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GROWTH OF A POPULATION OF BACTERIA IN A DYNAMICAL HOSTILE ENVIRONMENT

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Abstract

We study the growth of a population of bacteria in a dynamical hostile environment corresponding to the immune system of the colonized organism. The immune cells evolve as subcritical open clusters of oriented percolation and are perpetually reinforced by an immigration process, while the bacteria try to grow as a supercritical oriented percolation in the remaining empty space. We prove that the population of bacteria grows linearly when it survives. From this perspective, we build general tools to study dependent oriented percolation models issued from renormalization processes.

Keywords: Contact process; directed percolation; renormalization; random environment; stochastic domination; block construction; interacting particle system

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1. A growth model in a dynamical hostile environment

We consider the following discrete-time interacting particle system: at time $n = 0$, a particularly fertile bacterium (represented here by a type-1 particle) is submerged in a population of immune cells (type-2 particles) that are going to impede its development. The immune cells are not very fertile but benefit from a constant immigration process. Our aim is to find conditions that ensure that the growth of bacteria, when they survive, is linear.

Our system is described by a discrete-time Markov chain taking values in $\{0, 1, 2\}^{\mathbb{Z}^d}$, depending on the three parameters $p, q, \alpha \in (0, 1)$. The time is indexed by $\mathbb{N} = \{0, 1, 2, \dots\}$, and we also note $\mathbb{N}^* = \{1, 2, 3, \dots\}$. The transition between two states is in two steps. First, between time n and time $n + \frac{1}{2}$, each particle tries to colonize its neighbor sites: it succeeds with probability p if it is a type-1 particle, and with probability q if it is a type-2 particle. All events are independent and, in a case of conflict, the type-2 particle wins. Next, between time $n + \frac{1}{2}$ and time $n+1$, the immigration of type-2 particles occurs: on each site, a type-2 particle appears with probability $\alpha > 0$, possibly taking the place of the particle previously occupying the site. Once again, all events are independent.

In the degenerate case where $q = 0$ and $\alpha = 0$, we recover independent oriented percolation with parameter p, which provides a simple model for the spread of an infection. By classical arguments, there exists a critical probability $\overrightarrow{p_c}^{\text{alt}}(d + 1)$ that independent oriented percolation on $\mathbb{Z}^d \times \mathbb{N}$ grows infinitely. Of course, we choose $p > \overrightarrow{p_c}^{\text{alt}}(d + 1)$ to avoid the almost-sure extinction of the bacteria in the absence of immune cells. Hence, if $q = 0$ and $\alpha = 0$, we

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know that the bacteria survive with positive probability and, when they survive, their growth is linear. These results have been proved for the supercritical contact process by Bezuidenhout and Grimmett [1] and Durrett [5], and can readily be transposed for supercritical independent oriented percolation.

On the other hand, we choose $q < \overrightarrow{p_c}^{\mathrm{alt}}(d+1)$, which corresponds to the poor virulence of type-2 particles. However, the constant immigration rate α guarantees that type-2 particles are always present in the organism.

Let us now describe the model more formally. We work, for $d \geq 1$, on the following graph.

- The set of sites is $\mathbb{V}^{d+1} = \{(z, n) \in \mathbb{Z}^d \times \mathbb{N}\}$.
- We put an oriented edge from (z_1, n_1) to (z_2, n_2) if and only if $n_2 = n_1 + 1$ and $\|z_2 - z_1\|_1 \leq 1$; the set of these edges is denoted by $\overrightarrow{\mathbb{E}}^{d+1}_{\mathrm{alt}}$.

Define $\overrightarrow{\mathbb{E}}^d$ in the following way: in $\overrightarrow{\mathbb{E}}^d$, there is an oriented edge between two points, z_1 and z_2, in $\mathbb{Z}^d$ if and only if $\|z_1 - z_2\|_1 \leq 1$. The oriented edge in $\overrightarrow{\mathbb{E}}^{d+1}_{\mathrm{alt}}$ from (z_1, n_1) to (z_2, n_2) can be identified with the couple $((z_1, z_2), n_2) \in \overrightarrow{\mathbb{E}}^d \times \mathbb{N}^*$. Thus, we identify $\overrightarrow{\mathbb{E}}^{d+1}_{\mathrm{alt}}$ and $\overrightarrow{\mathbb{E}}^d \times \mathbb{N}^*$.

We set $\tilde{\Omega} = \{0, 1\}^{\overrightarrow{\mathbb{E}}^d} \times \{0, 1\}^{\overrightarrow{\mathbb{E}}^d} \times \{0, 1\}^{\mathbb{Z}^d}$, and we endow the set $\Omega = \tilde{\Omega}^{\mathbb{N}^*}$ with its Borel σ-algebra for the product topology. We consider the probability $\mathbb{P} = \mathbb{P}_{p,q,\alpha} = \nu^{\otimes \mathbb{N}^*}$, where

$$\nu = \nu_{p,q,\alpha} = \mathcal{B}(p)^{\otimes \overrightarrow{\mathbb{E}}^d} \otimes \mathcal{B}(q)^{\otimes \overrightarrow{\mathbb{E}}^d} \otimes \mathcal{B}(\alpha)^{\otimes \mathbb{Z}^d}$$

and where $\mathcal{B}(p)$ stands for the Bernoulli law with parameter p.

Starting from the initial configuration $x \in \{0, 1, 2\}^{\mathbb{Z}^d}$, we define the Markov chain $(\eta^x_n)_{n\geq 0}$ taking values in $\{0, 1, 2\}^{\mathbb{Z}^d}$ by

$$\eta^x_0 = x \quad \text{and} \quad \eta^x_{n+1} = f(\eta^x_n, \omega_{n+1}),$$

where $f\colon \{0, 1, 2\}^{\mathbb{Z}^d} \times \tilde{\Omega} \to \{0, 1, 2\}^{\mathbb{Z}^d}$ is defined as follows:

$$\begin{aligned} &f(x, ((\omega^e_1)_{e\in \overrightarrow{\mathbb{E}}^d}, (\omega^e_2)_{e\in \overrightarrow{\mathbb{E}}^d}, (\omega^k_3)_{k\in\mathbb{Z}^d})) \\ &\quad = \begin{pmatrix} \max\{2\omega^k_3, 2\max(\omega^{(i,k)}_2 \colon \|i-k\|_1 \leq 1,\ x_i = 2), \\ \qquad\qquad \max(\omega^{(i,k)}_1 \colon \|i-k\|_1 \leq 1,\ x_i = 1)\} \end{pmatrix}_{k\in\mathbb{Z}^d}. \end{aligned}$$

Note that type-2 particles do not see type-1 particles in their evolution, which explains why type-2 particles are assimilated to an environment. Considering two disjoint subsets E_1, E_2 of $\mathbb{Z}^d$ that represent the initial sets occupied by type-1 and type-2 particles, we also use the notation $\eta^{E_1,E_2}_n = \eta^{\mathbf{1}_{E_1}+2\mathbf{1}_{E_2}}_n$. We respectively denote by $\eta^{E_1,E_2}_{1,n}$ and $\eta^{E_2}_{2,n}$ the sets of sites occupied by type-1 particles and by type-2 particles at time n, and we consider the evolution of the bacteria population $(\eta^{\{0\},\varnothing}_{1,n})_{n\geq 0}$. Can this process survive? Does it grow linearly when it survives? We naturally introduce the following extinction time and hitting times:

$$\begin{aligned} \tau^{E_1,E_2}_1 &= \inf\{n \geq 0 \colon \eta^{E_1,E_2}_{1,n} = \varnothing\}, \\ t^{E_1,E_2}_1(y) &= \inf\{n \geq 0 \colon y \in \eta^{E_1,E_2}_{1,n}\} \quad \text{for all } y \in \mathbb{Z}^d. \end{aligned}$$

Note that $\alpha \mapsto \mathbb{P}_{p,q,\alpha}(\tau^{E_1,E_2}_1 = +\infty)$ is nonincreasing and exhibits a phase transition. We first prove that this phase transition does not depend on the initial configuration E_2 of the environment.

Theorem 1. *For every $p > \overrightarrow{p_c}^{\,\mathrm{alt}}(d+1)$ and every $q < \overrightarrow{p_c}^{\,\mathrm{alt}}(d+1)$,*

$$\mathbb{P}_{p,q,\alpha}(\tau_1^{0,\mathbb{Z}^d\setminus\{0\}} = +\infty) > 0 \quad \text{for all } \alpha \in [0,1] \quad \Longleftrightarrow \quad \mathbb{P}_{p,q,\alpha}(\tau_1^{0,\varnothing} = +\infty) > 0.$$

We thus define $\alpha_c(p,q) = \sup\{\alpha \geq 0 \colon \mathbb{P}_{p,q,\alpha}(\tau_1^{0,\varnothing} = +\infty) > 0\}$.

Our main result is the following.

Theorem 2. *For every $p > \overrightarrow{p_c}^{\,\mathrm{alt}}(d+1)$ and every $q < \overrightarrow{p_c}^{\,\mathrm{alt}}(d+1)$,*

$$0 < \alpha_c(p,q) < 1.$$

Moreover, for every $\alpha < \alpha_c(p,q)$, there exist positive constants A, B, and C such that, for every $E \subset \mathbb{Z}^d \setminus \{0\}$, $x \in \mathbb{Z}^d$, and $t > 0$,

$$\mathbb{P}_{p,q,\alpha}(\tau_1^{0,E} = +\infty) > 0, \tag{1}$$

$$\mathbb{P}_{p,q,\alpha}(\tau_1^{0,E} = +\infty,\ t_1^{0,E}(x) \geq C\|x\|_1 + t) \leq A\exp(-Bt), \tag{2}$$

$$\mathbb{P}_{p,q,\alpha}(t < \tau_1^{0,E} < +\infty) \leq A\exp(-Bt). \tag{3}$$

We thus prove that if the immigration of type-2 particles is not too important, the bacteria population survives with positive probability, and, when it survives, it grows linearly, as is the case in the absence of immune cells. We can also explain this model in terms of dependent oriented percolation: on the oriented graph $\mathbb{Z}^d \times \mathbb{N}$, for each site $(z,n) \in \mathbb{Z}^d \times \mathbb{N}$, with probability α, we erase the finite cluster of oriented percolation with parameter q starting from (z,n). The remaining random oriented graph is then given to the type-1 particle, which tries to develop as an oriented percolation with parameter p. Thus, the growth of type-1 particles can be seen as a dependent oriented percolation model, with an unbounded but exponentially fast decreasing dependence. Our result ensures the linear growth of this oriented percolation when it percolates.

A natural question concerning the existence of an asymptotic shape result arises.

Conjecture 1. *For every $p > \overrightarrow{p_c}^{\,\mathrm{alt}}(d+1)$, every $q < \overrightarrow{p_c}^{\,\mathrm{alt}}(d+1)$, and every $\alpha \in (0, \alpha_c(p,q))$, there exists a norm μ on $\mathbb{R}^d$ such that, for any two disjoints subsets E_1 and E_2 in $\mathbb{Z}^d$ with $E_1 \neq \varnothing$, we have, for $\varepsilon > 0$, $\mathbb{P}_{p,q,\alpha}(\cdot \mid \tau_1^{E_1,E_2} = +\infty)$ almost surely, and every large enough t,*

$$(1-\varepsilon)B_\mu(0,1) \subset \frac{1}{t}B_t \subset (1+\varepsilon)B_\mu(0,1),$$

where $B_t = \{x \in \mathbb{Z}^d \colon t_1^{E_1,E_2}(x) \leq t\} + [-\frac{1}{2}, \frac{1}{2}]^d$.

This conjecture is consistent with out simulations (see Figure 1). We think that this result can be proved with subadditive methods similar to those used in the case of the contact process in a random environment; see [11].

We can find a certain number of similar competition mechanisms in the literature under the name of hierarchical competition (see [8]), of contact process (or oriented percolation) in a dynamical random environment (see [3], [18], [19], and [20]), or without any specific denomination (see [9] and [10]). The common characteristic of these models is that one type of particle (here type-2 particles) evolves in a Markovian way, and that the second type evolves as a contact process or an oriented percolation in the remaining empty space.

In this paper we use renormalization techniques. This is not surprising: the efficiency of such techniques in the study of particle systems has long been known (see, for instance, [2],

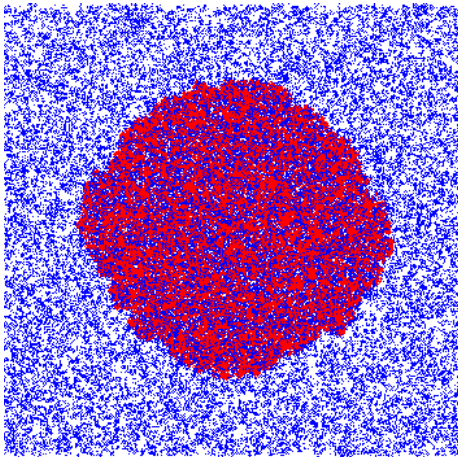

FIGURE 1: A simulation with $p = 0.7$, $q = 0.25$, and $\alpha = 10^{-3}$.

[6], or [7]), and the use of renormalization is usual to prove that survival occurs with positive probability. However, studying the system conditioned to survive can be subtle. Indeed, the renormalizationprocedures tend to destroy the independence properties given by the Markovianity, and the tried and tested restart arguments described in [4] must be adapted with some care. While the general idea remains simple, the implementation is quite technical and, for the moment, there are no ready-made tools for this kind of situation. The tools we build are in the spirit of the theorem of Liggett *et al.* [16], but in the context of dependent oriented percolations resulting from renormalization procedures; see Theorem 3.

2. Comparison and coupling results

While the setting of static renormalization can be defined quite formally, there are other types of renormalization that are harder to classify: they all consider local events that cannot be defined in an absolute way, but depend on a local component and also on the past of the renormalization process. This past can be associated to a time line as in [1] and [5], or to a sequence of spatial boxes as in [14].

After renormalization, we are led to study a dependent oriented percolation process. The fact that this process survives with positive probability can be proved quite directly from the comparison result of Liggett *et al.* [16]. However, when one wants to study the oriented percolation process conditioned to survive, things are more intricate: our Theorem 3 thus gives a general setting to ensure that 'conditioned on its survival, the oriented percolation process on $\mathbb{Z}^d \times \mathbb{N}$ built from the renormalization process stochastically dominates an independent oriented percolation process with parameter as large as we want'. The aim is of course to transfer the properties of the supercritical independent percolation process to the dependent percolation process.

We work on the graph $\mathbb{Z}^d \times \mathbb{N}$, as defined in the introduction. We consider $\Omega = \{0, 1\}^{\vec{\mathbb{E}}^{d+1}_{\text{alt}}}$ endowed with its Borel σ-algebra and the probability

$$\mathbb{P}_p = \mathcal{B}(p)^{\otimes \vec{\mathbb{E}}^{d+1}_{\text{alt}}};$$

the edges such that $\omega_e = 1$ are said to be open, with all other edges closed. For two sites v, w in $\mathbb{Z}^d \times \mathbb{N}$, we denote by $v \to w$ the existence of an open oriented path from v to w.

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The critical probability is denoted by $\overrightarrow{p_c}^{\text{alt}}(d+1)$. The time translations θ_n on Ω are defined by $\theta_n((\omega_{(e,k)})_{e\in\overrightarrow{\mathbb{E}}^d,\,k\geq 1}) = (\omega_{(e,k+n)})_{e\in\overrightarrow{\mathbb{E}}^d,\,k\geq 1}$. We set, for $n\in\mathbb{N}$ and $x\in\mathbb{Z}^d$,

$$\begin{aligned}
\xi_n^x &= \{y\in\mathbb{Z}^d\colon (x,0)\to(y,n)\},\\
\xi_n^{\mathbb{Z}^d} &= \bigcup_{x\in\mathbb{Z}^d}\xi_n^x,\\
\tau^x &= \min\{n\in\mathbb{N}\colon \xi_n^x=\varnothing\},\\
H_n^x &= \bigcup_{0\leq k\leq n}\xi_k^x,\\
K_n^x &= (\xi_n^x\Delta\xi_n^{\mathbb{Z}^d})^c = \xi_n^x\cup(\mathbb{Z}^d\setminus\xi_n^{\mathbb{Z}^d}).
\end{aligned}$$

As for the contact process, $(H_n^x)_{n\geq 0}$ and $(K_n^x\cap H_n^x)_{n\geq 0}$ grow linearly in the case of survival.

Lemma 1. *We consider independent oriented percolation on $\mathbb{Z}^d\times\mathbb{N}$. For every $p>\overrightarrow{p_c}^{\text{alt}}(d+1)$, there exist strictly positive constants A, B, and C such that, for every $x\in\mathbb{Z}^d$, and every $L,n>0$,*

$$\begin{aligned}
&\mathbb{P}_p(\tau^x=+\infty,\ [-L,L]^d\not\subset K_{CL+n}^x)\leq A\mathrm{e}^{-Bn}\\
&\mathbb{P}_p(\tau^x=+\infty,[-L,L]^d\not\subset H_{CL+n}^x)\leq A\mathrm{e}^{-Bn}.
\end{aligned}$$

Proof. For the contact process, Durrett [5] showed how to deduce an analogous result from the construction of Bezuidenhout and Grimmett [1]. As explained in [1], the proofs remain valid for oriented percolation, which is the discrete-time analog of the contact process.

We now recall the comparison theorem of Liggett *et al.* [16]. In the following, for two edges e and f in $\overrightarrow{\mathbb{E}}^d$, we denote by $d(e,f)$ the distance $\|\cdot\|_1$ between the centers of e and f.

Proposition 1. *Let $d\geq 1$ be fixed. For every $M\geq 1$, there exists a function g_M from $[0,1]$ to $[0,1]$ with $\lim_{q\to 1}g_M(q)=1$ and such that if μ is a probability measure on $\Omega=\{0,1\}^{\overrightarrow{\mathbb{E}}^d}$ satisfying $\mu(\omega_e=1\mid\omega_f,\ d(e,f)\geq M)\geq q$ for $q\in[0,1]$ and every $e\in\overrightarrow{\mathbb{E}}^d$, then μ stochastically dominates a product of Bernoulli law with parameter $g_M(q)$:*

$$\mu\succeq\mathcal{B}(g_M(q))^{\otimes\overrightarrow{\mathbb{E}}^d}.$$

Relying on this result, we prove analogous results for a certain class of dependent oriented percolations.

Definition 1. Let $d\geq 1$ be fixed, let M be a positive integer, and let $q\in(0,1)$. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space endowed with a filtration $(\mathcal{G}_n)_{n\geq 0}$. We assume that, on this probability space, a random field $(W_e^n)_{e\in\overrightarrow{\mathbb{E}}^d,\,n\geq 1}$ taking values in $\{0,1\}$ is defined. This field gives the states—open or closed—of the edges in $\overrightarrow{\mathbb{E}}^{d+1}_{\text{alt}}$. We say that the law of the field $(W_e^n)_{e\in\overrightarrow{\mathbb{E}}^d,\,n\geq 1}$ is in $\mathcal{C}_d(M,q)$ if it satisfies the following two conditions.

- For all $n\geq 1$ and all $e\in\overrightarrow{\mathbb{E}}^d$, $W_e^n\in\mathcal{G}_n$.
- For all $n\geq 0$ and all $e\in\overrightarrow{\mathbb{E}}^d$, $\mathbb{P}(W_e^{n+1}=1\mid\mathcal{G}_n\vee\sigma(W_f^{n+1},\ d(e,f)\geq M))\geq q$.

Here $\sigma(W_f^{n+1},\ d(e,f)\geq M)$ is the σ-field generated by the random variables W_f^{n+1}, with $d(e,f)\geq M$.

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First, we give a stochastic comparison between fields in $\mathcal{C}_d(M, q)$ and Bernoulli product measures.

Lemma 2. *Let $d, M \geq 1$ be positive integers, and let $q \in (0, 1)$. If the distribution of $(W_e^n)_{e\in\vec{\mathbb{E}}^d, n\geq 1}$ belongs to $\mathcal{C}_d(M, q)$ then the distribution of the field $(W_e^{n+k})_{e\in\vec{\mathbb{E}}^d, k\geq 1}$ conditioned by $\mathcal{G}_n$ stochastically dominates $\mathcal{B}(g_M(q))^{\otimes\vec{\mathbb{E}}^{d+1}_{\mathrm{alt}}}$ for each n, where the function g_M is as defined in Proposition 1.*

In other words, for each $n \geq 0$, each $A \in \mathcal{G}_n$, and each nondecreasing bounded function f, we have

$$\mathbb{E}_W[\mathbf{1}_A(f \circ \theta_n)] \geq \mathbb{P}(A) \int_{\{0,1\}^{\vec{\mathbb{E}}^{d+1}_{\mathrm{alt}}}} f \, \mathrm{d}\mathcal{B}(g_M(q))^{\otimes\vec{\mathbb{E}}^{d+1}_{\mathrm{alt}}},$$

where θ_n is the translation operator on Ω that has been defined previously.

Proof. Let $E = \{0, 1\}^{\vec{\mathbb{E}}^d}$ and $q' = g_M(q)$, and fix $n \geq 1$. We will show that, for each nonnegative integer k and every nondecreasing bounded function f that depends only on the first k time coordinates, we have

$$\mathbb{E}[\mathbf{1}_A\, f(W^{n+1}, W^{n+2}, \dots, W^{n+k})] \geq \mathbb{P}(A) \int f \, \mathrm{d}\mathcal{B}(q')^{\otimes\vec{\mathbb{E}}^{d+1}_{\mathrm{alt}}}.$$

When $k = 0$, f is constant and the result is obvious.

Suppose that the result holds for k and let us prove it for $k + 1$. Let h be a nondecreasing bounded function on E^{k+1}, and consider $A \in \mathcal{G}_n$. Since we work on a Polish space, we can disintegrate $\mathbb{P}$ with respect to the σ-field $\mathcal{G}_{n+k}$ (see, e.g. [22, p. 256]). Then, we have, with the notation of Stroock [22],

$$\begin{aligned}
&\mathbb{E}[\mathbf{1}_A\, h(W^{n+1}, \dots, W^{n+k+1})] \\
&\quad = \mathbb{E}[\mathbf{1}_A\, \mathbb{E}[h(W^{n+1}, \dots, W^{n+k+1}) \mid \mathcal{G}_{n+k}]] \\
&\quad = \int_A \int_\Omega h(W^{n+1}(\omega'), \dots, W^{n+k}(\omega'), W^{n+k+1}(\omega')) \, \mathrm{d}\mathbb{P}_\omega^{\mathcal{G}_{n+k}}(\omega') \, \mathrm{d}\mathbb{P}(\omega) \\
&\quad = \int_A \int_\Omega h(W^{n+1}(\omega), \dots, W^{n+k}(\omega), W^{n+k+1}(\omega')) \, \mathrm{d}\mathbb{P}_\omega^{\mathcal{G}_{n+k}}(\omega') \, \mathrm{d}\mathbb{P}(\omega).
\end{aligned}$$

Since we supposed that the distribution of $(W_e^n)_{e\in\vec{\mathbb{E}}^d, n\geq 1}$ belongs to $\mathcal{C}_d(M, q)$, the distribution of $(W_e^{n+k+1})_{e\in\vec{\mathbb{E}}^d}$ under $\mathbb{P}_\omega^{\mathcal{G}_{n+k}}$ satisfies, for every fixed ω, the assumptions of the Liggett–Schonmann–Stacey comparison theorem (Theorem 1). Thus, it stochastically dominates $\mathcal{B}(q')^{\otimes\vec{\mathbb{E}}^d}$, which gives

$$\begin{aligned}
&\int_\Omega h(W^{n+1}(\omega), \dots, W^{n+k}(\omega), W^{n+k+1}(\omega')) \, \mathrm{d}\mathbb{P}_\omega^{\mathcal{G}_{n+k}}(\omega') \\
&\qquad \geq \int_E h(W^{n+1}(\omega), \dots, W^{n+k}(\omega), x) \, \mathrm{d}\mathcal{B}(q')^{\otimes\vec{\mathbb{E}}^d}(x) \\
&\qquad = f(W^{n+1}(\omega), \dots, W^{n+k}(\omega)),
\end{aligned}$$

where f is defined by

$$f(y_1, \dots, y_k) = \int_E h(y_1, \dots, y_k, x) \, \mathrm{d}\mathcal{B}(q')^{\otimes\vec{\mathbb{E}}^d}(x). \tag{4}$$

Thus, we obtain

$$\mathbb{E}[\mathbf{1}_A\, h(W^{n+1},\dots,W^{n+k})] \geq \int_A f(W^{n+1},\dots,W^{n+k})\,\mathrm{d}\mathbb{P}.$$

By the induction assumption,

$$\int_A f(W^{n+1},\dots,W^{n+k})\,\mathrm{d}\mathbb{P} \geq \mathbb{P}(A)\int_{E^k} f(y_1,\dots,y_k)\,\mathrm{d}(\mathcal{B}(q')^{\otimes \vec{\mathbb{E}}^d})^{\otimes k},$$

which, using (4), yields the desired result.

We now associate to every $\{0,1\}$-valued random field $(W_e^n)_{e\in\vec{\mathbb{E}}^d,\,n\geq 1}$ an oriented percolation process $(\xi_n^0(W))_{n\geq 1} = (\xi_n^0)_{n\geq 1}$ starting from $(0_{\mathbb{Z}^d},0)$ and defined in the usual way:

$$\xi_0^0 = \{0\}, \qquad \xi_{n+1}^0 = \{x\in\mathbb{Z}^d : \text{there exists } y\in\xi_n^0,\ W_{(y,x)}^{n+1} = 1\}.$$

For simplicity, we will often say 'oriented percolation in $\mathcal{C}_d(M,q)$' instead of 'oriented percolation associated to a field $\chi\in\mathcal{C}_d(M,q)$'.

We define the extinction time of the oriented percolation associated to W and starting from $(0_{\mathbb{Z}^d},0)$ as

$$\tau^0(W) = \tau^0 = \inf\{n\geq 1 : \xi_n^0 = \varnothing\}.$$

The following result allows a coupling between surviving dependent percolation in $\mathcal{C}_d(M,q)$ and supercritical Bernoulli percolation.

Theorem 3. *Let $d, M\geq 1$ be fixed positive integers, and let $q\in(0,1)$ be such that $g_M(q) > \vec{p}_{\mathrm{c}}^{\,\mathrm{alt}}(d+1)$.*

There exist positive constants β and γ such that, for each field $\chi\in\mathcal{C}_d(M,q)$, we can find a probability space where live a field $W = (W_e^n)_{e\in\vec{\mathbb{E}}^d,\,n\geq 1}$, a field $(W_e'^n)_{e\in\vec{\mathbb{E}}^d,\,n\geq 1}$, both taking both values in $\{0,1\}$, an $\mathbb{N}$-valued random variable T, and a $\mathbb{Z}^d$-valued random variable D such that:

- *$\|D\|_1 \leq T$ and $\mathbb{E}[\exp(\beta T)] \leq \gamma$;*
- *the field $(W_e^n)_{e\in\vec{\mathbb{E}}^d,\,n\geq 1}$ follows the distribution χ and $\mathbb{P}(\tau^0(W)=\infty) > 0$;*
- *$T = \tau^0(W)$ on the event $\{\tau^0(W) < +\infty\}$;*
- *conditioning on $\{\tau^0(W) = +\infty\}$, the open cluster issued from $(0_{\mathbb{Z}^d},0)$ of the field $(W_e'^n)_{e\in\vec{\mathbb{E}}^d,\,n\geq 1}$ has the same distribution as the open cluster issued from $(0_{\mathbb{Z}^d},0)$ conditioned on survival in independent oriented percolation with parameter $g_M(q)$; moreover, on $\{\tau^0(W) = +\infty\}$, we have*

$$\xi_{T+n}^0(W) \supset D + \xi_n^0(W') \quad \textit{for all } n\geq 0.$$

In fact, this theorem contains two results:

- it ensures the existence of an embedded, independent infinite cluster in the dependent infinite cluster, and controls its position;
- when the dependent cluster is finite, it also controls its height.

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Proof of Theorem 3. Define $q' = g_M(q)$. Let $E_1, \dots, E_n$ be finite subsets of $\mathbb{Z}^d$. We define $E = (E_1, \dots, E_n)$ and $|E| = n$. The event

$$A_E = \bigcap_{i=1}^{|E|} \{\xi_i^0 = E_i\}$$

is in $\mathcal{G}_n$; on this event, the history of the directed percolation process starting from $(0_{\mathbb{Z}^d}, 0)$ up to time n is characterized by E.

From now on, we only consider histories satisfying $\chi(A_E) > 0$; for such a history, we define a probability measure m_E on $\{0,1\}^{\overrightarrow{\mathbb{E}}^d \times \mathbb{N}^*}$ by

$$m_E(B) = \chi((W^{|E|+k})_{k \geq 1} \in B \mid A_E);$$

we call it the law of the dependent oriented percolation with history E. Thanks to Lemma 2, the probability measure m_E stochastically dominates $\mathcal{B}(q')^{\otimes \overrightarrow{\mathbb{E}}^d \times \mathbb{N}^*}$.

Strassen's theorem (see [21] and also [17]) allows us to build a law ν_E on $(\{0,1\}^{\overrightarrow{\mathbb{E}}^d \times \mathbb{N}^*})^2$ with marginals m_E and $\mathcal{B}(q')^{\otimes \overrightarrow{\mathbb{E}}^d \times \mathbb{N}^*}$, concentrated on $\{x \geq y\}$, with

$$x \geq y \quad \text{for all } (x, y) \in (\{0,1\}^{\overrightarrow{\mathbb{E}}^d \times \mathbb{N}^*})^2 \quad \Leftrightarrow \quad x_e \geq y_e \quad \text{for all } e \in \overrightarrow{\mathbb{E}}^d \times \mathbb{N}^*.$$

For every history E, the law ν_E allows us to establish a coupling between the states of the bonds in dependent and independent oriented percolations with common history E. Now, on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we can construct a family of $(\{0,1\}^{\overrightarrow{\mathbb{E}}^d \times \mathbb{N}^*})^2$-valued independent processes $({}^E\eta, {}^E\eta')_E$, which are indexed by the collection of all histories E in such a way that, for every history E,

$$({}^E\eta_e^n, {}^E\eta_e'^n)_{e \in \overrightarrow{\mathbb{E}}^d, n \geq 1} \overset{\text{law}}{=} \nu_E.$$

We denote by ${}^E\tau^x$ the time for the independent directed percolation related to ${}^E\eta'$ and starting from x (and not from the whole history E) to die. We write $\xi_n({}^E\eta)$ to denote the state at time n of the dependent percolation process with history E; thus, $\xi_0({}^E\eta) = E_{|E|}$. We also denote by ${}^EN^x = (\xi_1({}^E\eta), \dots, \xi_{{}^E\tau^x}({}^E\eta))$ the sequence of configurations occupied up to time ${}^E\tau^x$ by the dependent percolation process associated to η with history E and denote by ${}^EL^x$ its terminal configuration.

In words, given a history E and a point x, we run the coupling between the independent percolation associated to ${}^E\eta$ and the dependent percolation associated to ${}^E\eta'$ up to time ${}^E\tau^x$ when the cluster issued from x in the independent percolation dies out. We then store the new history of the dependent percolation in ${}^EN^x$ and its final state in ${}^EL^x$. Note that:

- the percolation fields both have history E;
- we define the whole percolation fields, not only the clusters issued from a specific set;
- we run the coupling until time ${}^E\tau^x$ when the open cluster issued from x in the independent percolation dies out;
- if the terminal configuration ${}^EL^x$ of the dependent percolation is empty then, by a stochastic comparison, ${}^E\tau^x$ is also the lifetime of the dependent percolation after history E.

We then build three sequences: a sequence of sites (x_n), a sequence of times (t_n), and a sequence of compatible histories (ε_n) which we recursively define as follows. Denote by Δ a cemetery point added to $\mathbb{Z}^d$, and let $\varepsilon_0 = \{0\}$, $t_0 = 0$, and $x_0 = 0$.

- If $x_i = \Delta$ then $t_{i+1} = +\infty$, $x_{i+1} = \Delta$, and $\varepsilon_{i+1} = \varepsilon_i$.

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- If $x_i \neq \Delta$ (and, thus, $t_i < +\infty$) then $t_{i+1} = t_i + {}^{\varepsilon_i}\tau^{x_i}$; if, moreover, ${}^{\varepsilon_i}\tau^{x_i} < +\infty$ and ${}^{\varepsilon_i}L^{x_i} \neq \varnothing$, then
$$x_{i+1} = \min {}^{\varepsilon_i}L^{x_i} \quad \text{and} \quad \varepsilon_{i+1} = (\varepsilon_i, {}^{\varepsilon_i}N^{x_i}),$$
where min denotes the lexicographic order on $\mathbb{Z}^d$. Otherwise, set $x_{i+1} = \Delta$ and $\varepsilon_{i+1} = \varepsilon_i$.

Then define

$$K = \min\{k \geq 1 \colon t_{k+1} = +\infty\}, \qquad T = t_K, \quad \text{and} \quad D = x_K.$$

For $i \leq K$ and $e \in \overrightarrow{\mathbb{E}}^d$, put $W_e^n = {}^{\varepsilon_i}\eta_e^{n-t_i}$ for $n \in [t_i, t_{i+1})$. Finally, for each $n \geq 1$ and each $e \in \overrightarrow{\mathbb{E}}^d$, define $W_e'^n = {}^{\varepsilon_K}\eta_{e-x_K}'^n$.

This procedure, which is close to the classical so-called ‘restart argument’, can be described as follows: starting from 0, we construct a coupling ${}^{\{0\}}\nu$ between dependent and independent percolations up to time $t_1 = {}^{\{0\}}\tau^0$ when independent percolation dies. Then we record the history of the dependent percolation in ε_1, and pick some point x_1 occupied by the dependent percolation process in the terminal configuration. We then construct another coupling ${}^{\varepsilon_1}\nu$ between the dependent percolation and some new independent percolation process starting from x_1, following this coupling until time t_2 when the new independent percolation also dies. We can complement the history of the dependent percolation and get ε_2, then choose x_2 occupied by the dependent percolation process in the terminal configuration, and so on.

We will soon see that K is almost surely finite; hence, $t_K < +\infty$ and $t_{K+1} = +\infty$. This can occur for one of the following reasons:

- ${}^{\varepsilon_K}\tau^{x_K} = +\infty$, which means that the independent oriented percolation starting from x_K at time t_K lives forever (and so does the dependent oriented percolation by stochastic domination);
- ${}^{\varepsilon_K}\tau^{x_K} < +\infty$ and ${}^{\varepsilon_K}L^{x_K} = \varnothing$, which means that the dependent oriented percolation died exactly at the same time as the independent oriented percolation starting from x_K at time t_K.

This procedure stops either because we find a time t_K when our Kth independent percolation process survives, or because the dependent percolation process died together with the independent percolation process.

Let us denote by $\mathcal{T}_n$ the σ-field generated by the $({}^E\eta, {}^E\eta')_{|E| \leq n}$. We have, for $\alpha > 0$,

$$\begin{aligned}
\mathbb{E}[\exp(\alpha^{\varepsilon_n}\tau^{x_n})\,\mathbf{1}_{\{K>n\}} \mid \mathcal{T}_{t_n}] &= \mathbb{E}[\exp(\alpha^{\varepsilon_n}\tau^{x_n})\,\mathbf{1}_{\{t_{n+1}<+\infty\}} \mid \mathcal{T}_{t_n}] \\
&= \mathbb{E}[\exp(\alpha^{\varepsilon_n}\tau^{x_n})\,\mathbf{1}_{\{t_n<+\infty,\, {}^{\varepsilon_n}L^{x_n}\neq\varnothing,\, {}^{\varepsilon_n}\tau^{x_n}<+\infty\}} \mid \mathcal{T}_{t_n}] \\
&\leq \mathbf{1}_{\{K>n-1\}} \int \mathbf{1}_{\{\tau^0<+\infty\}} \exp(\alpha\tau^0)\,\mathrm{d}\mathcal{B}(q')^{\otimes \overrightarrow{\mathbb{E}}^d \times \mathbb{N}^*}.
\end{aligned}$$

Thus, since $q' > \overrightarrow{p_c}^{\mathrm{alt}}(d+1)$, if we put $r = \int \mathbf{1}_{\{\tau^0<+\infty\}} \exp(\alpha\tau^0)\,\mathrm{d}\mathcal{B}(q')^{\otimes \overrightarrow{\mathbb{E}}^d \times \mathbb{N}^*}$, we can choose $\alpha > 0$ small enough to have $r < 1$. Then

$$\begin{aligned}
\mathbb{E}[\exp(\alpha t_{n+1})\,\mathbf{1}_{\{K=n+1\}}] &\leq \mathbb{E}[\exp(\alpha({}^{\varepsilon_0}\tau^{x_0} + \cdots + {}^{\varepsilon_n}\tau^{x_n}))\,\mathbf{1}_{\{K>n\}}] \\
&\leq r\mathbb{E}[\exp(\alpha t_n)\,\mathbf{1}_{\{K>n-1\}}] \\
&\leq r^{n+1};
\end{aligned}$$

thus, $\mathbb{E}[\exp(\alpha T)] \leq \sum_{i=0}^{+\infty} r^{i+1} = r/(1-r)$.

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Since $K \le T$, we obtain the existence of exponential moments for K, and the fact that K is almost surely finite.

Stacking up the conditional laws, we can check that the field W has the desired distribution.

Assume that $\tau^0(W) < +\infty$ and $K = k$. Then $t_k < +\infty$ and $t_{k+1} = +\infty$. For each $n \in [t_k, +\infty)$, we have, by construction, $(W_e^n)_{e\in\vec{\mathbb{E}}^d} = ({}^{\varepsilon_k}\eta_e^{n-t_k})_{e\in\vec{\mathbb{E}}^d}$. If ${}^{\varepsilon_k}L^{x_k} \neq \varnothing$ then $t_{k+1} = t_k + {}^{\varepsilon_k}\tau^{x_k} = +\infty$, implying that ${}^{\varepsilon_k}\tau^{x_k} = +\infty$, which cannot happen because $\tau^0(W) < +\infty$. Thus, ${}^{\varepsilon_k}L^{x_k} = \varnothing$, so $\tau^0(W) \le t_k = T$. The inequality $\tau^0(W) \ge t_k$ directly follows from the inclusion between independent and dependent percolations. Finally, if $\tau^0(W) < +\infty$ then $T = \tau^0(W)$.

On the event $\{\tau^0(W) = +\infty\}$, we have, by construction, $D \in \xi_T^0(W)$, so the inclusion property gives $\xi_{T+n}^0(W) \supset D + \xi_n^0(W')$ for all $n \ge 0$. Let B be any Borel set B in $\{0,1\}^{\vec{\mathbb{E}}_{\mathrm{alt}}^{d+1}}$, and define, for $x \in \mathbb{Z}^d$, $x \cdot B = \{(\eta_{e+x}^n)_{e\in\vec{\mathbb{E}}^d, n\ge 1} : \eta \in B\}$. Noting that $\{\tau^0(W) = +\infty, K = n, \varepsilon_n = E, x_n = x\} \subset \{{}^E\tau^x = +\infty\}$, we obtain, by independence,

$$\begin{aligned}
&\mathbb{P}(\tau^0(W) = +\infty,\ K = n,\ \varepsilon_n = E,\ x_n = x,\ W' \in B)\\
&\quad= \mathbb{P}(\tau^0(W) = +\infty,\ K = n,\ \varepsilon_n = E,\ x_n = x,\ {}^E\eta' \in (-x)B)\\
&\quad= \mathbb{P}(\tau^0(W) = +\infty,\ K \ge n,\ \varepsilon_n = E,\ x_n = x,\ {}^E\eta' \in (-x)B,\ {}^E\tau^x = +\infty)\\
&\quad= \mathbb{P}(\tau^0(W) = +\infty,\ K \ge n,\ \varepsilon_n = E,\ x_n = x)\mathbb{P}({}^E\tau^x = +\infty,\ {}^E\eta' \in (-x)B)\\
&\quad= \mathbb{P}(\tau^0(W) = +\infty,\ K \ge n,\ \varepsilon_n = E,\ x_n = x)\mathbb{P}_{q'}(\tau^0 = +\infty, B).
\end{aligned}$$

Summing over all possible values for E, n, and x, we obtain the existence of c such that

$$\mathbb{P}(\tau^0(W) = +\infty,\ W' \in B) = c\mathbb{P}_{q'}(\tau^0 = +\infty, B) \quad \text{for all } B \in \mathbb{B}(\{0,1\}^{\vec{\mathbb{E}}_{\mathrm{alt}}^{d+1}}).$$

The constant c is identified by taking $B = \Omega$, so we obtain $\mathbb{P}(W' \in B \mid \tau^0(W) = +\infty) = \mathbb{P}_{q'}(B \mid \tau^0 = +\infty)$.

3. Some properties of dependent oriented percolation

The coupling theorem, Theorem 3, permits the transfer of some properties from supercritical independent oriented percolations to dependent oriented percolations in $\mathcal{C}_d(M, q)$ for q close to 1. In practice, such processes often arise after the use of a dynamical renormalization scheme.

As a by-product of the proof of Theorem 3, we obtain information on the exponential moments of the extinction times. For oriented Bernoulli percolation, a Peierls-like argument shows that

$$\lim_{p\to 1} \inf_{\beta>0} \int \mathbf{1}_{\{\tau^0<+\infty\}} \exp(\beta\tau^0)\, \mathrm{d}\mathbb{P}_p = 0, \tag{5}$$

which can be transposed to the dependent fields of $\mathcal{C}_d(M, q)$ as follows.

Corollary 1. *Let $\varepsilon > 0$ and $M > 1$. There exist $\beta > 0$ and $q < 1$ such that, for each $\chi \in \mathcal{C}_d(M, q)$,*

$$\mathbb{E}_\chi[\mathbf{1}_{\{\tau^0<+\infty\}} \exp(\beta\tau^0)] \le \varepsilon.$$

Proof. We observed in the proof of Theorem 3 that $T = \tau^0$ when $\{\tau^0 < +\infty\}$. We also have the bound

$$\mathbb{E}_\chi[\mathbf{1}_{\{\tau^0<+\infty\}} \exp(\beta\tau^0)] \le \mathbb{E}_\chi[\mathrm{e}^{\beta T}] \le \frac{r}{1-r},$$

with $r = \int \mathbf{1}_{\{\tau^0<+\infty\}} \exp(\beta\tau^0)\, \mathrm{d}\mathbb{P}_{g_M(q)}$; the result then follows from (5).

As a direct application of the coupling theorem, Theorem 3, the linear growth of the set H_n of points reached before time n, given in Lemma 1 for independent directed percolation, can be transposed to any dependent percolation in $\mathcal{C}_d(M, q)$.

Corollary 2. *Let $d, M \geq 1$ be fixed positive integers, and let $q \in (0, 1)$ be such that $g_M(q) > \overrightarrow{p_c}^{\text{alt}}(d+1)$. There exist positive constants β, D_1, and D_2, and random variables $(S^y)_{y\in\mathbb{Z}^d}$ such that*

$$\mathbb{E}[\mathrm{e}^{\beta S^y}] \leq D_2 \quad \textit{for all } y \in \mathbb{Z}^d,$$

and such that, for each field $\chi \in \mathcal{C}_d(M, q)$, on the event $\{\tau^y = +\infty\}$, the directed percolation associated to χ satisfies

$$y + [-D_1 n, D_1 n]^d \subset H^y_{S^y+n} \quad \textit{for all } n \in \mathbb{N}.$$

Having in mind an accurate study of certain particle systems, it could be interesting to have estimates on the density of bi-infinite points in the dependent oriented percolation. Thus, we define

$$G(x, y) = \{k \in \mathbb{N}, (x, 0) \to (y, k) \to \infty\},$$
$$\gamma(\theta, x, y) = \inf\{n \in \mathbb{N} \colon |\{0, \dots, k\} \cap G(x, y)| \geq \theta k \text{ for all } k \geq n\}.$$

Corollary 3. *Let $M > 1$. There exist $q_0 < 1$ and positive constants A, B, θ, and β such that, for each $\chi \in \mathcal{C}_d(M, q_0)$, we have*

$$\mathbb{P}(+\infty > \gamma(\theta, x, y) > \beta\|x - y\|_1 + n) \leq A\mathrm{e}^{-Bn} \quad \textit{for all } x, y \in \mathbb{Z}^d \textit{and all } n \geq 0.$$

Such estimates allow us to study the large deviations of the asymptotic shape of the contact process [13]. Considering Theorem 3, Lemma 3 easily follows from the independent case. We define

$$\tilde{I}_\infty = \{(x, n) \in \mathbb{Z}^d \times \mathbb{N} \colon \mathbb{Z}^d \times \{0\} \to (x, n) \to \infty\}.$$

Note that if $x \in K^0_k$ and $(x, k) \in \tilde{I}_\infty$, then, by the definition of the coupled region K^0_k, $(0_{\mathbb{Z}^d}, 0) \to (x, k) \to \infty$.

Lemma 3. *Consider independent directed percolation on $\mathbb{Z}^d \times \mathbb{N}$. For each $\rho \in (0, 1)$, there exists $p_0(\rho) < 1$ such that, for each $p > p_0(\rho)$,*

$$\mathbb{P}_p(A \cap \tilde{I}_\infty = \varnothing) \leq 16\rho^{|A|-2} \quad \textit{for all finite } A \subset \{0\} \times \mathbb{N}.$$

Proof. Note first that, by inclusion, it is sufficient to prove the lemma for $d = 1$; when $d = 1$, we can use contour arguments. The oriented graph we defined is not the classical graph for oriented percolation in dimension 2: our graph has more edges. However, once again, by inclusion, it is sufficient to prove the lemma for the classical oriented percolation model in dimension 2 (see, for example, [4]) for which the dual graph is particularly simple. So we consider independent and identically distributed (i.i.d.) percolation with parameter p on the following oriented graph $\mathcal{L}_+$.

- The set of sites is $\mathcal{V} = \{(z, n) \in \mathbb{Z} \times \mathbb{Z} \colon |z| + n \text{ is even}\}$.
- There is an oriented edge from (z_1, n_1) to (z_2, n_2) if and only if $n_2 = n_1 + 1$ and $|z_2 - z_1| = 1$.

The critical probability for this model is denoted by $\overrightarrow{p_c}$.

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We define $\mathcal{L}_-$ by simply reversing the oriented edges of $\mathcal{L}_+$. The state—open or closed—of an edge is the same in the two graphs. We respectively denote by '$\to_+$' and '$\to_-$' the events of being linked by an open oriented path in $\mathcal{L}_+$ and in $\mathcal{L}_-$. As before, we define, for $\varepsilon \in \{+, -\}$,

$$\begin{aligned}
\xi^x_{\varepsilon,n} &= \{y \in \mathbb{Z}\colon (y, n) \in \mathcal{L}_\varepsilon(1),\ (x, 0) \to_\varepsilon (y, n)\},\\
\tau^x_\varepsilon &= \max\{\varepsilon n \in \mathbb{N}\colon \xi^x_{\varepsilon,n} \neq \varnothing\},\\
I^\varepsilon_\infty &= \{(x, n) \in \mathcal{L}_\varepsilon\colon \tau^x_\varepsilon \circ \theta_{\varepsilon n} = +\infty\},\\
I_\infty &= I^+_\infty \cap I^-_\infty.
\end{aligned}$$

As $\tilde{I}_\infty \supset I_\infty$, it is sufficient to prove the lemma when we replace $\tilde{I}_\infty$ by I_∞. Let A be a fixed finite subset of $\{0\} \times 2\mathbb{N}$ and let n be the smallest integer larger than $|A|/2$. Then

$$\begin{aligned}
\mathbb{P}_p(A \cap I_\infty = \varnothing) &\le \mathbb{P}_p(\text{there exists } B \subset A;\, |B| = n;\, B \cap I^+_\infty = \varnothing)\\
&\quad + \mathbb{P}_p(\text{there exists } B \subset A;\, |B| = n;\, B \cap I^-_\infty = \varnothing)\\
&\le 2\mathbb{P}_p(\text{there exists } B \subset A;\, |B| = n;\, B \cap I^+_\infty = \varnothing)\\
&\le 2 \sum_{B \subset A,\, |B| = n} \mathbb{P}_p(B \cap I^+_\infty = \varnothing).
\end{aligned}$$

We work from now on with the graph $\mathcal{L}_+$. We fix a finite set $B \subset A$. For $v \in \mathcal{V}$, denote by $C(v)$ the open cluster starting from v:

$$C(v) = \{w \in \mathcal{V}\colon v \to_+ w\}.$$

We set $C^f(v) = C(v)$ if $C(v)$ is finite and $C^f(v) = \varnothing$ otherwise. We also set

$$C^f(B) = \bigcup_{v \in B} C^f(v).$$

If $C \subset \mathcal{V}$ is a finite set of vertices, we denote by $\partial_{\mathrm{e}} C$ the set of edges entering in or exiting from C and by $\partial^*_{\mathrm{e}} C$ the union of the segment lines corresponding to the dual edges of $\partial_{\mathrm{e}} C$: it is a union of circuits. Note that

$$|\partial_{\mathrm{e}} C| \ge 2|C \cap (\{0\} \times 2\mathbb{Z})|.$$

Thus, as $B \subset A \subset \{0\} \times 2\mathbb{Z}$,

$$\{B \cap I^+_\infty = \varnothing\} \subset \{B \subset C^f(B)\} \subset \{|\partial_{\mathrm{e}} C^f(B)| \ge 2|B|\},$$

and so

$$\mathbb{P}_p(B \cap I^+_\infty = \varnothing) \le \sum_{i \ge |B|/2} \mathbb{P}(|\partial_{\mathrm{e}} C^f(B)| = 4i) = \sum_{i \ge |A|/4} \mathbb{P}(|\partial_{\mathrm{e}} C^f(B)| = 4i).$$

Let i be a fixed integer, and assume that $|\partial_{\mathrm{e}} C^f(B)| = 4i$. Note first that all edges exiting $C^f(B)$ must be closed. Looking on a 'diagonal line', we see that there are at least as many edges exiting $C^f(B)$ as edges entering $C^f(B)$ (here we count an edge which is both entering $C^f(B)$ and exiting $C^f(B)$ as an exiting edge), and, thus, at least half of the edges in $C^f(B)$ must be closed. Next, $\partial^*_{\mathrm{e}} C^f(B)$ is composed of at most i circuits. In $C^f(B)$, consider the set of minima for the order relation '$\to$': all edges entering B in these points are necessarily in $\partial_{\mathrm{e}} C^f(B)$, which allows us to root the circuits of $\partial_{\mathrm{e}}^* C^f(B)$ to some points in B. So,

$$\mathbb{P}_p(\partial_{\mathrm{e}} C^f(B)) = 4i) \le \binom{|B|}{i} 4^{4i} \mathbb{P}\bigg(\sum_{k=1}^{4i} X_k \le 2i\bigg) \le 2^n 4^{4i} \mathbb{P}\bigg(\sum_{k=1}^{4i} X_k \le 2i\bigg),$$

where $(X_k)_{k\ge 1}$ are i.i.d. random variables with Bernoulli law of parameter p. Now, large deviation inequalities imply that, for every $r \in (0, 1)$, there exists $p(r) \in (0, 1)$ such that, for all $p \ge p(r)$,

$$\mathbb{P}\bigg(\sum_{k=1}^{4i} X_k \le 2i\bigg) \le r^{4i}.$$

Let $\rho \in (0, 1)$ be fixed, and apply the previous estimate for $r = \rho/4\sqrt{2} \in (0, 1)$. This gives, for every $p \ge p(r)$,

$$\begin{aligned}
\mathbb{P}_p(A \cap I_\infty = \varnothing) &\le 2 \sum_{B\subset A,\, |B|=n} \mathbb{P}_p(B \cap I_\infty^+ = \varnothing) \\
&\le 2 \sum_{i\ge |A|/4} \mathbb{P}(\partial_{\mathrm{e}} C^f(B) = 4i) \\
&\le 4 \times 2^{|A|/2} \sum_{i\ge|A|/4} (4r)^{4i} \\
&\le \frac{4}{1-4r} (4\sqrt{2} r)^{|A|} \\
&\le 16 \rho^{|A|}.
\end{aligned}$$

Lemma 4. *We consider independent directed percolation on $\mathbb{Z}^d \times \mathbb{N}$. There exist positive constants A, B, θ, β, and $p < 1$ such that, for every $x, y \in \mathbb{Z}^d$,*

$$\mathbb{P}_p(\tau^x = +\infty,\ \gamma(\theta, x, y) \ge \beta\|y - x\|_\infty + n) \le A\mathrm{e}^{-Bn} \quad \textit{for all } n \in \mathbb{N}. \tag{6}$$

Proof. We actually prove the following simpler result: there exists p close to 1, and positive constants A, B, C', and θ such that, for all $x \in \mathbb{Z}^d$ and all $n \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{P}_p(\tau^0 = +\infty,\ |k \in \{C'\|x\|_\infty, \dots, C'\|x\|_\infty + n \colon (0_{\mathbb{Z}^d}, 0) \to (x, k) \to \infty\}| \le \theta n) \\
&\quad \le A\mathrm{e}^{-Bn}.
\end{aligned} \tag{7}$$

Let us show that (7) implies (6). We note that $\gamma(\theta, x, y)$ has the same distribution as $\gamma(\theta, 0, y - x)$ and that $\theta < 1$. Then, using (7),

$$\begin{aligned}
&\mathbb{P}_p\bigg(\tau^0 = +\infty,\ \gamma(\theta, 0, x) \ge \frac{C'}{\theta}\|x\|_\infty + n\bigg) \\
&\quad = \mathbb{P}_p\bigg(\tau^0 = +\infty,\ \text{there exists } k \ge \frac{C'}{\theta}\|x\|_\infty + n, \\
&\qquad\qquad |\{l \in \{0, \dots, k\} \colon (0_{\mathbb{Z}^d}, 0) \to (x, l) \to +\infty\}| \le \theta k\bigg) \\
&\quad \le \mathbb{P}_p(\tau^0 = +\infty,\ \text{there exists } k \ge n, \\
&\qquad\qquad |\{l \in \{C'\|x\|_\infty, \dots, C'\|x\|_\infty + k\} \colon (0_{\mathbb{Z}^d}, 0) \to (x, l) \to +\infty\}| \le \theta k)
\end{aligned}$$

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$$\leq \sum_{k\geq n} \mathbb{P}_p(\tau^0 = +\infty,$$

$$|\{l \in \{C'\|x\|_\infty, \dots, C'\|x\|_\infty + k\} \colon (0_{\mathbb{Z}^d}, 0) \to (x, l) \to +\infty\}| \leq \theta k)$$

$$\leq \sum_{k\geq n} A\exp(-Bk).$$

Taking $\beta = C'/\theta$, this proves (6).

Let us now prove (7). We define

$$\tilde{I}_\infty = \{(x,n) \in \mathbb{V}^{d+1} \colon \mathbb{Z}^d \times \{0\} \to (x,n) \to \infty\}.$$

Note that if $x \in K_k^0$ and $(x,k) \in \tilde{I}_\infty$, then, by the definition of the coupled region, $(0_{\mathbb{Z}^d}, 0) \to (x,k) \to \infty$. We take $C' = \lceil 1/C \rceil$, where C is given in Lemma 1: we choose any θ with $0 < \theta < \frac{1}{4}$. Then

$$\begin{aligned}
&\mathbb{P}_p(\tau^0 = +\infty,\ |\{k \in \{C'\|x\|_\infty, \dots, C'\|x\|_\infty + n\} \colon (0_{\mathbb{Z}^d}, 0) \to (x,k) \to \infty\}| \leq \theta n) \\
&\quad \leq \mathbb{P}_p\biggl(\tau^0 = +\infty,\ \text{there exists } k \geq C'\|x\|_\infty + \frac{n}{2},\ K_k^0 \not\supset [-CC'\|x\|_\infty, CC'\|x\|_\infty]^d\biggr) \\
&\qquad + \mathbb{P}_p\biggl(\biggl|\biggl\{k \in \biggl\{C'\|x\|_\infty + \frac{n}{2}, \dots, C'\|x\|_\infty + n\biggr\} \colon (x,k) \in \tilde{I}_\infty\biggr\}\biggr| \leq \theta n\biggr).
\end{aligned}$$

For the first term, using Lemma 1,

$$\begin{aligned}
&\mathbb{P}_p(\tau^0 = +\infty,\ \text{there exists } k \geq C'\|x\|_\infty + \frac{n}{2},\ K_k^0 \not\supset [-CC'\|x\|_\infty, CC'\|x\|_\infty]^d) \\
&\quad \leq \sum_{k\geq n/2} \mathbb{P}_p(\tau^0 = +\infty,\ K_{C'\|x\|_\infty + k} \not\supset [-CC'\|x\|_\infty, CC'\|x\|_\infty]^d) \\
&\quad \leq \sum_{k\geq n/2} A\exp(-Bk).
\end{aligned}$$

To control the second term, we use Lemma 3. Choosing $0 < \rho < 1$ such that $2\rho^{1/2} < 1$, we obtain, for $p \geq p_0(\rho)$,

$$\mathbb{P}_p\biggl(\biggl|k \in \{C'\|x\|_\infty + \frac{n}{2}, \dots, C'\|x\|_\infty + n\} \colon (x,k) \in \tilde{I}_\infty\biggr| \leq \theta n\biggr) \leq 2^{n/2+1} 16 \rho^{n/2 - \theta n - 3}.$$

This concludes the proof of (7), and therefore of the lemma.

4. An abstract restart procedure

In this section we formalize the restart procedure for Markov chains. Let E be the state space on which our Markov chains $(X_n^x)_{n\geq 0}$ evolve, where $x \in E$ denotes the starting point of the chain. We suppose that we have at our disposal a set $\tilde{\Omega}$, an update function $f \colon E \times \tilde{\Omega} \to E$, and a probability measure ν on $\tilde{\Omega}$ such that, on the probability space $(\Omega, \mathcal{F}, \mathbb{P}) = (\tilde{\Omega}^{\mathbb{N}^*}, \mathcal{B}(\tilde{\Omega}^{\mathbb{N}^*}), \nu^{\otimes \mathbb{N}^*})$, endowed with the natural filtering $(\mathcal{F}_n)_{n\geq 0}$ given by $\mathcal{F}_n = \sigma(\omega \mapsto \omega_k \colon k \leq n)$, the chains $(X_n^x)_{n\geq 0}$ starting from the different states satisfy the following representation:

$$X_0^x(\omega) = x, \qquad X_{n+1}^x(\omega) = f(X_n^x(\omega), \omega_{n+1}).$$

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As usual, we define $\theta\colon \Omega \to \Omega$ which maps $\omega = (\omega_n)_{n\geq 1}$ to $\theta\omega = (\omega_{n+1})_{n\geq 1}$. We assume that, for each $x \in E$, we have defined an $(\mathcal{F}_n)_{n\geq 0}$-adapted stopping time T^x, an $\mathcal{F}_{T^x}$-measurable function G^x, and an $\mathcal{F}$-measurable function F^x. We are interested in the following quantities:

$$T_0^x = 0 \quad \text{and} \quad T_{k+1}^x = \begin{cases} +\infty & \text{if } T_k^x = +\infty, \\ T_k^x + T^{x_k}(\theta_{T_k^x}) & \text{with } x_k = X^x_{\theta_{T_k^x}} \text{ otherwise;} \end{cases}$$

$$K^x = \inf\{k \geq 0\colon T_{k+1}^x = +\infty\}, \qquad M^x = \sum_{k=0}^{K^x-1} G^{x_k}(\theta_{T_k^x}) + F^{X^{x_K}}(\theta_{T_K^x}).$$

We wish to control the exponential moments of the M^xs with the help of exponential bounds for G^x and F^x. In numerous applications of directed percolation or the contact process, T^x is the extinction time of the process (or of some embedded process) starting from the smallest point (in the lexicographic order) in the configuration x.

Lemma 5. *Suppose that there exist real numbers $A > 0$, $c < 1$, $p > 0$, and $\beta > 0$, and that the real-valued functions $(G^x)_{x\in E}$ and $(F^x)_{x\in E}$ defined above are such that, for all $x \in E$,*

$$\begin{aligned} \mathbb{G}(x) &= \mathbb{E}[\exp(\beta G^x)\,\mathbf{1}_{\{T^x<+\infty\}}] \leq c, \\ \mathbb{F}(x) &= \mathbb{E}[\mathbf{1}_{\{T^x=+\infty\}}\exp(\beta F^x)] \leq A, \\ \mathbb{T}(x) &= \mathbb{P}(T^x = +\infty) \geq p. \end{aligned}$$

Then, for each $x \in E$, K^x is $\mathbb{P}$-almost surely finite and

$$\mathbb{E}[\exp(\beta M^x)] \leq \frac{A}{1-c} < +\infty.$$

Before presenting the proof, we note that we could give a statement about Markov chains avoiding the use of an update function, by working directly with the trajectory space of the Markov chain rather than with the generic underlying space: in this way, $\mathbb{P}(T^x = +\infty)$ would be replaced by $\mathbb{P}^x(T = +\infty)$ and many of the formulae would be simpler. However, the processes we plan to apply this lemma to are often built from a graphical construction (here the Ω where the growth model lives), and the functions $G^{\cdot}$ and $H^{\cdot}$ we plan to apply the lemma to are defined from the graphical representation, and not from the Markov chain.

Proof of Lemma 5. We can assume without loss of generality that $\beta = 1$. Let $x \in E$ be fixed. At first, we have, for each $n \geq 0$,

$$\begin{aligned} \mathbb{P}(K^x > n \mid \mathcal{F}_{T_n^x}) &= \mathbb{P}(T_{n+1}^x < +\infty \mid \mathcal{F}_{T_n^x}) \\ &= \mathbb{P}(T_n^x < +\infty,\ T^{x_n}(\theta_{T_n^x}) < +\infty \mid \mathcal{F}_{T_n^x}) \\ &= \mathbf{1}_{\{T_n^x<+\infty\}}(1 - \mathbb{T}(x_n)) \\ &\leq (1-p)\,\mathbf{1}_{\{T_n^x<+\infty\}} \\ &= \mathbf{1}_{\{K^x>n-1\}}(1-p). \end{aligned}$$

Then $\mathbb{P}(K^x > n) \leq (1-p)\mathbb{P}(K^x > n-1)$, which ensures that K^x is $\mathbb{P}$-almost surely finite.

Let $S_{-1}^x = 1$, and, for $k \geq 0$, put

$$S_k^x = \exp\left(\sum_{i=0}^{k} G^{x_i}(\theta_{T_i^x})\right)\mathbf{1}_{\{T_{k+1}^x<+\infty\}}.$$

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We note that S_k^x is $\mathcal{F}_{T_{k+1}^x}$-measurable. For $k \geq 0$, we have

$$\exp(M^x)\,\mathbf{1}_{\{K^x=k\}} = S_{k-1}^x\,\mathbf{1}_{\{T^{x_k}\circ\theta^{T_k^x}=+\infty\}}\exp(F^{x_k});$$

hence, by the strong Markov property, $\mathbb{E}[\exp(M^x)\,\mathbf{1}_{\{K^x=k\}} \mid \mathcal{F}_{T_k^x}] = S_{k-1}^x\mathbb{F}(x_k)$. Then

$$\mathbb{E}[\exp(M^x)\,\mathbf{1}_{\{K^x=k\}}] \leq A\mathbb{E}[S_{k-1}^x].$$

For $k \geq 1$, the strong Markov property again gives

$$\mathbb{E}[S_{k+1}^x \mid \mathcal{F}_{T_{k+1}^x}] = S_k^x \times \mathbb{G}(x_{k+1}).$$

Then $\mathbb{E}[S_{k+1}^x] \leq c\mathbb{E}[S_k^x]$ and $\mathbb{E}[\exp(M^x)\,\mathbf{1}_{\{K^x=k\}}] \leq Ac^k$. We conclude the proof by summing over k.

5. Application to the model

5.1. Dependence to initial conditions

We first prove that the positivity of the probability of survival for the bacteria does not depend on the initial condition of the environment. We note that Steif and Warfheimer [20] proved a similar result for the model introduced by Broman [3].

Proof of Theorem 1. Let $p > \overrightarrow{p_c}$, $q < \overrightarrow{p_c}$, and $\alpha > 0$ such that $\mathbb{P}_{p,q,\alpha}(\tau_1^{0,\varnothing} = +\infty) > 0$. We want to show that $\mathbb{P}_{p,q,\alpha}(\tau_1^{0,\mathbb{Z}^d\setminus\{0\}} = +\infty) > 0$. Let us denote by C_n the event 'there exists $x \in [-n, n]^d$ such that $\mathbb{Z}^d \times \{0\}$ is linked to (x, n) by open bonds of directed oriented percolation with parameter q'. By a time reversal argument, we obtain

$$\mathbb{P}_{p,q,\alpha}(C_n) \leq (2n+1)^d\mathbb{P}_q(T > n) \leq A\exp(-Bn),$$

where T is the extinction time of some subcritical oriented percolation process with parameter q. We conclude that, if $A_N = \bigcap_{k\geq N} C_k^c$,

$$\lim_{N\to+\infty}\mathbb{P}_{p,q,\alpha}(A_{N-1}\circ\theta_1) = \lim_{N\to+\infty}\mathbb{P}_{p,q,\alpha}(A_{N-1}) = 1,$$

whence

$$\lim_{N\to+\infty}\mathbb{P}_{p,q,\alpha}(\tau_1^{0,\varnothing} = +\infty,\ A_{N-1}\circ\theta_1) = \mathbb{P}_{p,q,\alpha}(\tau_1^{0,\varnothing} = +\infty).$$

In particular, there exists N such that $\mathbb{P}_{p,q,\alpha}(\tau_1^{0,\varnothing} = +\infty,\ A_{N-1}\circ\theta_1) > 0$. Let us denote by B the event 'all the oriented edges issued from $[-3N, 3N]^d \times \{0\}$ are closed for the percolation with parameter q'. By independence, we have

$$\mathbb{P}_{p,q,\alpha}(\tau_1^{0,\varnothing} = +\infty,\ A_{N-1}\circ\theta_1,\ B) = \mathbb{P}_{p,q,\alpha}(\tau_1^{0,\varnothing} = +\infty,\ A_{N-1}\circ\theta_1)\mathbb{P}_{p,q,\alpha}(B) > 0.$$

It remains to prove that $\tau_1^{0,\mathbb{Z}^d\setminus\{0\}} = +\infty$ holds on this event. It is sufficient to prove that the processes $(\eta_{1,n}^{0,\mathbb{Z}^d\setminus\{0\}})_{n\geq 0}$ and $(\eta_{1,n}^{0,\varnothing})_{n\geq 0}$ coincide on this event; but, because of the definition of the dynamics, it is sufficient to note that, on the event $(A_{N-1}\circ\theta_1)\cap B$, we have

$$\eta_{2,n}^{\varnothing}\cap[-n,n]^d = \eta_{2,n}^{\mathbb{Z}^d\setminus\{0\}}\cap[-n,n]^d \quad \text{for all } n \geq 1,$$

which completes the proof.

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5.2. Outline of the proof of Theorem 2

The idea of the proof is to define a local block event with probability close to 1, which expresses the fact that if the bacterium occupies a sufficiently large area at a given place, it will presumably extend itself a bit further. If the associated block process percolates then the linear growth is ensured by Theorem 3. With a restart argument, we find a point in space–time, not too far from the origin, where the bacterium occupies a sufficiently large area and where the associated block process percolates, which will give the desired result.

The statement of Theorem 2 actually contains two facts that must be proved separately: the fact that $\alpha_c > 0$ and the fact that, for $\alpha < \alpha_c$, the process, when surviving, grows linearly. We can find in the literature many examples of block events similar to those we use. Most of these papers take inspiration from the Bezuidenhout and Grimmett article [1]. We think that, for this kind of dynamical renormalization scheme, the existence of a coupling between the dependent oriented percolation of blocks and a Bernoulli oriented percolation conditioned to survive is barely explained in the literature. This led us to write Theorem 3. In Subsection 5.3 we focus on the case where α is small and the renormalization event simpler. The construction for $\alpha < \alpha_c$, which is technically more subtle, is explained in Subsection 5.4.

5.3. Positivity of α_c (the case of small α)

We prove here that, when α is small enough, $\mathbb{P}_{p,q,\alpha}(\tau_1^{0,\mathbb{Z}^d\setminus\{0\}} = +\infty) > 0$ and the growth is linear on the event $\{\tau_1^{0,\mathbb{Z}^d\setminus\{0\}} = +\infty\}$.

5.3.1. *The block event.* Let $I, L \in \mathbb{N}^*$ with $I < L$. Recall that the constant C is given in Lemma 1. We let

$$T = 6CL \quad \text{and} \quad J = 2(L + T).$$

For $\bar{k} \in \mathbb{Z}^d$, $x \in [-L, L)^d$, and $u \in \mathbb{Z}^d$ such that $\|u\|_1 \leq 1$, we define the following event:

$$A(\bar{k}, x, u) = \Bigg\{\text{there exists } s \in [-L, L)^d,$$
$$2L(\bar{k} + u) + s + [-I, I]^d \subset \eta_{1,T}^{2L\bar{k}+x+[-I,I]^d,\mathbb{Z}^d\setminus(2L\bar{k}+[-J,J]^d)},$$
$$\eta_{2,T}^{2L\bar{k}+x+[-I,I]^d,\mathbb{Z}^d\setminus(2L\bar{k}+[-J,J]^d)} \cap (2L(\bar{k}+u) + [-J, J]^d) = \varnothing,$$
$$2L\bar{k} + [-L, L]^d \subset \bigcup_{0\leq t\leq T} \eta_{1,t}^{2L\bar{k}+x+[-I,I]^d,\mathbb{Z}^d\setminus(2L\bar{k}+[-J,J]^d)}\Bigg\}.$$

If $A(\bar{k}, x, u)$ holds, we denote by $s(\bar{k}, x, u)$ an element s satisfying the condition above.

Let us briefly explain the significance of the event $A(\bar{k}, x, u)$. Obviously, $\eta_{1,T}^{A,B}$ is nondecreasing with respect to A and nonincreasing with respect to B, whereas $\eta_{2,T}^{B}$ is nondecreasing with respect to B. Thus, if $A(\bar{k}, x, u)$ holds and if one knows that at time 0 the block $2L\bar{k} + x + [-I, I]^d$ is full of '1s' and the block $2L\bar{k} + [-J, J]^d$ contains no '2', then one knows that analogous conditions will be fulfilled around $2L(\bar{k} + u)$ at time T. Of course, the idea is to follow a chain of such events in an oriented percolation and to draw a path ensuring the development of the bacteria.

Lemma 6. *For each $p > \overrightarrow{p_c}^{\mathrm{alt}}(d + 1)$, each $q < \overrightarrow{p_c}^{\mathrm{alt}}(d + 1)$, and each $\varepsilon > 0$, we can find large enough integers $I < L$ and small enough $\alpha \in (0, 1)$ such that, for every $\bar{k} \in \mathbb{Z}^d$, $x \in [-L, L)^d$, and $u \in \mathbb{Z}^d$ such that $\|u\|_1 \leq 1$,*

$$\mathbb{P}_{p,q,\alpha}(A(\bar{k}, x, u)) \geq 1 - \varepsilon.$$

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Moreover, as soon as $\|\bar{k}-\bar{l}\|_1 > 4+18C$, *for every* $x, y \in [-L, L]^d$, *every* $u, v \in \mathbb{Z}^d$ *such that* $\|u\|_1 \le 1$, *and* $\|v\|_1 \le 1$, *the events* $A(\bar{k}, x, u)$ *and* $A(\bar{l}, y, v)$ *are independent.*

Proof. First note that $\mathbb{P}_{p,q,\alpha}(A(\bar{k}, x, u)) = \mathbb{P}_{p,q,\alpha}(A(\bar{0}, x, u))$, which allows us to consider only the $\bar{n} = \bar{0}$ case.

Under $\mathbb{P}_{p,q,\alpha}$, the collections of random variables $\omega_1 = (\omega^e_{1,n})_{e\in\overrightarrow{\mathbb{E}}^d, n\in\mathbb{N}^*}$ and $\omega_2 = (\omega^e_{2,n})_{e\in\overrightarrow{\mathbb{E}}^d, n\in\mathbb{N}^*}$ have the law of the bonds of an independent directed percolation with parameters p and q, respectively. We realize these percolation structures on Ω, keeping the notation introduced in the introduction: thus, under $\mathbb{P}_{p,q,\alpha}$, $(\xi^A_n(\omega_1))_{n\ge 0}$ is a directed Bernoulli percolation process with parameter p starting from the set A and $(\tau^x_1(\omega_2))_{n\ge 0}$ is the extinction time for a directed Bernoulli percolation process with parameter q starting from x. Under $\mathbb{P}_{p,q,\alpha}$, the collection of random variables $\omega_3 = (\omega^e_{3,n})_{e\in\overrightarrow{\mathbb{E}}^d, n\in\mathbb{N}^*}$ are independent Bernoulli with parameter α. They represent the immigration of immune cells.

Let $\varepsilon > 0$. We choose two integers I and L with $I < L$—their values will be fixed later. Define

$$\begin{aligned} B(x,u) = \Big\{&\text{there exists } s \in [-L, L)^d,\ 2Lu + s + [-I, I]^d \subset \xi_T^{x+[-I,I]^d}(\omega_1),\\ &\text{for all } (y,n) \in [-(4L+2T), (4L+2T)]^d \times \{1,\dots,T\},\ \omega_3^{y,n} = 0,\\ &\text{and } \tau_1^y \circ \theta_n(\omega_2) \le \frac{T}{2}, [-L, L]^d \subset \bigcup_{0\le t\le T} \xi_t^{x+[-I,I]^d}(\omega_1)\Big\}. \end{aligned}$$

We will show that $B(x, u) \subset A(\bar{0}, x, u)$ and also that one can choose I and L in such a way that $\mathbb{P}_{p,q,\alpha}(B(x, u)) \ge 1 - \varepsilon$, which will give the desired result. The advantage of using B isthat it does not deal with the competition process, using only the directed percolation and the immigration processes. Thus, it is easier to estimate its probability.

Step 1*: show that* $B(x, u) \subset A(\bar{0}, x, u)$. The existence of a convenient s for the condition of $A(\bar{0}, x, u)$ is given by $B(x, u)$ for the oriented percolation with parameter p embedded in the model. We need to verify that our event ensures that the type-2 particles cannot disturb the progress of type-1 particles.

Note that $A = x + [-I, I]^d$ and $B = \mathbb{Z}^d \setminus [-J, J]^d$. At time 0, the smallest distance between points in $\eta^{A,B}_{1,0}$ and $\eta^B_{2,0}$ is at least $2L + 2T - (L + I) > 2T$. In the zone $[-J, J]^d$, there is no immigration between time 0 and time T, so $\eta^{A,B}_{1,t}$ and $\eta^B_{2,t}$ get closer at a speed that does not exceed 2 per time unit; thus, by time T, the type-2 particles have not disturbed the movement of type-1 particles.

It remains to see that $\eta_{2,T}$ cannot reach $2Lu + [-J, J]^d$. Remember that there is no immigration between time 0 and time T in the area $[-(4L + 2T), (4L + 2T)]^d$. Moreover, type-2 particles that are outside $[-(4L + 2T), (4L + 2T)]^d$ at time 0 do not have enough time to reach $2Lu + [-J, J]^d$ at time T, so only type-2 particles that were already inside $[-(4L + 2T), (4L + 2T)]^d$ at time 0 must be considered. But these particles are all dead at time $T/2$. This completes the proof of the inclusion.

Step 2*: bound the probability of* $B(x, u)$ *from below.* Recall that $\mathbb{P} = \mathbb{P}_{p,q,\alpha}$. We first choose an integer I large enough to have

$$\mathbb{P}(\tau_1^{x+[-I,I]^d}(\omega_1) = +\infty) \ge 1 - \frac{\varepsilon}{12} \quad \text{for all } x \in \mathbb{Z}^d. \tag{8}$$

By the Fortuin–Kasteleyn–Ginibre inequality, $\mathbb{P}(\text{for all } y \in [-I, I]^d,\ \tau^y_1(\omega_1) = +\infty) > 0$.

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Translation invariance and ergodicity of $\mathbb{P}$ then give

$$\lim_{L\to+\infty} \mathbb{P}(\text{there exists } n \in [0, L]\colon \text{for all } y \in nu + [-I, I]^d,\ \tau_1^y(\omega_1) = +\infty) = 1.$$

Let $L_1 > I$ be such that, for each $L \geq L_1$,

$$\mathbb{P}(\text{there exists } n \in [0, L]\colon \text{for all } y \in nu + [-I, I]^d,\ \tau_1^y(\omega_1) = +\infty) > 1 - \frac{\varepsilon}{12}.$$

Let $L \geq L_1$. By a time reversal argument, we have, for each $t > 0$,

$$\begin{aligned}
&\mathbb{P}(\text{there exists } n \in [0, L]\colon nu + [-I, I]^d \subset \xi_t^{\mathbb{Z}^d}(\omega_1)) \\
&\quad = \mathbb{P}(\text{there exists } n \in [0, L]\colon \text{for all } y \in nu + [-I, I]^d, \tau_1^y(\omega_1) \geq n) \\
&\quad \geq 1 - \frac{\varepsilon}{12}.
\end{aligned} \tag{9}$$

Now, Lemma 1 gives the existence of some $L_2 \geq L_1$ such that, for each $L \geq L_2$, we have simultaneously

$$\begin{aligned}
&\mathbb{P}(\text{there exists } y \in [-2L, 2L]^d\colon \tau_1^y(\omega_1) = +\infty,\ Lu + [-2L, 2L]^d \not\subset K_{6CL}^y(\omega_1)) \\
&\quad \leq (4L+1)^d \mathbb{P}(\tau_1^0(\omega_1) = +\infty,\ [-5L, 5L]^d \not\subset K_{6CL}^0(\omega_1)) \\
&\quad \leq \frac{\varepsilon}{12},
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
&\mathbb{P}(\text{there exists } y \in [-2L, 2L]^d\colon \tau_1^y(\omega_1) = +\infty,\ [-L, L]^d \not\subset H_{6CL}^y(\omega_1)) \\
&\quad \leq (4L+1)^d \mathbb{P}(\tau_1^0(\omega_1) = +\infty,\ [-3L, 3L]^d \not\subset H_{6CL}^y(\omega_1)) \\
&\quad \leq \frac{\varepsilon}{12}.
\end{aligned} \tag{11}$$

With (9) and (10), we obtain

$$\begin{aligned}
&\mathbb{P}(\tau_1^{x+[-I,I]^d}(\omega_1) = +\infty,\ \text{for all } n \in [0, L],\ Lu + nu + [-I, I]^d \not\subset \xi_T^{x+[-I,I]^d}(\omega_1)) \\
&\quad \leq \mathbb{P}(\text{there exists } y \in x + [-I, I]^d\colon \tau_1^y(\omega_1) = +\infty,\ Lu + [-2L, 2L]^d \not\subset K_T^y(\omega_1)) \\
&\qquad + \mathbb{P}(\text{for all } n \in [0, L],\ Lu + nu + [-I, I]^d \not\subset \xi_T^{\mathbb{Z}^d}(\omega_1)) \\
&\quad \leq \frac{\varepsilon}{6}.
\end{aligned}$$

With (8) and (11), we conclude that, for each $x \in [-L, L]^d$,

$$\begin{aligned}
&\mathbb{P}(H_T^{x+[-I,I]}(\omega_1) \supset [-L, L]^d,\ \text{there exists } n \in [0, L], \\
&\quad (L+n)u + [-I, I]^d \subset \xi_T^{x+[-I,I]}(\omega_1)) \\
&\qquad \geq 1 - \frac{\varepsilon}{3}.
\end{aligned} \tag{12}$$

Since $q < \overrightarrow{p_c}^{\text{alt}}(d+1)$, there exist positive constants A and B such that, for each L,

$$\begin{aligned}
&\mathbb{P}\bigg(\text{there exists } y \in [-(4L+2T), (4L+2T)]^d\colon \tau_1^y(\omega_2) > \frac{T}{2}\bigg) \\
&\quad \leq (8L+4T+1)^d A \exp\bigg(-\frac{BT}{2}\bigg).
\end{aligned}$$

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We deduce that there exists some integer $L_3 \geq L_2$ such that, for each $L \geq L_3$,

$$\mathbb{P}\left(\text{there exists } y \in [-(4L+2T),(4L+2T)]^d : \tau_1^y(\omega_2) > \frac{T}{2}\right) \leq \frac{\varepsilon}{3}. \tag{13}$$

Now fix $L \geq L_3$ and choose $\alpha > 0$ small enough such that

$$\mathbb{P}(\text{there exists } (y,n) \in [-(4L+2T),(4L+2T)]^d \times \{1,\dots,T\}, \omega_{3,n}^y = 1) \leq \frac{\varepsilon}{3}. \tag{14}$$

We conclude by putting (12), (13), and (14) together.

5.3.2. *Block events percolation.* Let $I < L$ be fixed integers. First, for each $x \in \mathbb{Z}^d$, we build a field $({}^x W^n_{(z,u)})_{n\geq 1,\, z\in\mathbb{Z}^d,\, \|u\|_1\leq 1}$ from the events defined above. The random variable $W^{n+1}_{(z,u)}$ will give the state of the oriented bond between the macroscopic sites (z,n) and $(z+u,n+1)$; these sites correspond to the coordinates of the boxes $(2Lz, nT) + [-L,L]^d \times [1,T]$ and $(2L(z+u),(n+1)T) + [-L,L]^d \times [1,T]$. The field $({}^x W^n_{(z,u)})_{n\geq 1,\, z\in\mathbb{Z}^d,\, \|u\|_1\leq 1}$ then defines a macroscopic dynamical dependent oriented percolation.

For $x \in \mathbb{Z}^d$, we denote by $[x]_{2L} \in \mathbb{Z}^d$ the unique integer such that $x \in 2L[x]_{2L} + [-L,L)^d$, and we set $\{x\}_{2L} = x - 2L[x]_{2L} \in [-L,L)^d$. We set $d_0^x([x]_{2L}) = \{x\}_{2L}$ and also $d_0^x(\bar{k}) = +\infty$ for every $\bar{k} \in \mathbb{Z}^d$ that is not equal to $[x]_{2L}$. Then, for each $\bar{k} \in \mathbb{Z}^d$, each $u \in \mathbb{Z}^d$ with $\|u\|_1 \leq 1$, and each $n \geq 1$:

- if $d_n^x(\bar{k}) = +\infty$, define ${}^x W^{n+1}_{(\bar{k},u)} = 1$;
- otherwise, define

$$\begin{aligned} {}^x W^{n+1}_{(\bar{k},u)} &= \mathbf{1}_{A(\bar{k},d_n^x(\bar{k}),u)} \circ \theta_{nT}, \\ d_{n+1}^x(\bar{k}) &= \min\{s(\bar{k}-u, d_n^x(\bar{k}-u), u) \circ \theta_{nT} : \|u\|_1 \leq 1,\ d_n^x(\bar{k}-u) \neq +\infty\}. \end{aligned}$$

Let $\mathcal{G}_n = \sigma(\omega_1^{e,k}, \omega_2^{e,k}, \omega_3^{x,k}, e \in \overrightarrow{\mathbb{E}}^d, x \in \mathbb{Z}^d,\ k \leq nT)$. Note that, conditionally on $\mathcal{G}_n$, the random variables ${}^x W^{n+1}_{(\bar{k},u)}$ and ${}^x W^{n+1}_{(\bar{l},v)}$ are independent as soon as $\|\bar{k}-\bar{l}\|_1 > 4 + 18C$. Then we take $M = 5 + 18C$, and prove the following lemma.

Lemma 7. *For each $p > \overrightarrow{p_c}^{\mathrm{alt}}(d+1)$, $q < \overrightarrow{p_c}^{\mathrm{alt}}(d+1)$, and $q_0 < 1$, we can find some integers $I < L$ and a parameter $\alpha > 0$ such that, for each $x \in \mathbb{Z}^d$,*

$$\textit{the law of } ({}^x W^n_e)_{n\geq 0,\, e\in\overrightarrow{\mathbb{E}}^d} \textit{ under } \mathbb{P}_{p,q,\alpha} \textit{ belongs to } \mathcal{C}(M,q_0).$$

Proof. Note that, for every $x, \bar{k} \in \mathbb{Z}^d$ and each $n \geq 1$, the variable $d_n^x(\bar{k})$ is $\mathcal{G}_n$-measurable, and so is ${}^x W^n_{(\bar{k},u)}$.

Let us now consider $x, \bar{k} \in \mathbb{Z}^d$, $n \geq 0$, and $u \in \mathbb{Z}^d$ such that $\|u\|_1 \leq 1$: Lemma 6 ensures that

$$\begin{aligned} &\mathbb{E}_{p,q,\alpha}[{}^x W^{n+1}_{(\bar{k},u)} \mid \mathcal{G}_n \vee \sigma({}^x W^{n+1}_{(\bar{l},v)},\ \|v\|_1 \leq 1,\ \|\bar{l}-\bar{k}\|_1 \geq M)] \\ &\quad = \mathbb{E}_{p,q,\alpha}[{}^x W^{n+1}_{(\bar{k},u)} \mid \mathcal{G}_n] \\ &\quad = \mathbf{1}_{\{d_n^x(\bar{k})=+\infty\}} + \mathbf{1}_{\{d_n^x(\bar{k})<+\infty\}} \mathbb{P}_{p,q,\alpha}({}^x W^{n+1}_{(\bar{k},u)} = 1 \mid d_n^x(\bar{k}) < +\infty) \\ &\quad = \mathbf{1}_{\{d_n^x(\bar{k})=+\infty\}} + \mathbf{1}_{\{d_n^x(\bar{k})<+\infty\}} \mathbb{P}_{p,q,\alpha}(A(\bar{k}, d_n^x(\bar{k}), u)). \end{aligned}$$

Using Lemma 6, we can find some integers $I < L$ and a parameter $\alpha > 0$ such that

$$\mathbb{E}_{p,q,\alpha}[{}^x W^{n+1}_{(\bar{k},u)} \mid \mathcal{G}_n \vee \sigma({}^x W^{n+1}_{(\bar{l},v)},\ \|v\|_1 \leq 1,\ \|\bar{l}-\bar{k}\|_1 \geq M)] \geq q_0.$$

This completes the proof.

5.3.3. *From the macroscopic to microscopic scale.*

Proof of Theorem 2 for small α. The inequality $\alpha_{\rm c} \le 1 - 1/(2d+1)$ easily follows from a counting argument. Let $p > \overrightarrow{p_{\rm c}}^{\rm alt}(d+1)$ and $q < \overrightarrow{p_{\rm c}}^{\rm alt}(d+1)$, and take $M = 5 + 18C$ as previously. By Lemma 1, we can find $q_0 < 1$ with $g_M(q_0) > \overrightarrow{p_{\rm c}}^{\rm alt}(d+1)$ and $\beta_0 > 0$ such that, for each field $\chi \in \mathcal{C}_d(M, q_0)$,

$$\mathbb{E}_\chi[\mathbf{1}_{\{\tau_1^0 < +\infty\}} \exp(\beta_0 \tau_1^0)] \le \tfrac{1}{2}. \tag{15}$$

We choose I, L, and α to satisfy the conditions given in Lemma 7. We will prove that, for this α, the survival of the bacteria is possible, as well as for the other announced estimates.

Let $x \in \{0,1,2\}^{\mathbb{Z}^d}$ be some configuration; we denote by $E_1(x)$ the set of sites occupied by type-1 particles in configuration x. If $E_1(x) \ne \varnothing$, we denote by $j(x)$ the smallest point in $E_1(x)$ (in the lexicographic order). Note that there exists $c > 0$ such that

$$\mathbb{P}_{p,q,\alpha}(\eta^x_{1,4dT} \supset j(x) + [-4T, 4T]^d) \ge c \quad \text{for all } x \in \{0,1,2\}^{\mathbb{Z}^d}. \tag{16}$$

Indeed, it is sufficient to open in ω_1 every bond in

$$B = (j(x), 0) + [-4dT - 1, 4dT + 1]^d \times [0, 4T],$$

to close in ω_2 every bond in B, and to forbid in ω_3 every birth of type-2 in B: all of this corresponds to fixing a finite number of coordinates in ω, which can be done with a positive probability.

If the event in (16) happens, we have at time $4dT$ a large box $j(x) + [-4T, 4T]^d$ occupied by type-1 particles. From this box, we can start the macroscopic percolation by building the random field ${}^xW = ({}^{j(x)}W_e^n \circ \theta_{4dT})_{e \in \overrightarrow{\mathbb{E}}^d,\, n \ge 0}$. Our choices of I, L, and α, and Lemma 7, ensure that ${}^xW$ belongs to $\mathcal{C}_d(M, q_0)$. Since $g_M(q_0) > \overrightarrow{p_{\rm c}}^{\rm alt}(d+1)$, (16) gives

$$\mathbb{P}_{p,q,\alpha}(\tau_1^x = +\infty) \ge \mathbb{P}_{p,q,\alpha}(\eta^x_{1,4dT} \supset j(x) + [-4T, 4T]^d)\mathbb{P}_{g_M(q_0)}(\tau_1^0 = +\infty) > 0,$$

which proves (1).

To show the exponential estimates, we will apply Lemma 5. If $E_1(x) = \varnothing$, we let $T^x = +\infty$; otherwise, let

$$T^x = \begin{cases} 4dT & \text{if the event in (16) does not occur,} \\ 4dT + T \times \tau_1^{[j(x)]_{2L}} \circ \theta_{4dT} & \text{otherwise,} \end{cases}$$

where $\tau_1^{[j(x)]_{2L}}$ represents the extinction time in the percolation ${}^xW$ starting from the macroscopic site $[j(x)]_{2L}$ containing $j(x)$.

For each $x \in \{0,1,2\}^{\mathbb{Z}^d}$ such that $E_1(x) \ne \varnothing$, we have

$$\mathbb{P}_{p,q,\alpha}(T^x = +\infty) \ge c\mathbb{P}_{g_M(q_0)}(\tau_1^0 = +\infty).$$

We take $G^x = T^x$; for $0 < \beta_1 < \beta_0$, inequality (15) gives

$$\mathbb{E}_{p,q,\alpha}[\mathrm{e}^{\beta_1 T^x}\,\mathbf{1}_{\{T^x < +\infty\}}] \le \mathrm{e}^{\beta_1 4dT} \sup_{\chi \in \mathcal{C}(M,q_0)} \mathbb{E}_\chi[\mathbf{1}_{\{\tau_1^0 < +\infty\}}\,\mathrm{e}^{\beta_0 \tau_1^0}] \le \frac{\mathrm{e}^{\beta_1 4dT}}{2} \le \frac{2}{3},$$

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provided β_1 is small enough. We take $F^{\varnothing} = 0$ and, for $x \neq \varnothing$,

$$F^x = T \times S^{[j(x)]_{2L}} \circ \theta_{4dT},$$

where S is as defined in Corollary 2. This corollary moreover gives the existence of exponential moments for S. Thus, the restart lemma ensures that the variable

$$M^x = T_K^x + F^{\eta^x_{T_K^x}} \circ \theta_{T_K^x}$$

admits exponential moments.

Let us begin to work on the event $\{\tau_1^x = +\infty\}$. In this case, η^x_{1,T_K^x} is nonempty and, at time $T_K^x + 4dT$, the bacteria occupy a large box $j(\eta^x_{T_K^x+4dT}) + [-4T, 4T]^d$, from which the macroscopic percolation lives forever; moreover,

$$M^x = T_K^x + T \times S^{[j(\eta^x_{T_K^x+4dT})]_{2L}} \circ \theta_{T_K^x+4dT}.$$

By the definition of the macroscopic percolation, if the bond

$${}^{j(\eta^x_{T_K^x+4dT})}W^n_{\bar{k},u} \circ \theta_{T_K^x+4dT}$$

is open then every point in the box $2L\bar{k} + [-L, L]^d$ is visited by the bacteria between time $T_K^x + 4dT + nT$ and time $T_K^x + 4dT + (n+1)T$. In particular, using Corollary 2, it follows that

$$2L[j(\eta^x_{T_K^x+4dT})]_{2L} + [-2nD_1L, 2nD_1L]^d \subset \bigcup_{0 \le m \le n+M^x+4dT} \eta^x_{1,m} \quad \text{for all } n \in \mathbb{N}.$$

We can then deduce (2) and the existence of exponential moments for M^x and T_K^x.

Finally, since $\{\tau_1^x < +\infty\} \subset \{\tau_1^x \le M^x\}$, (3) follows from the bound for the exponential moments of M^x given in Lemma 5; this completes the proof of Theorem 2 for small α.

5.4. The case $\alpha < \alpha_c(p, q)$: the Bezuidenhout–Grimmett way

We fix p, q, and α such that

$$\mathbb{P}_{p,q,\alpha}(\tau_1^{0,\mathbb{Z}^d\setminus\{0\}} = +\infty) = \mathbb{P}(\tau_1^{0,\mathbb{Z}^d\setminus\{0\}} = +\infty) > 0,$$

or, in other words, such that $\alpha < \alpha_c(p, q)$.

The proof for the linear growth of the bacteria conditioned to survive is, as in the case of small α, based on a renormalization process leading to the construction of a d-dimensional supercritical oriented percolation.

In the previous case, when building the local block event, we could choose α small enough for our model to behave nearly as independent oriented percolation. This is no longer the case when α is close to α_c. Instead, we adapt the strategy developed by Bezuidenhout and Grimmett [1] for the supercritical contact process on $\mathbb{Z}^d$, which is also the strategy followed by Steif and Warfheimer [20] in the case of a contact process where the death rate depends on a dynamical environment. The key point is the following proposition (which corresponds to Proposition 2.22 of [15] or Lemma 4.10 of [20]). For the sake of brevity, the proof is omitted. We provide a complete proof of the proposition in the preprint version of this work that can be found online; see [12]. We denote by V the set of $e \in \mathbb{Z}^d$ with $\|e\|_1 \le 1$.

5.4.1. *The block event.*

Proposition 2. *Let $\varepsilon > 0$ and $k \geq 1$ be fixed. There exist n, a, and b with $n < a$ such that, for every $u \in V$, every $\bar{n_0} \in \mathbb{Z}^d$, every $x_0 \in [-a, a]^d$, and every $t_0 \in [0, b]$, we can define the random variables $Y(\bar{n_0}, u, x_0, t_0) \in \mathbb{Z}^d$ and $S(\bar{n_0}, u, x_0, t_0) \in \mathbb{N} \cup \{+\infty\}$ such that*

- $Y(\bar{n_0}, u, x_0, t_0) \in [-a, a]^d$*;*
- $S(\bar{n_0}, u, x_0, t_0) \in [5kb, (5k+1)b] \cup \{+\infty\}$*;*
- $y + 2ka(\bar{n_0} + u) + [-n, n]^d \subset \eta_{1,s-t_0}^{x_0+2ka\bar{n_0}+[-n,n]^d, \mathbb{Z}^d \setminus (x_0+2ka\bar{n_0}+[-n,n]^d)} \circ \theta_{t_0}$ *on the event* $\{Y(\bar{n_0}, u, x_0, t_0) = y,\ S(\bar{n_0}, u, x_0, t_0) = s\}$*;*
- $\mathbb{P}_{p,q,\alpha}(S(\bar{n_0}, u, x_0, t_0) < +\infty) \geq 1 - \varepsilon$*;*
- *the event $\{Y(\bar{n_0}, u, x_0, t_0) = y,\ S(\bar{n_0}, u, x_0, t_0) = s\}$ belongs to the σ-algebra generated by the background random variables related to the space–time area*

$$\left(\bigcup_{j=0}^{k-1} ([-5a, 5a]^d \times [0, 6b]) + (2jau, 5jb)\right) \cap (\mathbb{Z}^d \times [t_0, s]).$$

5.4.2. *Dependent macroscopic percolation.* Note that $T = 5b$. For $\bar{n_0} \in \mathbb{Z}^d$, $x_0 \in [-a, a)^d$, $t_0 \in [0, b]$, and $u \in \mathbb{Z}^d$ such that $\|u\|_1 \leq 1$, we define $A(\bar{n_0}, u, x_0, t_0) = \{S(\bar{n_0}, u, x_0, t_0) < +\infty\}$ and $\Psi(\bar{n_0}, u, x_0, t_0) = (S(\bar{n_0}, u, x_0, t_0), Y(\bar{n_0}, u, x_0, t_0)) \in \mathbb{N} \times \mathbb{Z}^d$.

We now build a field $({}^{\bar{n_0}}W^n_{(\bar{k},u)})_{n\geq 0,\, \bar{k}\in\mathbb{Z}^d,\, \|u\|_1\leq 1}$ in an analogous manner to the small α case. That is, we let $d_0(\bar{y}) = 0$ for each $\bar{y} \in \mathbb{Z}^d$, $t_0(\bar{n_0}) = 0$, and also $t_0(\bar{y}) = +\infty$ for every $\bar{y} \in \mathbb{Z}^d$ that differs from 0. Then, for each $\bar{y} \in \mathbb{Z}^d$, each $u \in \mathbb{Z}^d$ such that $\|u\|_1 \leq 1$, and each $n \geq 0$,

- if $t_n(\bar{y}) = +\infty$, define ${}^{\bar{n_0}}W^{n+1}_{(\bar{y},u)} = 1$;
- otherwise, define ${}^{\bar{n_0}}W^{n+1}_{(\bar{y},u)} = \mathbf{1}_{\{S(\bar{y},u,d_n(\bar{y}),t_n(\bar{y}))<+\infty\}} \circ \theta_{nT}$.

Then

$$\begin{aligned}&(t_{n+1}(\bar{y}), d_{n+1}(\bar{y}))\\ &\qquad = \min\{\Psi(\bar{y}+u, -u, d_n(\bar{y}+u), t_n(\bar{y}+u)) \circ \theta_{nT} : \|u\|_1 \leq 1,\ t_n(\bar{y}+u) \neq +\infty\}.\end{aligned}$$

To specify what 'min' means, choose the smallest t in the natural order, and then the smallest s in the lexical order. If the set is empty, the min is $(+\infty, 0)$. Then $(t_{n+1}(\bar{y}), d_{n+1}(\bar{y}))$ represents the relative position of the entrance area for the ${}^{\bar{n_0}}W^{n+1}_{(\bar{y},u)}$s, with $\|u\|_1 \leq 1$.

Note that $nT + t_{n+1}(\bar{y})$ is an $(\mathcal{F}_k)_{k\geq 0}$-stopping time.

It is now time to put the pieces together: take $M = 2$, and choose $q_0 < 1$ such that $g_M(q_0) > \overrightarrow{p_c}^{\mathrm{alt}}$ and q_0 satisfies the conclusion of Corollary 3 with $M = 2$.

Using Proposition 2 with $1 - \varepsilon = q_0$ and $k > 7$, we can build an oriented percolation process $({}^{\bar{n_0}}W^n_{(\bar{k},u)})_{n\geq 0,\, \bar{k}\in\mathbb{Z}^d,\, \|u\|_1\leq 1}$. Among open bonds, only those corresponding to the realization of good events are relevant for the propagation of type-1 particles. Let us note however that the percolation cluster starting at $\bar{n_0}$ only contains bonds that are effectively used by the process.

Let us denote by $\chi^{\bar{n_0}}$ the law of the field $({}^{\bar{n_0}}W^n_{(\bar{k},u)})_{n\geq 0,\, \bar{k}\in\mathbb{Z}^d,\, \|u\|_1\leq 1}$ under $\mathbb{P}_{p,q,\alpha}$.

Lemma 8. *We can choose the construction parameters a, b, and n of Proposition 2 such that $\chi^{\bar{n_0}}$ belongs to $\mathcal{C}_d(M, q_0)$.*

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The proof, which is quite technical, is omitted, but uses only tested methods. It can be found in the related technical report [12]. Note that, as the exit points are at random heights, the σ-field $\mathcal{G}_n$ must be chosen with care to ensure that $\chi^{\bar{n_0}}$ belongs to $\mathcal{C}_d(M, q_0)$.

5.4.3. *From macroscopic to microscopic scale.* The proof of Theorem 2 can be split into two parts.

1. Prove that if the epidemy survives then points not far from x will often be occupied at a reasonable time.
2. Deduce that x itself will be hit at a reasonable time.

The first part can be formalized as follows.

Lemma 9. *Let $E \subset \mathbb{Z}^d \setminus \{0\}$. There exist $a \in \mathbb{N}$ and positive constants C_1, C_2, A, and B such that, if we define $R_n^a(x)$ with $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$ by $R_0^a(x) = 0$ and*

$$R_i^a(x) = \inf\{t \geq R_{i-1}^a;\ \textit{there exists } y \in x + [-a, a]^d;\ y \in \eta_{1,t}^{0,E}\},$$

then we have

$$\mathbb{P}_{p,q,\alpha}(\tau_1^{0,E} = +\infty,\ R_n^a(x) \geq C_1\|x\| + C_2 n) \leq A\mathrm{e}^{-Bn} \quad \textit{for all } x \in \mathbb{Z}^d \textit{ and all } n \geq 0. \tag{17}$$

Thanks to our tools for dependent oriented percolation, Lemma 9 is a consequence of Proposition 2. Before presenting its proof, we show now it can be used to prove Theorem 2.

Proof of Theorem 2. Let $E \subset \mathbb{Z}^d \setminus \{0\}$, fix $x \in \mathbb{Z}^d$, and define $T_0' = 0$, and for $i \geq 1$,

$$\begin{aligned} T_i &= T_i(x) = \inf\{t \geq T_{i-1}';\ \text{there exists } y \in x + [-a, a]^d;\ y \in \eta_{1,t}^{0,E}\} \\ T_i' &= T_i + a + 1. \end{aligned}$$

Consider the event $B = \{\text{for all } y \in x + [-a, a]^d;\ \text{there exists } t \leq a;\ x \in \eta_{1,t}^{y,\mathbb{Z}^d\setminus\{y\}}\}$ and also, for $n \geq 1$

$$A_n = \bigcap_{i=0}^{n} \{T_i < +\infty,\ \theta^{-T_i}(B^c)\}.$$

Note that, by construction, $\theta^{-T_i}(B)$ is $\mathcal{F}_{T_{i+1}}$-measurable, so

$$\mathbb{P}_{p,q,\alpha}(A_n \mid \mathcal{F}_{T_n}) = \mathbf{1}_{A_{n-1}\cap\{T_n<+\infty\}}\,\mathbb{P}_{p,q,\alpha}(B^c).$$

It is easy to see that $\mathbb{P}(B) \geq c$ for some c does not depend on x. It follows that $\mathbb{P}_{p,q,\alpha}(A_N) \leq (1-c)^n$ for each $n \geq 1$. Note that the sequence $(T_k(x))_{k\geq 1}$ does not consider all infections around x, but it is not difficult to see that $T_n(x) \leq R_{(a+1)n}^a(x)$. So, Lemma 9 gives

$$\mathbb{P}_{p,q,\alpha}(\tau_1^{0,E} = +\infty,\ t(x) \geq C_1\|x\| + C_2(a+1)n + a + 1) \leq A\mathrm{e}^{-B(a+1)n} + (1-c)^n,$$

completing the proof.

Proof of Lemma 9. Note that the events in (17) and Corollary 3 control the density of times where a point (or a neighborhood of a point) is occupied.

Using the events described in Proposition 2, we exhibit (after a restart procedure) a macroscopic percolation that satisfies the assumptions of Corollary 3. This will prove (17) and, hence, the lemma.

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Assume that $\tau_1^{0,E} = +\infty$. Take $M = 2$, and choose $q_0 < 1, \theta, \beta$ such that $g_M(q_0) > \overrightarrow{p_c}^{\text{alt}}$ and q_0,θ,β satisfies the conclusion of Corollary 3 with $M = 2$. By Lemma 8, we can choose the parameters a, b, and n in Proposition 2 to ensure that the distribution of the macroscopic oriented percolation is in $\mathcal{C}_d(M, q_0)$. Then, using the events of Proposition 2, the construction of Subsection 5.4.2, and Theorem 3, a restart argument gives the existence of some $(Y, T) \in \mathbb{Z}^d \times \mathbb{N}$ such that

- $Y + [-2a, 2a]^d \subset \eta_{1,T}^{0,E}$;
- for every $k \geq 1$, $\|Y\| \leq T \leq k$ with probability at least $1 - A\mathrm{e}^{-Bk}$;
- a macroscopic oriented percolation $({}^{\bar{Y}}W^n_{(\bar{x},u)})_{n\geq 0,\, \bar{x}\in\mathbb{Z}^d,\, \|u\|_1\leq 1} \circ \theta_T$ which almost surely survives starts from $Y + [-2a, 2a]^d$ at time T. More precisely, the distribution of the field $({}^{\bar{Y}}W^n_{(\bar{x},u)})_{n\geq 0,\, \bar{x}\in\mathbb{Z}^d,\, \|u\|_1\leq 1} \circ \theta_T$ is $\chi^{\bar{Y}}(\cdot \mid \bar{\tau}_{\bar{Y}} = +\infty)$. Recall that $\chi^{\bar{n_0}}$ is the law of the field $({}^{\bar{n_0}}W^n_{(\bar{k},u)})_{n\geq 0,\, \bar{k}\in\mathbb{Z}^d,\, \|u\|_1\leq 1}$ under $\mathbb{P}_{p,q,\alpha}$, which was defined in Subsection 5.4.2.

Then, Lemma 3 says that $\gamma(\theta, \bar{Y}, \bar{x}) \leq \beta\|\bar{x} - \bar{Y}\| + k$ with probability at least $1 - A\mathrm{e}^{-Bk}$, where $\bar{x}$ and $\bar{Y}$ respectively stand for the coordinates of macroscopic blocks containing x and Y.

By the very definition of $\gamma(\cdot)$, we have

$$R_k^a(x) \leq T + \frac{6b}{\theta} \max(\gamma(\theta, \bar{Y}, \bar{x}) \circ \theta_T, k).$$

This leads to

$$\mathbb{P}_{p,q,\alpha}\Big(R_k^a(x) \leq k + \frac{6b}{\theta}(\beta(\|\bar{x}\| + k) + k)\Big) \geq 1 - 2A\mathrm{e}^{-Bk},$$

which completes the proof.

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