum_{i=0}^{n-1} \delta_{y_i}\bigg) &\le \frac{1}{n}\sum_{i=0}^{n-1} \rho(\delta_{x_i}, \delta_{y_i}) \\ &= \frac{1}{n}\bigg(\sum_{i \in \Lambda} \rho(\delta_{x_i}, \delta_{y_i}) + \sum_{i \notin \Lambda} \rho(\delta_{x_i}, \delta_{y_i})\bigg) \\ &\le \frac{1}{n}[\varepsilon|\Lambda| + \mathrm{diam}(X)(n - |\Lambda|)] \\ &\le \varepsilon + \delta \mathrm{diam}(X).\end{aligned}$$

□

LEMMA 2.3. *Suppose* $\{a_i : i \in \Lambda\}$ *is a finite subset of* $\{x \in \mathbb{R} : |x| \le 1\}$, *then for every non-empty set* $S \subset \Lambda$, *one has*

$$\bigg|\frac{1}{|S|}\sum_{i \in S} a_i - \frac{1}{|\Lambda|}\sum_{i \in \Lambda} a_i\bigg| \le \frac{2(|\Lambda| - |S|)}{|\Lambda|}.$$

Proof. Clearly,

$$\begin{aligned}\bigg|\frac{1}{|S|}\sum_{i \in S} a_i - \frac{1}{|\Lambda|}\sum_{i \in \Lambda} a_i\bigg| &= \bigg|\frac{|\Lambda| - |S|}{|\Lambda||S|}\sum_{i \in S} a_i - \frac{1}{|\Lambda|}\sum_{i \in \Lambda \setminus S} a_i\bigg| \\ &\le \frac{|\Lambda| - |S|}{|\Lambda||S|}\sum_{i \in S} |a_i| + \frac{1}{|\Lambda|}\sum_{i \in \Lambda \setminus S} |a_i| \\ &\le \frac{2(|\Lambda| - |S|)}{|\Lambda|}.\end{aligned}$$

This completes the proof of Lemma 2.3. □

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2.2. *Pseudo-expansive.* The goal of this subsection is to show that pseudo-expansiveness is weaker than expansiveness.

LEMMA 2.4. *Suppose that T is a homeomorphism and (X, T) is expansive with an expansive constant c, then for any $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that for every $x, y \in X$ with $d(T^i x, T^i y) \le c$ for any $-N \le i \le N$, we have $d(x, y) < \varepsilon$.*

Proof. Otherwise, there exists $\varepsilon_0 > 0$ such that for any $n \in \mathbb{N}$, there exists $x_n, y_n \in X$ with $d(T^i x_n, T^i y_n) \le c$ for any $-n \le i \le n$ and $d(x_n, y_n) \ge \varepsilon_0$. Since X is compact, we can suppose that there exist $0 < n_1 < n_2 < \cdots$ such that $\lim_{j\to\infty} x_{n_j} = x$ and $\lim_{j\to\infty} y_{n_j} = y$, then $d(x, y) \ge \varepsilon_0$. Fix $k \in \mathbb{Z}$, there exists $l \in \mathbb{N}$ such that for any $j \ge l$, we have that $n_j \ge |k|$, and hence, $d(T^k x_{n_j}, T^k y_{n_j}) \le c$, so letting $j \to \infty$, we have that $d(T^k x, T^k y) \le c$. By the expansiveness of (X, T), we have that $x = y$, which contradicts with $d(x, y) \ge \varepsilon_0$. □

LEMMA 2.5. *Suppose that T is a homeomorphism and (X, T) is expansive with an expansive constant c, then (X, T) is pseudo-expansive with a pseudo-expansive constant c.*

Proof. From the definition of pseudo-expansiveness, we only need to show that $\lim_{i\to\infty} d(T^i x, T^i y) = 0$ provided that $d(T^n x, T^n y) \le c$ for any $n \in \mathbb{N}_0$. For such x, y, given $\varepsilon > 0$, let $N = N(\varepsilon)$ be chosen as in Lemma 2.4, then for any $n \ge N$, we have that $d(T^i(T^n x), T^i(T^n y)) \le c$ for any $-N \le i \le N$, and hence, $d(T^n x, T^n y) < \varepsilon$. As a result, $\lim_{i\to\infty} d(T^i x, T^i y) = 0$. □

2.3. *Closability and linkability.* The (n, ε) Bowen ball at x is denoted by

$$B_n(x, \varepsilon) := \{y \in X : d(T^i x, T^i y) < \varepsilon \text{ for } 0 \le i \le n-1\}.$$

Given $K \subset \mathrm{Per}(T)$, denote

$$\mathcal{M}_T^{\mathrm{co}}(K) := \{\delta_x^n : x \in \mathrm{Per}_n(T) \cap K, n \ge 1\}.$$

Definition 2.6. [12, Definition 4.2] A point $x \in X$ is *closable* with respect to a non-empty set $K \subset \mathrm{Per}(T)$, or simply K-closable if, for every $\varepsilon > 0$ and $N > 0$, there exist $p = p(x, \varepsilon, N)$, $q = q(x, \varepsilon, N) \in \mathbb{N}$ such that there is $y \in B_p(x, \varepsilon) \cap K$ satisfying $T^q(y) = y$ and $N \le p \le q \le (1+\varepsilon)p$.

Definition 2.7. [12, Definition 4.5] A measure $\mu \in \mathcal{M}_T(X)$ is K-closable if some generic point of μ is K-closable. A dynamical system (X, T) is K-closable if every $\mu \in \mathcal{M}_T^{erg}(X)$ is K-closable.

LEMMA 2.8. [12, Theorem 4.11] *If (X, T) is K-closable for some $K \subset \mathrm{Per}(T)$, then $\mathcal{M}_T^{\mathrm{co}}(K)$ is dense in $\mathcal{M}_T^{\mathrm{erg}}(X)$.*

Definition 2.9. [12, Definition 4.12] A set $K \subset \mathrm{Per}(T)$ is *linkable* if for every $y_1, y_2 \in K$, $\varepsilon > 0$, and $\lambda \in [0, 1]$, there exist $p_1, p_2, q_1, q_2 \in \mathbb{N}$ and $z \in K$ with $T^{q_2} z = z$ satisfying:

(1) $\lambda - \varepsilon \le p_1/(p_1 + p_2) \le \lambda + \varepsilon$;

(2) $p_j \le q_j - q_{j-1} \le (1+\varepsilon)p_j$ and $T^{q_{j-1}} z \in B_{p_j}(y_j, \varepsilon)$ for $q_0 = 0$, $j = 1, 2$.

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LEMMA 2.10. [12, Theorem 6.1] *Suppose that* (X, T) *is a dynamical system,* $K \subset \mathrm{Per}(T)$ *is linkable, and* $\mathcal{M}_T^{\mathrm{co}}(K)$ *is dense in* $\mathcal{M}_T^{\mathrm{erg}}(X)$*, then:*

(1) $\mathcal{M}_T^{\mathrm{co}}(K)$ *is dense in* $\mathcal{M}_T(X)$*;*

(2) *for every non-empty compact connected subset* $V \subset \mathcal{M}_T(X)$*,* $C_T(X) = \overline{G_V \cap C_T(X)}$*.*

2.4. *Metric entropy and topological entropy*

2.4.1. *Metric entropy.* We call $(X, \mathcal{B}, \mu)$ a probability space if $\mathcal{B}$ is a Borel σ-algebra on X and μ is a probability measure on X. For a finite measurable partition $\xi = \{A_1, \ldots, A_n\}$ of $(X, \mathcal{B}, \mu)$, define

$$H_\mu(\xi) = -\sum_{i=1}^{n} \mu(A_i) \log \mu(A_i).$$

Let $T : X \to X$ be a continuous map preserving μ. We denote by $\bigvee_{i=0}^{n-1} T^{-i}\xi$ the partition whose elements are the sets $\bigcap_{i=0}^{n-1} T^{-i} A_{j_i}$, $1 \le j_i \le n$. Then the following limit exists:

$$h_\mu(T, \xi) = \lim_{n\to\infty} \frac{1}{n} H_\mu\bigg(\bigvee_{i=0}^{n-1} T^{-i}\xi\bigg)$$

and we define the metric entropy of μ as

$$h_\mu(T) := \sup\{h_\mu(T, \xi) : \xi \text{ is a finite measurable partition of } X\}.$$

2.4.2. *Topological entropy.* Given $n \in \mathbb{N}$ and $\varepsilon > 0$, a subset $E \subset X$ is said to be (n, ε)-separated if for any two distinct points $x, y \in E$, there exists $0 \le k \le n-1$ such that $d(T^k x, T^k y) > \varepsilon$. The largest cardinality of an (n, ε)-separated subset of X is denoted by $s(n, \varepsilon)$, then the topological entropy of (X, T) is defined as

$$h_{\mathrm{top}}(T) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s(n, \varepsilon).$$

2.5. *Symbolic dynamics.* Let A be a finite alphabet, $A^{\mathbb{N}_0} := \{x_0x_1 \cdots : x_i \in A, i \in \mathbb{N}_0\}$, the one-side full shift over A is $(A^{\mathbb{N}_0}, T)$, where $T(x_0x_1x_2\ldots) = x_1x_2\ldots$ for any $x_0x_1x_2\ldots \in A^{\mathbb{N}_0}$.

2.5.1. *Words.* A member w of $A^{\{i,i+1,\ldots,j\}}$ for some $0 \le i \le j$ is called a word over A and its length $j - i + 1$ is denoted by $|w|$. Let $A^* := \bigcup_{0\le i\le j} A^{\{i,i+1,\ldots,j\}}$ denote the set of all words over A. For $n \in \mathbb{N}$, denote $A^n := A^{\{0,\ldots,n-1\}}$. Given a subset $\mathcal{F} \subset A^*$, we can define the set $X = X(\mathcal{F}) := \{x \in A^{\mathbb{N}_0} : x_ix_{i+1} \cdots x_j \notin \mathcal{F} \text{ for any } 0 \le i \le j\}$, and it is well known that X is a subshift of $A^{\mathbb{N}_0}$ and $\mathcal{F}$ is called the forbidden words of X. Given $w \in A^*$ and $n \in \mathbb{N}_0 \cup \{\infty\}$, we define $w^n := \underbrace{ww\cdots w}_{n \text{ items}}$.

2.5.2. *Language.* The language of a subshift X, denoted by $\mathcal{L}(X)$, is the set of all words which appear in points of X. For any $n \in \mathbb{N}$, denote $\mathcal{L}_n(X) := \mathcal{L}(X) \cap A^n$.

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2.5.3. *Cylinder sets.* Given a subshift X and a word $w \in \mathcal{L}_n(X)$, the cylinder set $[w]$ is the set of all $x \in X$ with $x_0 x_1 \cdots x_{n-1} = w$.

2.5.4. *Topological entropy.* The topological entropy of a subshift X can be calculated by using the language of X,

$$h_{\mathrm{top}}(T) = \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{L}_n(X)|.$$

2.6. *Gap functions.* In this subsection, we study the gap function $M(n, \varepsilon)$ appearing in the definition of non-uniform specification property.

LEMMA 2.11. *If $|X| \geq 2$ and (X, T) satisfies the non-uniform specification property with $M(n, \varepsilon)$, then*

$$\lim_{\varepsilon\to 0} M(1, \varepsilon) = \infty.$$

Proof. Recall that $M(1, \varepsilon)$ is non-increasing with ε, suppose for a contradiction that $\lim_{\varepsilon\to 0} M(1, \varepsilon) < \infty$. Choose a positive integer N sufficiently large such that $M(1, \varepsilon) \leq N$ for all $0 < \varepsilon < 1$.

Fix $x, y \in X$ such that $T^N x \neq y$. Let $\delta = \frac{1}{3} d(T^N x, y)$. Since X is compact, there exists $0 < \varepsilon < \delta$ such that

$$d(T^N w_1, T^N w_2) < \delta$$

whenever $d(w_1, w_2) < 2\varepsilon$. By the non-uniform specification property, there exists a $z \in X$ such that

$$d(x, z) \leq \varepsilon \quad \text{and} \quad d(y, T^N z) \leq \varepsilon,$$

and thus

$$d(T^N x, T^N z) < \delta \quad \text{and} \quad d(y, T^N z) < \delta.$$

This implies

$$\delta = \tfrac{1}{3} d(T^N x, y) \leq \tfrac{1}{3}(d(T^N x, T^N z) + d(T^N z, y)) < \tfrac{2}{3}\delta,$$

which is a contradiction. This completes the proof of Lemma 2.11. □

LEMMA 2.12. *Let $M(n, \varepsilon) : \mathbb{N} \times (0, \infty) \to \mathbb{N}$ be a function such that $M(n, \varepsilon)$ is non-decreasing with n and non-increasing with ε. Suppose*

$$\lim_{\varepsilon\to 0} M(1, \varepsilon) = \infty$$

and

$$\sup_{\varepsilon>0} \liminf_{n\to\infty} \frac{M(n, \varepsilon)}{n} > 0.$$

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Then there exists a non-increasing sequence of positive real numbers $\{\varepsilon_k\}_{k\geq 1}$ *with* $\varepsilon_k \to 0$ *such that*

$$\inf_{n\geq 1} \min_{1\leq k\leq n} \frac{M(n-k+1,\varepsilon_k)-k-1}{n+1} > 0.$$

Proof. Since $\lim_{\varepsilon\to 0} M(1,\varepsilon)=\infty$, we can define

$$\varepsilon_n = \min\{\varepsilon_0, \sup\{\varepsilon : M(1,\varepsilon) > (n+1)^4\}\},$$

where $0<\varepsilon_0<1$ is a positive number such that

$$\liminf_{n\to\infty} \frac{M(n,2\varepsilon_0)}{n+1} > 0.$$

Clearly ε_n is non-increasing and $\varepsilon_n \to 0$. We claim that

$$\inf_{n\geq 1} \min_{1\leq k\leq n} \frac{M(n-k+1,\varepsilon_k)-k-1}{n+1} > 0.$$

Let $s=\inf_{n\geq 1}(M(n,2\varepsilon_0)/(n+1))$. It is easy to check that $s>0$. Since $M(n,\varepsilon)$ is non-increasing with ε, we obtain that

$$\inf_{n\geq 1} \frac{M(n,\varepsilon_k)}{n+1} \geq s$$

for all $k\in\mathbb{N}$. Choose a positive integer $N>4/s^2$. Then for $n\geq 4N$ and $1\leq k\leq \sqrt{n}-1\leq \lfloor\sqrt{n}\rfloor$, one has

$$\begin{aligned}
\frac{M(n-k+1,\varepsilon_k)-k-1}{n+1} &\geq \frac{M(n-k+1,\varepsilon_1)-\sqrt{n}}{n+1}\\
&\geq \frac{n-\lfloor\sqrt{n}\rfloor+2}{n+1}\frac{M(n-\lfloor\sqrt{n}\rfloor+1,\varepsilon_1)}{n-\lfloor\sqrt{n}\rfloor+2} - \frac{1}{\sqrt{n}}\\
&\geq \frac{1}{2}s - \frac{s}{4} = \frac{s}{4} > 0,
\end{aligned}$$

and for $n\geq 4N\geq 4$ and $\sqrt{n}-1<k\leq n$, one has

$$\frac{M(n-k+1,\varepsilon_k)-k-1}{n+1} \geq \frac{M(1,\varepsilon_k)-k-1}{n+1} \geq \frac{(k+1)^4}{n+1}-1 \geq \frac{n^2}{n+1}-1\geq 1.$$

This implies

$$\inf_{n\geq 1} \min_{1\leq k\leq n} \frac{M(n-k+1,\varepsilon_k)-k-1}{n+1} \geq c,$$

where

$$c=\min\left\{1,\frac{s}{4},\min_{1\leq k\leq n\leq 4N}\left\{\frac{M(n-k+1,\varepsilon_k)-k-1}{n+1}\right\}\right\}.$$

By the definition of ε_k, we obtain that $M(n,\varepsilon_k)>k+1$ for all $n\in\mathbb{N}$ and thus $c>0$. This completes the proof of Lemma 2.12. $\square$

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2.7. *DC2 chaos and Devaney chaos*

2.7.1. *DC2 chaos*

Definition 2.13. A pair $x, y \in X$ is said to be $DC2$-scrambled if the following two conditions hold:

$$\text{for all } t > 0, \quad \limsup_{n\to\infty} \frac{1}{n}|\{0 \le i \le n-1 : d(f^i(x), f^i(y)) < t\}| = 1;$$

$$\text{there exists } t_0 > 0, \quad \liminf_{n\to\infty} \frac{1}{n}|\{0 \le i \le n-1 : d(f^i(x), f^i(y)) < t_0\}| < 1.$$

A set S is called a DC2-scrambled set if any pair of its distinct points is DC2-scrambled. Additionally, (X, T) is said to be DC2 chaotic if it has an uncountable DC2-scrambled set.

LEMMA 2.14. [10, Theorem 1.1] *Suppose that (X, T) is a dynamical system with $h_{\text{top}}(T) > 0$, then (X, T) is DC2 chaotic.*

2.7.2. *Devaney chaos*

Definition 2.15. (X, T) is said to be Devaney chaotic, if it satisfies the following conditions:

(1) (X, T) is topological transitive;
(2) $\overline{\text{Per}(T)} = X$;
(3) (X, T) has sensitive dependence on initial conditions, that is, there exists $\varepsilon > 0$ such that for any $\delta > 0$ and $x \in X$, there exists $y \in X$ and $n \in \mathbb{N}$ such that $d(x, y) < \delta$ and $d(T^n x, T^n y) > \varepsilon$.

It was shown by Banks *et al* [2] that condition (3) can be deduced from conditions (1) and (2), provided that $|X| = \infty$.

2.8. *Irregular sets*

LEMMA 2.16. *Suppose that (X, T) is a dynamical system with $\mathcal{C}(T) \neq \emptyset$, then*

$$G^{\overline{\mathcal{M}_T^{\text{erg}}(X)}} \subset \bigcap_{f\in\mathcal{C}(T)} I_f(T).$$

Proof. Fix $f \in C(X)$ with $I_f(T) \neq \emptyset$, then there exist $\mu_1, \mu_2 \in V_T(x)$ with $\int_X f\, d\mu_1 < \int_X f\, d\mu_2$ and, by ergodic decomposition theorem, there exist $\nu_1, \nu_2 \in \mathcal{M}_T^{\text{erg}}(X)$ with $\int_X f\, d\nu_1 < \int_X f\, d\nu_2$. Hence, $G^{\overline{\mathcal{M}_T^{\text{erg}}(X)}} \subset I_f(T)$. As a result, $G^{\overline{\mathcal{M}_T^{\text{erg}}(X)}} \subset \bigcap_{f\in\mathcal{C}(T)} I_f(T)$. □

2.9. *Some basic properties of ergodic measures*

LEMMA 2.17. [8, Proposition 5.7] *Suppose that (X, T) is a dynamical system, then $\mathcal{M}_T^{\text{erg}}(X)$ is a G_δ subset of $\mathcal{M}_T(X)$.*

LEMMA 2.18. [28, p. 2192] *Suppose that $\mathcal{M}_T^{\mathrm{erg}}(X)$ is dense in $\mathcal{M}_T(X)$, then $\mathcal{M}_T^{\mathrm{erg}}(X)$ is connected.*

2.10. *Approximate product property and entropy-dense property.* After the large deviation estimates were obtained Eizenberg *et al* [11], the approximate product property, a weaker form of specification property, was first introduced by Pfister and Sullivan [25] to obtain similar results for more general dynamical systems, such as the β-shifts. In some sense, it is weaker than the non-uniform specification property (Lemma 3.5).

Definition 2.19. (X, T) is said to satisfy the approximate product property if for any $\varepsilon > 0$, $\delta_1 > 0$, and $\delta_2 > 0$, there exists $N = N(\varepsilon, \delta_1, \delta_2) \in \mathbb{N}$ such that for any $n \geq N$ and any sequence $\{x_i\}_{i=1}^{+\infty}$ of X, there exist a sequence of integers $\{h_i\}_{i=1}^{+\infty}$ and $x \in X$ satisfying $h_1 = 0$, $n \leq h_{i+1} - h_i \leq n(1 + \delta_2)$, and

$$|\{0 \leq j \leq n-1 : d(T^{h_i+j}x, T^j x_i) > \varepsilon\}| \leq \delta_1 n.$$

Definition 2.20. (X, T) is said to satisfy the entropy-dense property if for any $\mu \in \mathcal{M}_T(X)$, any $\eta < h_\mu(T)$, and any $\varepsilon > 0$, there exists $\nu \in \mathcal{M}_T^{\mathrm{erg}}(X)$ such that $\rho(\mu, \nu) < \varepsilon$ and $h_\nu(T) > \eta$.

LEMMA 2.21. [25, Theorem 2.1] *Suppose that (X, T) is a dynamical system satisfying the approximate product property, then it is entropy-dense.*

LEMMA 2.22. *Suppose that (X, T) is a dynamical system satisfying the approximate product property, then it satisfies properties (R3), (R4), and (R5).*

Proof. By Lemma 2.21, it is entropy-dense. From the definition of entropy-dense property, $\mathcal{M}_T^{\mathrm{erg}}(X)$ is dense in $\mathcal{M}_T(X)$, and hence by Lemma 2.18, $\mathcal{M}_T^{\mathrm{erg}}(X)$ is connected. Additionally, by Lemma 2.17, $\mathcal{M}_T^{\mathrm{erg}}(X)$ is residual in $\mathcal{M}_T(X)$. $\square$

3. *Non-uniform specification property*

Given $\mu \in \mathcal{M}(X)$ and $\varepsilon > 0$, denote

$$B(\mu, \varepsilon) := \{\nu \in \mathcal{M}(X) : \rho(\mu, \nu) < \varepsilon\}.$$

LEMMA 3.1. [14, Theorem 2.1] *Suppose that (X, T) is a topological dynamical system, given $K \subset \mathcal{M}_T(X)$. If $\{x \in X : V_T(x) \cap B(\mu, \varepsilon) \neq \emptyset\}$ is dense in X for any $\varepsilon > 0$ and any $\mu \in K$, then $G^{\overline{K}}$ is residual in X.*

THEOREM 3.2. *Suppose that (X, T) satisfies the non-uniform periodic specification property with $M(n, \varepsilon)$, then we have the following:*

(1) $\overline{\mathrm{Per}(T)} = X$ *and* $C_T(X) = X$;

(2) (X, T) *is topologically mixing;*

(3) *if (X, T) is pseudo-expansive, then*

$$h_{\mathrm{top}}(T) \geq \limsup_{n\to\infty} \frac{1}{n} \log |\mathrm{Per}_n(T)|;$$

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(4) *for every compact subset $K \subset \mathcal{M}_T(X)$ with $G^K \neq \emptyset$, we have G^K is residual in X. If further, (X, T) is pseudo-expansive, then for every non-empty compact connected subset $K \subset \mathcal{M}_T(X)$ with $G_K \neq \emptyset$, G_K is dense in X.*

If further, $|X| \geq 2$, then

(5) $h_{\text{top}}(T) > 0$ *and for any* $\delta > 0$,

$$\limsup_{n\to\infty} \frac{1}{n} \log |\text{Per}_n(T)| \geq \limsup_{n\to\infty} \frac{1}{n} \log |\text{Per}_{n,\delta}(T)| > 0;$$

(6) (X, T) *is DC2 chaotic;*

(7) (X, T) *is Devaney chaotic;*

(8) $\mathcal{C}(T) \neq \emptyset$ *and* $\bigcap_{f\in\mathcal{C}(T)} I_f(T)$ *is residual in X.*

Proof. (1) From the definition, we have $\overline{\text{Per}(T)} = X$ and thus by [8, Proposition 21.12], $C_T(X) = X$.

(2) When $\sup_{\varepsilon>0} \lim_{n\to\infty}(M(n, \varepsilon)/n) = 0$, the result is directly from [16, Theorem 17(4)]. In fact, their proof still works without the assumption that $\sup_{\varepsilon>0} \lim_{n\to\infty}(M(n, \varepsilon)/n) = 0$, and we put it here. Given a pair of non-empty open subsets U and V of X, there exist $x \in U$, $y \in V$, and $\varepsilon > 0$ such that $B(x, 2\varepsilon) \subset U$ and $B(y, 2\varepsilon) \subset V$. Let $N = M(1, \varepsilon)$, then for every $n \geq N$, there exists $z \in X$ such that $d(x, z) \leq \varepsilon$ and $d(y, T^n z) \leq \varepsilon$. Hence, $f^n(U) \cap V \neq \emptyset$. As a result, (X, T) is topologically mixing.

(3) Suppose that c is a pseudo-expansive constant for (X, T), then $\text{Per}_n(T)$ is (n, c)-separated, and hence $h_{\text{top}}(T) \geq \limsup_{n\to\infty}(1/n) \log s(n, c) \geq \limsup_{n\to\infty}(1/n) \log |\text{Per}_n(T)|$.

(4) (i) Since $G^K \neq \emptyset$, we can choose $y \in G^K$. Given $\varepsilon > 0$ and $\mu \in K$, by Lemma 3.1, to show that G^K is residual in X, it is enough to show that $\{x \in X : V_T(x) \cap B(\mu, \varepsilon) \neq \emptyset\}$ is dense in X. Given $x \in X$ and $0 < \delta < \varepsilon/2$, let $N = M(1, \delta)$. For any $n \in \mathbb{N}$, by non-uniform periodic specification property, we can find $z_n \in X$ such that $d(x, z_n) \leq \delta$ and $d(T^i y, T^{N+i} z_n) \leq \delta$ for any $0 \leq i \leq n-1$. Since X is compact, we can choose an accumulation point z of $\{z_n : n \geq 1\}$. Then $d(x, z) \leq \delta$ and $d(T^i y, T^{N+i} z) \leq \delta$ for any $i \geq 0$. Hence, $\rho(\delta^n_{T^N z}, \delta^n_y) \leq \delta$ for any $n \geq 1$. As a result, $V_T(T^N z) \cap B(\mu, \varepsilon) \neq \emptyset$ and thus $V_T(z) \cap B(\mu, \varepsilon) \neq \emptyset$. Therefore, $\{x \in X : V_T(x) \cap B(\mu, \varepsilon) \neq \emptyset\}$ is dense in X.

(ii) When (X, T) is pseudo-expansive, let c be a pseudo-expansive constant. Since $G_K \neq \emptyset$, we can choose $y \in G_K$. Given $x \in X$ and $0 < \delta \leq c$, let $N = M(1, \delta)$. Following the same line in part (i), we can find $z \in X$ such that $d(z, x) \leq \delta$ and $d(T^i y, T^{N+i} z) \leq \delta$ for any $i \geq 0$. By the pseudo-expansiveness of (X, T), $\lim_{n\to\infty} d(T^{N+n} z, T^n y) = 0$, since $y \in G_K$, we have $T^N z \in G_K$. Hence, $z \in G_K$. As a result, G_K is dense in X.

(5) (i) Since $|X| \geq 2$, we can choose $\varepsilon > 0$ and $x, y \in X$ with $d(x, y) > 3\varepsilon$. Given $n \in \mathbb{N}$ and an n-tuple $(z_0, z_1, \ldots, z_{n-1}) \in \{x, y\}^n$, let $N = M(1, \varepsilon)$. Then there exists $z \in X$ with $d(z_i, T^{iN} z) \leq \varepsilon$ for any $0 \leq i \leq n-1$. Hence, for every two different n-tuples, the points we find are (nN, ε)-separated. As a result, $s(nN, \varepsilon) \geq 2^n$. Therefore,

$$h_{\text{top}}(T) \geq \limsup_{n\to\infty} \frac{1}{nN} \log s(nN, \varepsilon) \geq \lim_{n\to\infty} \frac{1}{nN} \cdot n \log 2 > 0.$$

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(ii) Given $\delta > 0$, by item (2), (X, T) is topologically mixing. Choose a transitive point $z_* \in X$ and $N_0 \in \mathbb{N}$ with

$$d_H(\{z_*, Tz_*, \ldots, T^{N_0-1}z_*\}, X) < \frac{\delta}{2}.$$

Since $|X| \geq 2$, we can choose $0 < \varepsilon < \delta/2$ and $x, y \in X$ with $d(x, y) > 3\varepsilon$. Let $N_1 = M(1, \varepsilon)$ and $N_2 = M(N_0, \varepsilon)$, then given $n \in \mathbb{N}$ and a $(n+1)$-tuple $(z_0, z_1, \ldots, z_{n-1}, z_*)$, there exists $z \in X$ with $d(z_i, T^{iN_1}z) \leq \varepsilon$ for any $0 \leq i \leq n-1$, $d(T^j z_*, T^{nN_1+j}z) \leq \varepsilon$ for any $0 \leq j \leq N_0 - 1$ and $T^{m_n}z = z$, where $m_n = nN_1 + N_0 - 1 + N_2$. Hence, $z \in \mathrm{Per}_{m_n,\delta}(T)$. Since for every two different $(n+1)$-tuples, the points we find are different, we have $|\mathrm{Per}_{m_n,\delta}(T)| \geq (1/m_n)2^n$. As a result,

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \log|\mathrm{Per}_{n,\delta}(T)| &\geq \limsup_{n\to\infty} \frac{1}{m_n} \log|\mathrm{Per}_{m_n,\delta}(T)| \\ &\geq \limsup_{n\to\infty} \left(\frac{n}{m_n}\log 2 - \frac{1}{m_n}\log m_n\right) > 0.\end{aligned}$$

(6) By item (5), $h_{\mathrm{top}}(T) > 0$ and thus by Lemma 2.14, (X, T) is DC2 chaotic.

(7) Since $h_{\mathrm{top}}(T) > 0$, we have that $|X| = \infty$. By item (2), (X, T) is topological mixing and thus topological transitive. Combining with item (1), we have that (X, T) is Devaney chaotic.

(8) By item (1), $\overline{\mathrm{Per}(T)} = X$, and combining with $|X| = \infty$, we have that $|\mathcal{M}_T^{\mathrm{erg}}(X)| \geq 2$. Since $\overline{\mathcal{M}_T^{\mathrm{erg}}(X)}$ is compact, we can choose a countable subset $D = \{\mu_1, \mu_2, \ldots, \} \subset \mathcal{M}_T^{\mathrm{erg}}(X)$ with $\overline{D} = \overline{\mathcal{M}_T^{erg}(X)}$. By Theorem 3.2(4), G^{μ_i} is residual in *X* for any $i \geq 1$, and hence, G^D is residual in *X*. Since $V_T(x)$ is compact for any $x \in X$, we have that

$$G^{\overline{\mathcal{M}_T^{\mathrm{erg}}(X)}} = G^{\overline{D}} = G^D.$$

As a result, $G^{\overline{\mathcal{M}_T^{\mathrm{erg}}(X)}}$ is residual in *X*. Since $|\mathcal{M}_T^{\mathrm{erg}}(X)| \geq 2$, we can choose $f \in C(X)$ and $\nu_1, \nu_2 \in \mathcal{M}_T^{\mathrm{erg}}(X)$ with $\int_X f\, d\nu_1 < \int_X f\, d\nu_2$, and hence, $\emptyset \neq G^{\overline{\mathcal{M}_T^{\mathrm{erg}}(X)}} \subset I_f(T)$ and thus $\mathcal{C}(T) \neq \emptyset$. By Lemma 2.16, $\bigcap_{f\in\mathcal{C}(T)} I_f(T)$ is residual in *X*. □

From the proof of Theorem 3.2, for dynamical system (X, T) satisfying the non-uniform specification property with $M(n, \varepsilon)$, Theorem 3.2(2) and Theorem 3.2(4) still hold. If further $|X| \geq 2$, then $h_{\mathrm{top}}(T) > 0$; Theorems 3.2(6) and 3.2(8) still hold.

LEMMA 3.3. *Suppose (X, T) satisfies the non-uniform periodic specification property with $M(n, \varepsilon)$, where*

$$\sup_{\varepsilon>0} \liminf_{n\to\infty} \frac{M(n, \varepsilon)}{n} = 0.$$

Then (X, T) is Per(T)*-closable and* Per(T) *is linkable.*

Proof. (1) First, we show that (X, T) is Per(T)-closable. Fix $\mu \in \mathcal{M}_T^{\mathrm{erg}}(X)$, $x \in G_\mu$, $\varepsilon > 0$, and $N > 0$. Choose $p \geq N$ large enough such that

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$$\frac{M(p,\varepsilon/2)}{p} < \frac{\varepsilon}{2} \quad \text{and} \quad \frac{M(1,\varepsilon/2)}{p} < \frac{\varepsilon}{2}. \tag{3.2}$$

Let $x_1 = x_2 = x$, $a_1 = 0$, $b_1 = p - 1$, $b_2 = a_2 = M(p, \varepsilon/2) + p - 1$, and $q = M(p, \varepsilon/2) + p - 1 + M(1, \varepsilon/2)$, then by the non-uniform periodic specification property, there exists $y \in \mathrm{Per}_q(T)$ such that $y \in B_p(x, \varepsilon)$. From equation (3.2), we have $q \le (1+\varepsilon)p$. Therefore, x is $\mathrm{Per}(T)$-closable. From the definition, we have that (X, T) is $\mathrm{Per}(T)$-closable.

(2) Now, we show that $\mathrm{Per}(T)$ is linkable. Fix $y_1, y_2 \in \mathrm{Per}(T)$ and $\lambda \in [0, 1]$.

(i) $\lambda = 0$, fix $\varepsilon > 0$, choose $p_1 \in \mathbb{N}$ with $M(p_1, \varepsilon/2)/p_1 < \varepsilon$, and choose p_2 large enough such that $M(p_2, \varepsilon/2)/p_2 < \varepsilon$ and $p_1/(p_1 + p_2) \le \varepsilon$. Let $x_1 = y_1, x_2 = y_2$, $a_1 = 0$, $b_1 = p_1 - 1$, $q_1 = p_1 - 1 + M(p_1, \varepsilon/2)$, $a_2 = q_1$, $b_2 = q_1 + p_2 - 1$, $q_2 = q_1 + p_2 - 1 + M(p_2, \varepsilon/2)$, then $p_j \le q_j - q_{j-1} \le (1+\varepsilon)p_j$ and by the non-uniform periodic specification property, there exists $z \in \mathrm{Per}_{q_2}(T)$ such that $T^{q_{j-1}}z \in B_{p_j}(y_j, \varepsilon)$ for $q_0 = 0$, $j = 1, 2$.

(ii) $\lambda = 1$, fix $\varepsilon > 0$, choose $p_2 \in \mathbb{N}$ with $(M(p_2, \varepsilon/2)/p_2) < \varepsilon$, and choose p_1 large enough such that $(M(p_1, \varepsilon/2)/p_1) < \varepsilon$ and $1 - \varepsilon \le p_1/(p_1 + p_2)$. Let $x_1 = y_1, x_2 = y_2$, $a_1 = 0$, $b_1 = p_1 - 1$, $q_1 = p_1 - 1 + M(p_1, \varepsilon/2)$, $a_2 = q_1$, $b_2 = q_1 + p_2 - 1$, $q_2 = q_1 + p_2 - 1 + M(p_2, \varepsilon/2)$, then $p_j \le q_j - q_{j-1} \le (1+\varepsilon)p_j$ and by the non-uniform periodic specification property, there exists $z \in \mathrm{Per}_{q_2}(T)$ such that $T^{q_{j-1}}z \in B_{p_j}(y_j, \varepsilon)$ for $q_0 = 0$, $j = 1, 2$.

(iii) $0 < \lambda < 1$, fix $0 < \varepsilon < \min\{\lambda, 1 - \lambda\}$ and choose $q \in \mathbb{N}$ with

$$\frac{1}{q} \le \min\left\{\frac{1}{\lambda+\varepsilon} - 1, \frac{1}{2(1/(\lambda+\varepsilon) - 1)}\right\}. \tag{3.3}$$

Choose $p \in \mathbb{N}$ large enough such that

$$\left(\frac{1}{\lambda+\varepsilon} - 1\right)\left\lceil \frac{p}{q} \right\rceil + 1 \le \left(\frac{1}{\lambda-\varepsilon} - 1\right)\left\lceil \frac{p}{q} \right\rceil,$$

$$\left(\frac{1}{\lambda+\varepsilon} - 1\right)\frac{p}{q} + \frac{1}{\lambda+\varepsilon} \le p \quad \text{and} \quad \frac{M(p,\varepsilon/2)}{p} < \frac{\varepsilon}{q^2}.$$

Let $p_1 = \lceil p/q \rceil \le p$, then $M(p_1, \varepsilon/2)/p_1 < \varepsilon$. Let $p_2 = \lceil (1/(\lambda+\varepsilon) - 1)\lceil p/q \rceil \rceil$, then $p_2 \ge (1/(\lambda+\varepsilon) - 1)(p/q) \ge p/q^2$ and

$$p_2 \le \left(\frac{1}{\lambda+\varepsilon} - 1\right)\left\lceil \frac{p}{q} \right\rceil + 1 \le \left(\frac{1}{\lambda+\varepsilon} - 1\right)\left(\frac{p}{q} + 1\right) + 1 \le p.$$

Hence, $M(p_2, \varepsilon/2)/p_2 < \varepsilon$. Since

$$\frac{1}{\lambda+\varepsilon} - 1 \le \frac{p_2}{p_1} \le \frac{1}{p_1}\left[\left(\frac{1}{\lambda+\varepsilon} - 1\right)p_1 + 1\right] \le \frac{1}{\lambda-\varepsilon} - 1,$$

we have $\lambda - \varepsilon \le p_1/(p_1 + p_2) \le \lambda + \varepsilon$. Let $x_1 = y_1, x_2 = y_2, a_1 = 0, b_1 = p_1 - 1$, $q_1 = p_1 - 1 + M(p_1, \varepsilon/2), a_2 = q_1, b_2 = q_1 + p_2 - 1, q_2 = q_1 + p_2 - 1 + M(p_2, \varepsilon/2)$, then $p_j \le q_j - q_{j-1} \le (1+\varepsilon)p_j$ and by the non-uniform periodic specification property, there exists $z \in \mathrm{Per}_{q_2}(T)$ such that $T^{q_{j-1}}z \in B_{p_j}(y_j, \varepsilon)$ for $q_0 = 0$, $j = 1, 2$.

Combining parts (i), (ii), and (iii), we have that $\mathrm{Per}(T)$ is linkable. □

THEOREM 3.4. *Suppose (X, T) satisfies the non-uniform periodic specification property with $M(n, \varepsilon)$, where*

$$\sup_{\varepsilon>0} \liminf_{n\to\infty} \frac{M(n, \varepsilon)}{n} = 0,$$

then:

(1) $\mathcal{M}_T^{\mathrm{co}}(X)$ *is dense in* $\mathcal{M}_T(X)$*;*

(2) *for any non-empty compact connected subset* $V \subset \mathcal{M}_T(X)$*,* G_V *is dense in X.*

Proof. By Lemma 3.3, (X, T) is Per(T)-closable and Per(T) is linkable. By Lemma 2.8, $\mathcal{M}_T^{\mathrm{co}}(X)$ is dense in $\mathcal{M}_T^{\mathrm{erg}}(X)$. Since $C_T(X) = X$, by Lemma 2.10, we have:

(1) $\mathcal{M}_T^{\mathrm{co}}(X)$ is dense in $\mathcal{M}_T(X)$;

(2) for any non-empty compact connected subset $V \subset \mathcal{M}_T(X)$, G_V is dense in X. □

LEMMA 3.5. *Suppose (X, T) satisfies the non-uniform specification property with $M(n, \varepsilon)$, where*

$$\sup_{\varepsilon>0} \liminf_{n\to\infty} \frac{M(n, \varepsilon)}{n} = 0,$$

then (X, T) satisfies the approximate product property.

Proof. When $\sup_{\varepsilon>0} \lim_{n\to\infty}(M(n, \varepsilon)/n) = 0$, the result is directly from [**16**, Remark 29]. In fact, when we only assume that $\sup_{\varepsilon>0} \liminf_{n\to\infty}(M(n, \varepsilon)/n) = 0$, the result still holds. Here is the proof.

Fix $\varepsilon > 0$, $\delta_1 > 0$, and $\delta_2 > 0$. Since $\liminf_{n\to\infty}(M(n, \varepsilon)/n) = 0$, we can choose L_1 large enough such that

$$\frac{M(L_1, \varepsilon) - 1}{L_1 + M(L_1, \varepsilon) - 1} \le \delta_1.$$

Let $L_2 = L_1 + M(L_1, \varepsilon) - 1$ and $L_3 = \max_{1\le i\le L_2} M(i, \varepsilon) = M(L_2, \varepsilon)$. Choose $N = N(\varepsilon, \delta_1, \delta_2)$ large enough such that $L_3/N \le \delta_2$. Now, we fix $n \ge N$ and a sequence $\{x_i\}_{i=1}^{\infty}$. Suppose that $n - 1 = kL_2 + p$, where $0 \le p \le L_2 - 1$. For $i \in \mathbb{N}$ and $0 \le j \le k$, denote $y_{i,j} = T^{jL_2}x_i$. Then for any $m \in \mathbb{N}$, consider the orbit segments:

$$\langle y_{1,0}, Ty_{1,0}, \ldots, T^{L_1-1}y_{1,0}\rangle, \langle y_{1,1}, Ty_{1,1}, \ldots, T^{L_1-1}y_{1,1}\rangle, \ldots, \langle y_{1,k-1}, Ty_{1,k-1}, \ldots, T^{L_1-1}y_{1,k-1}\rangle, \langle y_{1,k}, Ty_{1,k}, \ldots, T^{p}y_{1,k}\rangle;$$

$$\langle y_{2,0}, Ty_{2,0}, \ldots, T^{L_1-1}y_{2,0}\rangle, \langle y_{2,1}, Ty_{2,1}, \ldots, T^{L_1-1}y_{2,1}\rangle, \ldots, \langle y_{2,k-1}, Ty_{2,k-1}, \ldots, T^{L_1-1}y_{2,k-1}\rangle, \langle y_{2,k}, Ty_{2,k}, \ldots, T^{p}y_{2,k}\rangle;$$

$$\ldots$$

$$\langle y_{m,0}, Ty_{m,0}, \ldots, T^{L_1-1}y_{m,0}\rangle, \langle y_{m,1}, Ty_{m,1}, \ldots, T^{L_1-1}y_{m,1}\rangle, \ldots, \langle y_{m,k-1}, Ty_{m,k-1}, \ldots, T^{L_1-1}y_{m,k-1}\rangle, \langle y_{m,k}, Ty_{m,k}, \ldots, T^{p}y_{m,k}\rangle.$$

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By the non-uniform specification property, there exists $z_m \in X$ such that for $1 \le i \le m$, $0 \le j \le k-1$, and $0 \le l \le L_1 - 1$, we have

$$d(T^{jL_2+(i-1)(n-1+M(p+1,\varepsilon))+l} z_m, T^l y_{i,j}) \le \varepsilon$$

and for $1 \le i \le m$ and $0 \le l \le p$, we have

$$d(T^{kL_2+(i-1)(n-1+M(p+1,\varepsilon))+l} z_m, T^l y_{i,k}) \le \varepsilon.$$

For $i \in \mathbb{N}$, denote $h_i = (i-1)(n-1+M(p+1,\varepsilon))$, then $h_1 = 0$ and

$$n \le h_{i+1} - h_i = n - 1 + M(p+1,\varepsilon) \le n - 1 + L_3 \le n(1+\delta_2).$$

Since X is compact, we can choose z as an accumulation point of $\{z_m : m \ge 1\}$. Then for any $i \ge 1$,

$$|\{0 \le j \le n-1 : d(T^{h_i+j} z, T^j x_i) > \varepsilon\}| \le k(M(L_1,\varepsilon)-1) = \frac{k(M(L_1,\varepsilon)-1)}{n} n$$
$$= \frac{k(M(L_1,\varepsilon)-1)}{k(L_1+M(L_1,\varepsilon)-1)+p+1} n \le \frac{M(L_1,\varepsilon)-1}{L_1+M(L_1,\varepsilon)-1} n \le \delta_1 n.$$

Therefore, (X, T) satisfies the approximate product property. □

THEOREM 3.6. *Suppose (X, T) satisfies the non-uniform specification property with $M(n,\varepsilon)$, where*

$$\sup_{\varepsilon>0} \liminf_{n\to\infty} \frac{M(n,\varepsilon)}{n} = 0,$$

then (X, T) satisfies properties (R3), (R4), and (R5).

Proof. Combining with Lemmas 3.5 and 2.22 completes the proof. □

4. *Proof of Theorem A*

Let $\{\varepsilon_n\}_{n\ge1}$ be the non-increasing sequence of positive real numbers from Lemma 2.12 for $M(n,\varepsilon)$. Consider the one-side full shift $(\{0,1\}^{\mathbb{N}_0}, T)$. The metric of $\{0,1\}^{\mathbb{N}_0}$ is given by

$$d(x, y) = \sum_{n\ge0} \delta_n d'(x_n, y_n),$$

where $\delta_n = (\varepsilon_n/2^n) - (\varepsilon_{n+1}/2^{n+1}) > 0$ and d' is the discrete metric on $\{0, 1\}$.

Let

$$c_0 = \min\left\{1, \inf_{n\ge1} \min_{1\le k\le n} \frac{M(n-k+1,\varepsilon_k)-k-1}{n+1}\right\}.$$

By Lemma 2.12, we obtain that $c_0 > 0$. Given $0 < c \le c_0$, consider forbidden words

$$\mathcal{F}_c = \{1^s 0^t 1 : cs > t > 0\},$$

then $\mathcal{F} \ne \emptyset$. Define

$$X_c = X_c(\mathcal{F}_c) = \{x \in \{0,1\}^{\mathbb{N}_0} : x_i x_{i+1} \cdots x_j \notin \mathcal{F}_c \text{ for all } 0 \le i \le j\}.$$

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Then X_c is a subshift of $\{0, 1\}^{\mathbb{N}_0}$ and T_c is defined as $T_c := T|_{X_c}$. Fix $0 < c \le c_0$, for convenience, we rewrite (X_c, T_c) as (X, T).

It can be checked that (X, T) is a coded shift (that is, its language is freely generated by a countable set of words), so one could theoretically use existing literature such as [**6**] or [**16**] (where they also prove equivalence of some mixing properties for coded subshifts), but since the system is simple, we give self-contained proofs.

First, we will show that (X, T) satisfies the non-uniform specification property with $M(n, \varepsilon)$. Without loss of generality, we assume that $\varepsilon \le \varepsilon_0$. Consider any integer $k \ge 2$, any points $x^{(1)}, \ldots, x^{(k)} \in X$, any non-negative integers $a_1, b_1, \ldots, a_k, b_k$ with

$$a_1 \le b_1 < \cdots < a_k \le b_k$$

and

$$a_{i+1} - b_i \ge M(b_i - a_i + 1, \varepsilon) \quad \text{for } 1 \le i \le k-1,$$

and an integer p with $p \ge b_k - a_1 + M(b_k - a_k + 1, \varepsilon)$. Let l be the non-negative integer such that $\varepsilon_{l+1} < \varepsilon \le \varepsilon_l$. Since

$$M(b_i - a_i + 1, \varepsilon) \ge M(b_i - a_i + 1, \varepsilon_l) \ge (l+1)^4 \ge l + 1,$$

we obtain that

$$a_{i+1} - b_i \ge M(b_i - a_i + 1, \varepsilon) \ge l + 1 \quad \text{for } 1 \le i \le k-1$$

and $p \ge b_k - a_1 + M(b_k - a_k + 1, \varepsilon) \ge b_k - a_1 + l + 1$. So we can define

$$z' = (x_0^{(1)} \cdots x_{b_1 - a_1 + l}^{(1)} 0^{a_2 - b_1 - l - 1} x_0^{(2)} \cdots x_{b_2 - a_2 + l}^{(2)} \cdots x_0^{(k)} \cdots x_{b_k - a_k + l}^{(k)} 0^{p + a_1 - b_k - l - 1})^\infty$$

and let $z = T^{pa_1 - a_1} z'$. Clearly, $T^{a_1} z = z'$ and $T^p z = z$.

We claim that $z \in X$. For $w \in \{0, 1\}^*$ and $w \notin \mathcal{F}$, clearly $w0^\infty \in X$, so we just need to show that z does not contain any forbidden word. Since $\varepsilon \le \varepsilon_l$, we obtain by the definition of c that

$$M(b_i - a_i + 1, \varepsilon) - l - 1 \ge M(b_i - a_i + 1, \varepsilon_l) - l - 1 \ge c(b_i - a_i + l + 1). \tag{4.4}$$

Suppose for a contradiction that z contains a forbidden word $u = 1^s 0^t 1$ with $cs > t > 0$. Since $x^{(i)} \in X$ and $u \in \mathcal{F}$, u cannot be contained in any $x^{(i)}$. Therefore, 0^t must contain an entire $0^{a_{i+1} - b_i - l}$ or $0^{p + a_1 - b_k - l}$, and 1^s must be contained in some $x^{(i)}$. By equation (4.4), it means that

$$\begin{aligned} t \ge a_{i+1} - b_i - l - 1 &\ge M(b_i - a_i + 1, \varepsilon) - l - 1 \\ &\ge c(b_i - a_i + l + 1) \ge cs > t \end{aligned}$$

or

$$\begin{aligned} t \ge p + a_1 - b_k - l - 1 &\ge M(b_k - a_k + 1, \varepsilon) - l - 1 \\ &\ge c(b_k - a_k + l + 1) \ge cs > t, \end{aligned}$$

which is a contradiction. Therefore, $z \in X$.

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Now we will show that

$$d(T^{n-a_i}x^{(i)}, T^n z) \le \varepsilon \quad \text{for } a_i \le n \le b_i, 1 \le i \le k.$$

By the definition of z, it is easy to see that

$$z_n = x^{(i)}_{n-a_i} \quad \text{for } a_i \le n \le b_i + l, 1 \le i \le k,$$

so we obtain that

$$d(T^{n-a_i}x^{(i)}, T^n z) \le d(T^{b_i-a_i}x^{(i)}, T^{b_i} z) \le \sum_{k \ge l+1} \delta_k \le \varepsilon_{l+1} < \varepsilon$$

for $a_i \le n \le b_i, 1 \le i \le k$. Therefore, (X, T) satisfies the non-uniform specification property with $M(n, \varepsilon)$.

We will show now that statement (1) holds and start with

$$\overline{\mathcal{M}_T^{\mathrm{co}}(X)} = \overline{\mathcal{M}_T^{\mathrm{erg}}(X)}. \tag{4.5}$$

It is well known that $\mathcal{M}_T^{\mathrm{co}}(X) \subseteq \mathcal{M}_T^{\mathrm{erg}}(X)$ and $G_\mu \neq \emptyset$ for ergodic measure μ, so it suffices to show that

$$\overline{\mathcal{M}_T^{\mathrm{co}}(X)} \supseteq \{\mu \in \mathcal{M}_T(X) : G_\mu \neq \emptyset\}. \tag{4.6}$$

Fix any invariant measure μ with $G_\mu \neq \emptyset$ and given a generic point $x \in G_\mu$. If μ is a Dirac measure, then the argument is trivial. Now we assume that μ is not a Dirac measure. By the assumption, there are infinite 0 and 1 terms contained in $x_0, x_1, \ldots$. Write $x = 1^{s_1}0^{t_1}1^{s_2}0^{t_2}\ldots$ with $t_i > 0$ for $i \ge 1$ and $s_i > 0$ for $i > 1$. Given $\varepsilon > 0$, there exists $L \in \mathbb{N}$ such that $d(x, y) \le \varepsilon/3$ whenever $x_i = y_i$ for any $0 \le i \le L-1$. Since x is a generic point of μ, there exists j sufficiently large such that

$$\rho(\delta_x^N, \mu) < \frac{\varepsilon}{3}, \quad N \ge L \quad \text{and} \quad \frac{L}{N}\mathrm{diam}(X) \le \frac{\varepsilon}{3},$$

where $N = \sum_{i=1}^{j}(s_i + t_i)$. Define $y = (1^{s_1}0^{t_1}1^{s_2}0^{t_2}\cdots 1^{s_j}0^{t_j})^\infty$. Clearly, y is a periodic point of X with period N and $d(T^i x, T^i y) \le \varepsilon/3$ for any $0 \le i \le N-L$. By Lemma 2.2, we have

$$\rho(\delta_x^N, \delta_y^N) \le \frac{\varepsilon}{3} + \frac{L}{N}\mathrm{diam}(X) \le \frac{2\varepsilon}{3}$$

and thus $\rho(\mu, \delta_y^N) < \varepsilon/3 + 2\varepsilon/3 = \varepsilon$. This completes the proof of equation (4.6).

Now we will show that

$$\overline{\mathcal{M}_T^{\mathrm{co}}(X)} \subsetneq \mathcal{M}_T(X). \tag{4.7}$$

Arbitrarily consider a CO-measure $\nu = \delta_y^p$, where $y \in \mathrm{Per}(T)$ with minimal period $p \ge 2$. For $x \in X$, define

$$\chi_m(x) = \frac{1}{m}\sum_{n=0}^{m-1} x_n. \tag{4.8}$$

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Then $\chi_m \in C(X)$ and we claim that

$$\int_X \chi_m \, d\nu \leq \frac{1}{c+1} \quad \text{for all } m \in \mathbb{N}. \tag{4.9}$$

Since $y \in \mathrm{Per}_p(T)$, for every $m \in \mathbb{N}$, one has

$$\frac{1}{p}\sum_{i=0}^{p-1} \chi_m(T^i y) = \frac{1}{pm}\sum_{i=0}^{p-1}\sum_{n=0}^{m-1}(T^i y)_n = \frac{1}{pm}\sum_{n=0}^{m-1}\bigg(\sum_{i=0}^{p-1} y_{i+n}\bigg) = \frac{1}{p}\sum_{i=0}^{p-1} y_i,$$

and thus

$$\int_X \chi_m \, d\nu = \frac{1}{p}\sum_{i=0}^{p-1} y_i.$$

Since $p \geq 2$, we can assume that $y = 1^{s_1}0^{t_1}1^{s_2}0^{t_2}\cdots 1^{s_k}0^{t_k}$, where $s_i, t_i > 0$ with $cs_i \leq t_i$ for $i = 1, 2, \ldots, k$. Then we obtain that

$$\sum_{i=0}^{p-1} y_i = \sum_{i=1}^{k} s_i \leq \frac{1}{c+1}\sum_{i=1}^{k}(s_i + t_i) = \frac{p}{c+1}.$$

Hence,

$$\int_X \chi_m \, d\nu \leq \frac{1}{c+1} \quad \text{for all } m \in \mathbb{N}.$$

This completes the proof of equation (4.9), and we conclude that

$$\int_X \chi_m \, d\nu \leq \frac{1}{c+1}$$

for all $\nu \in \mathcal{M}_T^{\mathrm{co}}(X) \setminus \{\delta_{1^\infty}\}$ and $m \in \mathbb{N}$.

Now, consider $\mu = (c\delta_{0^\infty} + 2\delta_{1^\infty})/(c+2) \in \mathcal{M}_T(X)$. Clearly,

$$\int_X \chi_m \, d\mu = \frac{2}{c+2} \in \bigg(\frac{1}{c+1}, 1\bigg)$$

for all $m \in \mathbb{N}$, so $\mu \notin \overline{\mathcal{M}_T^{\mathrm{co}}(X)}$ and this completes the proof of equation (4.7).

We now prove that for $\delta > 0$ sufficiently small, one has

$$\overline{\mathcal{M}_{T,\delta}^{\mathrm{co}}(X)} \subsetneq \overline{\mathcal{M}_T^{\mathrm{co}}(X)}. \tag{4.10}$$

We have shown that for every $m \in \mathbb{N}$,

$$\int_X \chi_m \, d\nu \leq \frac{1}{c+1} \quad \text{for } \nu \in \mathcal{M}_T^{\mathrm{co}}(X) \setminus \{\delta_{1^\infty}\}$$

and obviously

$$\int_X \chi_m \, d\nu = 1 \quad \text{for } \nu = \delta_{1^\infty}.$$

Therefore, δ_{1^∞} is an isolated point of $\mathcal{M}_T^{\mathrm{co}}(X)$ and thus $d_H(S_{\delta_{1^\infty}}, X) = \varepsilon_0 > 0$, and we obtain that

$$\overline{\mathcal{M}_{T,\delta}^{\mathrm{co}}(X)} \subseteq \overline{\mathcal{M}_T^{\mathrm{co}}(X)} \setminus \{\delta_{1^\infty}\} \subsetneq \overline{\mathcal{M}_T^{\mathrm{co}}(X)}$$

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for $0 < \delta < \varepsilon_0$. This completes the proof of equation (4.10). Combining equations (4.5), (4.7), and (4.10), we conclude that statement (1) holds.

Now, we will show that statement (2) holds. According to Theorem 3.2, $h_{\text{top}}(T) > 0$. Given $\delta > 0$, it is easy to see that $|\text{Per}_{n,\delta}(T)| \leq |\mathcal{L}_n(X)|$ and thus

$$\limsup_{n\to\infty} \frac{1}{n} \log |\text{Per}_{n,\delta}(T)| \leq \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{L}_n(X)| = h_{\text{top}}(T). \tag{4.11}$$

For $n \geq 2$, let

$$\mathcal{P}_n := \{w \in \mathcal{L}_n(X) : w_0 = 1, w_{n-1} = 0, w^\infty \in \text{Per}_n(T)\}.$$

Define

$$\mathcal{N}_n = \{(r, s, t) \in \mathbb{Z}^3 : 0 \leq r, s, t \leq n, r + s + t \leq n - 2, cs \leq t\}.$$

For $(r, s, t) \in \mathcal{N}_n$, define $\pi(r, s, t) : \mathcal{P}_{n-(r+s+t)} \to \mathcal{L}_n(X)$ given by

$$\pi(r, s, t)(w) = 0^r w 1^s 0^t.$$

Clearly, $\pi(r, s, t)$ is injective for every $(r, s, t) \in \mathcal{N}_n \cup \{(0, 0, 0)\}$. This implies $|\mathcal{P}_k| \leq |\mathcal{L}_n(X)|$ for $k = 2, \ldots, n$ and thus

$$\sum_{k=2}^{n} |\mathcal{P}_k| \leq n|\mathcal{L}_n(X)|. \tag{4.12}$$

However, for each $u \in \mathcal{L}_n(X) \setminus \{0^{n-k-l}1^k0^l : 0 \leq k, l \leq n, k + l \leq n\}$, there exists (w, r, s, t) such that $u = \pi(r, s, t)(w)$. This implies

$$|\mathcal{L}_n(X)| - (n + 1)^2 \leq (n + 1)^3 \bigg(\sum_{k=2}^{n} |\mathcal{P}_k|\bigg). \tag{4.13}$$

Combining equations (4.12) and (4.13), we conclude that

$$\lim_{n\to\infty} \frac{1}{n} \log \bigg(\sum_{k=2}^{n} |\mathcal{P}_k|\bigg) = \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{L}_n(X)| = h_{\text{top}}(T). \tag{4.14}$$

Now, we will show that

$$h_{\text{top}}(T) \leq \liminf_{n\to\infty} \frac{1}{n} \log |\text{Per}_{n,\delta}(T)|.$$

Choose $m \in \mathbb{N}$ and $w \in \mathcal{P}_m$ such that for any $x \in \text{Per}_n(T)$ with $n \geq m$, we have $x \in \text{Per}_{n,\delta}(T)$ provided that w is a word of x. Now, fix $n \in \mathbb{N}$ with $n \geq 2 + m$. For $0 \leq r \leq n - (2 + m)$, define $\tilde{\pi}(r) : \mathcal{P}_{n-(r+m)} \to \text{Per}_{n,\delta}(T)$ given by

$$\tilde{\pi}(r)(u) = (0^r uw)^\infty.$$

Then $\tilde{\pi}(r_1)(u_1) \neq \tilde{\pi}(r_2)(u_2)$ whenever $(r_1, u_1) \neq (r_2, u_2)$. Hence,

$$|\text{Per}_{n,\delta}(T)| \geq \sum_{k=2}^{n-m} |\mathcal{P}_k|.$$

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By equation (4.14), we have

$$\liminf_{n\to\infty}\frac{1}{n}\log|\mathrm{Per}_{n,\delta}(T)|\geq\lim_{n\to\infty}\frac{1}{n}\log\bigg(\sum_{k=2}^{n-m}|\mathcal{P}_k|\bigg)=h_{\mathrm{top}}(T). \tag{4.15}$$

Combining equations (4.11) and (4.15), we conclude that

$$h_{\mathrm{top}}(T)=\lim_{n\to\infty}\frac{1}{n}\log|\mathrm{Per}_{n,\delta}(T)|.$$

This completes the proof of statement (2).

Finally, we prove statement (3). Since δ_{1^∞} is an isolated point of $\mathcal{M}_T^{\mathrm{co}}(X)$ and $\overline{\mathcal{M}_T^{\mathrm{co}}(X)}=\overline{\mathcal{M}_T^{\mathrm{erg}}(X)}$, δ_{1^∞} is also an isolated point of $\mathcal{M}_T^{\mathrm{erg}}(X)$, and hence, (X,T) does not satisfy property (R4). Since $\overline{\mathcal{M}_{T_c}^{\mathrm{erg}}(X_c)}\subsetneq\mathcal{M}_{T_c}(X_c)$, from the definition of entropy-dense property, (X,T) does not satisfy property (R5).

5. *Proof of Theorem B*

We consider the cluster of dynamical systems $\{(X_c,T_c)\}_{0<c\leq c_0}$ defined in Theorem A. Fix $0<c\leq c_0$. For convenience, we rewrite (X_c,T_c) as (X,T). We have shown that (X,T) satisfies the non-uniform specification property with $M(n,\varepsilon)$. Define

$$\mathcal{U}=\mathcal{M}_T(X)\setminus\overline{\mathcal{M}_T^{\mathrm{co}}(X)}.$$

By Theorem A, $\mathcal{U}$ is a non-empty open subset of $\mathcal{M}_T(X)$. We need to show that the statements (1)–(3) hold.

Statement (1) follows from the Proposition 5.1.

PROPOSITION 5.1. *Let X be the subshift defined in Theorem A and $\mathcal{U}$ be as above, then*

$$\mathcal{U}=\{\mu\in\mathcal{M}_T(X):G_\mu=\emptyset\}.$$

Proof. We have shown that

$$\{\mu\in\mathcal{M}_T(X):G_\mu\neq\emptyset\}\subseteq\overline{\mathcal{M}_T^{\mathrm{co}}(X)}$$

in the proof of Theorem A (see equation (4.6)) and thus

$$\mathcal{U}\subseteq\{\mu\in\mathcal{M}_T(X):G_\mu=\emptyset\}.$$

It suffices to show that

$$\overline{\mathcal{M}_T^{\mathrm{co}}(X)}\subseteq\{\mu\in\mathcal{M}_T(X):G_\mu\neq\emptyset\}. \tag{5.16}$$

Arbitrarily given $\mu\in\overline{\mathcal{M}_T^{\mathrm{co}}(X)}$, we wish to show that $G_\mu\neq\emptyset$. We only need consider the case that $\mu\in\overline{\mathcal{M}_T^{\mathrm{co}}(X)}\setminus\mathcal{M}_T^{\mathrm{co}}(X)$.

Since $\mathrm{Per}_1(T)$ is closed, we can choose $y^{(n)}\in X$ with minimal period $p_n\geq 2$ such that $\nu_n=\delta_{y^{(n)}}^{p_n}\to\mu$. Let $d_n=\rho(\nu_n,\mu)$. Denote by B_n the closed ball in $\mathcal{M}_T(X)$ with center ν_n and radius $r_n=d_n+2^{-n}$.

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It is easy to check that the sequence $\{B_n : n \in \mathbb{N}\}$ satisfies the following:

(a) $B_n \cap B_{n+1} \cap \{\mu\} \neq \emptyset$;

(b) $\bigcap_{k\geq 1} \bigcup_{n\geq k} B_n = \{\mu\}$;

(c) $\lim_{n\to\infty} r_n = 0$.

Without loss of generality, for every $n \in \mathbb{N}$, we assume that

$$y^{(n)} = (1^{s_1(n)}0^{t_1(n)} \cdots 1^{s_{k(n)}(n)}0^{t_{k(n)}(n)})^{\infty},$$

where $s_i(n), t_i(n) > 0$ and $cs_i(n) \leq t_i(n)$ for $1 \leq i \leq k(n)$. Define

$$w^{(n)} = 1^{s_1(n)}0^{t_1(n)} \cdots 1^{s_{k(n)}(n)}0^{t_{k(n)}(n)} \in \mathcal{L}_{p_n}(X).$$

Let M_j be sufficiently large such that $\operatorname{diam}[w] < 2^{-j-2}$ for all $w \in \mathcal{L}_{M_j}(X)$. Choose N_j sufficiently large such that

$$N_{j+1} > 4^{j+2}\bigg(\sum_{i=1}^{j}(p_iN_i + 2M_i + p_{i+1})\bigg) \quad \text{and} \quad \frac{M_j + p_{j+1}}{N_j} < 4^{-j-1} \tag{5.17}$$

for $j \in \mathbb{N}$. Let $0 \leq R_j < p_{j+1}$ such that p_{j+1} divides $\sum_{i=1}^{j}(p_iN_i + 2M_i + R_i)$ and define

$$x = (w^{(1)})^{N_1} y^{(1)}_0 \cdots y^{(1)}_{M_1-1} 0^{M_1+R_1} (w^{(2)})^{N_2} y^{(2)}_0 \cdots y^{(2)}_{M_2-1} 0^{M_2+R_2} \cdots. \tag{5.18}$$

Clearly, $x \in X$. We introduce x in another point if view. Let

$$\begin{aligned}
&a_1 = 0 && b_1 = p_1N_1\\
&a_2 = p_1N_1 + 2M_1 + R_1 && b_2 = a_2 + p_2N_2\\
&\cdots && \\
&a_{n+1} = \sum_{j=1}^{n}(p_jN_j + 2M_j + R_j) && b_{n+1} = a_{n+1} + p_{n+1}N_{n+1}.
\end{aligned}$$

It is easy to see that $a_{n+1} = b_n + 2M_n + R_n$. Hence, by equation (5.17), we can easily see that

$$\frac{a_{n+1} - b_n + p_{n+1}}{N_n} \leq \frac{a_{n+1} - b_n + a_n}{N_n} \leq 4^{-n} \quad \text{and} \quad \frac{a_n}{b_n} \leq 4^{-n-1}. \tag{5.19}$$

Let

$$\begin{cases} x_n = y^{(j)}_n & \text{if } a_j \leq n \leq b_j + M_j - 1 \text{ for some } j \in \mathbb{N},\\ x_n = 0 & \text{if } b_j + M_j \leq n < a_{j+1} \text{ for some } j \in \mathbb{N}, \end{cases}$$

and define $x = x_0x_1\cdots$. It is easy to check that the definition above is equivalent to equation (5.18). Note that

$$d(T^nx, T^ny^{(j)}) < 2^{-j-2} \quad \text{for } a_j \leq n \leq b_j - 1. \tag{5.20}$$

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Now we will show that $\delta_x^k \to \mu$. Let $i = i(k)$ be the positive integer such that $b_i \le k < b_{i+1}$. Define

$$\alpha_k = \begin{cases} \dfrac{b_i - a_i}{b_i - a_i + k - r - a_{i+1}} & \text{if } k > a_{i+1}, \\ 1 & \text{if } k \le a_{i+1}, \end{cases}$$

where $0 \le r < p_{i+1}$ is an integer such that p_{i+1} divides $k - r - a_{i+1}$ when $k > a_{i+1}$. Define $\gamma_k = \alpha_k \nu_i + (1 - \alpha_k)\nu_{i+1}$. Arbitrarily given $f \in C(X)$ with $\|f\| \le 1$, we claim that

$$\lim_{k\to\infty} \left| \int_X f \, d\gamma_k - \int_X f \, d\delta_x^k \right| = 0.$$

For any $\varepsilon > 0$, where we denote by $\omega_f(\varepsilon)$ the oscillation $\max\{|f(y) - f(z)| : d(y,z) \le \varepsilon\}$, choose a positive integer K sufficiently large such that for any $k \ge K$, $i = i(k)$, the following hold:

- $4^{-i+1} < \varepsilon$;
- $\omega_f(2^{-i-3}) \le \omega_f(2^{-i-2}) < \varepsilon/4$;
- $M_i/N_i < \varepsilon/16$.

By equation (5.20), we obtain that

$$\left| \frac{1}{b_i - a_i} \sum_{j=a_i}^{b_i-1} f(T^j y^{(i)}) - \frac{1}{b_i - a_i} \sum_{j=a_i}^{b_i-1} f(T^j x) \right| \le \omega_f(2^{-i-2}) < \frac{\varepsilon}{4}.$$

This implies

$$\left| \int_X f \, d\nu_i - \frac{1}{b_i - a_i} \sum_{j=a_i}^{b_i-1} f(T^j x) \right| < \frac{\varepsilon}{4}. \tag{5.21}$$

If $k \le a_{i+1}$, then $\gamma_k = \nu_i$. Since $\|f\| \le 1$, by equations (5.19), (5.21), and Lemma 2.3, we conclude that

$$\begin{aligned} \left| \int_X f \, d\delta_x^k - \int_X f \, d\gamma_k \right| &\le \left| \int_X f \, d\delta_x^k - \frac{1}{b_i - a_i} \sum_{j=a_i}^{b_i-1} f(T^j x) \right| \\ &\quad + \left| \frac{1}{b_i - a_i} \sum_{j=a_i}^{b_i-1} f(T^j x) - \int_X f \, d\nu_i \right| \\ &< \left| \frac{1}{k} \sum_{j=0}^{k-1} f(T^j x) - \frac{1}{b_i - a_i} \sum_{j=a_i}^{b_i-1} f(T^j x) \right| + \frac{\varepsilon}{4} \le \frac{\varepsilon}{4} + \frac{2(k - b_i + a_i)}{k} \\ &\le \frac{\varepsilon}{4} + \frac{2(a_{i+1} - b_i + a_i)}{N_i} < \frac{\varepsilon}{4} + 2 \cdot 4^{-i} < \varepsilon. \end{aligned}$$

If $k > a_{i+1}$, then one has

$$\int_X f \, d\nu_{i+1} = \frac{1}{k - r - a_{i+1}} \sum_{j=a_{i+1}}^{k-r-1} f(T^j y^{(i+1)}). \tag{5.22}$$

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By equation (5.20), we obtain that

$$\left|\frac{1}{k-r-a_{i+1}}\sum_{j=a_{i+1}}^{k-r-1} f(T^j y^{(i+1)}) - \frac{1}{k-r-a_{i+1}}\sum_{j=a_{i+1}}^{k-r-1} f(T^j x)\right| \le \omega_f(2^{-i-3}) < \frac{\varepsilon}{4}. \tag{5.23}$$

By equations (5.22) and (5.23), we obtain that

$$\left|\int_X f\, d\nu_{i+1} - \frac{1}{k-r-a_{i+1}}\sum_{j=a_{i+1}}^{k-r-1} f(T^j x)\right| < \frac{\varepsilon}{4}. \tag{5.24}$$

Define $S=([a_i, b_i-1]\cup[a_{i+1}, k-r-1])\cap\mathbb{Z}$. Clearly, $|S|=k-r-a_{i+1}+b_i-a_i$. Note that

$$\frac{1}{|S|}\sum_{j\in S} f(T^j x) = \frac{\alpha_k}{b_i-a_i}\sum_{j=a_i}^{b_i-1} f(T^j y^{(i)}) + \frac{1-\alpha_k}{k-r-a_{i+1}}\sum_{j=a_{i+1}}^{k-r-1} f(T^j y^{(i+1)}). \tag{5.25}$$

Since $\|f\|\le 1$, by equation (5.24) and Lemma 2.3, we conclude that

$$\begin{aligned}\left|\int_X f\, d\delta_x^k - \frac{1}{|S|}\sum_{j\in S} f(T^j x)\right| &= \left|\frac{1}{k}\sum_{j=0}^{k-1} f(T^j x) - \frac{1}{|S|}\sum_{j\in S} f(T^j x)\right| \\ &\le \frac{2(k-|S|)}{k} = \frac{2(a_{i+1}+r-b_i+a_i)}{k} \\ &\le \frac{2(a_{i+1}-b_i+p_{i+1})}{N_i} + \frac{a_i}{b_i}.\end{aligned}$$

Hence, we obtain by equation (5.19) that

$$\left|\int_X f\, d\delta_x^k - \frac{1}{|S|}\sum_{j\in S} f(T^j x)\right| < \frac{3}{4}\varepsilon. \tag{5.26}$$

Combining equations (5.21), (5.24), (5.25), and (5.26), we conclude that

$$\begin{aligned}\left|\int_X f\, d\delta_x^k - \int_X f\, d\gamma_k\right| \le{}& \left|\int_X f\, d\delta_x^k - \frac{1}{|S|}\sum_{j\in S} f(T^j x)\right| \\ &+ \alpha_k\left|\frac{1}{b_i-a_i}\sum_{j=a_i}^{b_i-1} f(T^j x) - \int_X f\, d\nu_i\right| \\ &+ (1-\alpha_k)\left|\frac{1}{k-r-a_{i+1}}\sum_{j=a_{i+1}}^{k-r-1} f(T^j x) - \int_X f\, d\nu_{i+1}\right| \\ <{}& \frac{3\varepsilon}{4} + \alpha_k\frac{\varepsilon}{4} + (1-\alpha_k)\frac{\varepsilon}{4} = \varepsilon.\end{aligned}$$

Therefore, for any $k\ge K$, one has

$$\left|\int_X f\, d\delta_x^k - \int_X f\, d\gamma_k\right| < \varepsilon.$$

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Hence,

$$\lim_{k\to\infty}\left|\int_X f\,d\gamma_k - \int_X f\,d\delta_x^k\right| = 0,$$

and thus

$$\mu = \lim_{k\to\infty}\gamma_k = \lim_{k\to\infty}\delta_x^k.$$

We have shown that $x \in G_\mu$. This implies equation (5.16) and completes the proof of Proposition 5.1. □

Since $G_\mu \neq \emptyset$, by Theorem 3.2, G_μ is dense in X. So statement (2) holds by Proposition 5.1.

All that is left is to show that statement (3) holds, namely there exists $\mu_0 \in \mathcal{U}$ with $G^{\mu_0} \neq \emptyset$. We just need to show that there exists $x \in X$ and $\mu_0 \in V_T(x)$ such that $G_{\mu_0} = \emptyset$.

Consider

$$x = 101^N0^N1^{N^2}0^{N^2}1^{N^3}0^{N^3}\ldots,$$

where $N \geq 2$ will be determined later. Since $c \leq 1$, we obtain that $x \in X$. We wish to find $\mu_0 \in V_T(x)$ such that $G_{\mu_0} = \emptyset$. Let $K_j = 2(1 + N + \cdots + N^{j-1}) + N^j$. We claim that

$$\frac{1}{c+1} < \lim_{j\to\infty}\int_X \chi_1\,d\delta_x^{K_j} < 1, \tag{5.27}$$

where $\chi_1(x) = x_0$ is defined by equation (4.8).

Clearly,

$$\int_X \chi_1\,d\delta_x^{K_j} = \frac{1 + N + \cdots + N^j}{K_j} = \frac{N^{j+1}-1}{N^{j+1}+N^j-2}.$$

Hence,

$$\lim_{j\to\infty}\int_X \chi_1\,d\delta_x^{K_j} = \lim_{j\to\infty}\frac{N^{j+1}-1}{N^{j+1}+N^j-2} = \frac{N}{N+1}.$$

Let $N > (1/c)$ and then

$$\lim_{j\to\infty}\int_X \chi_1\,d\delta_x^{K_j} = \frac{N}{N+1} \in \left(\frac{1}{c+1}, 1\right).$$

Let μ_0 be an accumulation point of $\delta_x^{K_j}$. By equation (5.27), we obtain that

$$\frac{1}{c+1} < \int_X \chi_1\,d\mu_0 < 1.$$

Now we show that $G_{\mu_0} = \emptyset$. Suppose for a contradiction that $G_{\mu_0} \neq \emptyset$. Arbitrarily given $x \in G_{\mu_0}$, clearly μ_0 is not a Dirac measure, so there are infinite 0 and 1 terms contained in $x_0, x_1, \ldots$. Suppose $x = 1^{s_1}0^{t_1}1^{s_2}0^{t_2}\cdots$ with $t_i > 0$ for $i \geq 1$ and $s_i > 0$ for $i > 1$. Let $n_i = s_i + t_i$ and $N_j = n_1 + \cdots + n_j$. Since $cs_i \leq t_i$, one has

$$\int_X \chi_1\,d\delta_x^{N_j} = \frac{1}{N_j}\sum_{n=0}^{N_j-1}\chi_1(T^nx) = \frac{1}{N_j}\sum_{i=1}^{j}s_i \leq \frac{1}{c+1}.$$

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This implies

$$\frac{1}{c+1} < \int_X \chi_1 \, d\mu_0 = \lim_{j\to\infty} \int_X \chi_1 \, d\delta_x^{N_j} \le \frac{1}{c+1},$$

which is a contradiction. Hence, $\mu_0 \in \mathcal{U}$ and $G^{\mu_0} \neq \emptyset$. By Theorem 3.2(8), G^{μ_0} is residual in X. This completes the proof of Theorem B.

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