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WEIGHTED ORLICZ ALGEBRAS ON LOCALLY COMPACT GROUPS

ALEN OSANÇLIOL and SERAP ÖZTOP✉

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Abstract

For a locally compact group G with left Haar measure and a Young function Φ, we define and study the weighted Orlicz algebra $L_w^\Phi(G)$ with respect to convolution. We show that $L_w^\Phi(G)$ admits no bounded approximate identity under certain conditions. We prove that a closed linear subspace I of the algebra $L_w^\Phi(G)$ is an ideal in $L_w^\Phi(G)$ if and only if I is left translation invariant. For an abelian G, we describe the spectrum (maximal ideal space) of the weighted Orlicz algebra and show that weighted Orlicz algebras are semisimple.

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1. Introduction

Let G be a locally compact group and $L^1(G)$ be its group algebra. The properties of $L^1(G)$ have been well studied over the last several decades. Important generalizations of $L^1(G)$ are the $L^p(G)$ spaces for $1 \le p < \infty$. The space $L^p(G)$ is a Banach space, even a Banach module over $L^1(G)$ with respect to convolution, but it is a Banach algebra only when G is compact for $1 < p < \infty$ [11].

One very natural phenomenon occurring in harmonic analysis is the appearance of a 'weight' on the group or a 'weighted norm' on the algebra in computations. A weight w on G is usually a continuous function from G into the positive reals. But, in almost all cases, the weights satisfy an extra condition, the submultiplicativity; that is, $w(xy) \le w(x)w(y)$ for all $x, y \in G$. For such a weight, one can extend the construction of $L^p(G)$ to the 'weighted' $L_w^p(G)$ spaces, which are nothing but the usual L^p spaces on G defined as

$$L_w^p(G) = \{f : fw \in L^p(G) \text{ and } \|f\|_{p,w} = \|fw\|_p\}.$$

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These spaces have interesting properties and numerous applications in harmonic analysis. One particular aspect of their behaviour is that they can form a Banach algebra with respect to the convolution. More precisely, when $p = 1$, it follows routinely that $L^1_w(G)$ is a Banach algebra, namely the weighted group algebra, which plays an important role in different areas of harmonic analysis; for more detail, see two excellent books [5, 10].

However, this is not true in general if $p > 1$. Kuznetsova [7, 8] studied certain properties of $L^p_w(G)$ and, in particular, she found conditions under which $L^p_w(G)$ is a Banach algebra with respect to the convolution. She also investigated some important properties of these spaces such as the existence of an approximate identity and, for abelian G, a description of the maximal ideal space of $L^p_w(G)$.

In mathematical analysis, an Orlicz space is a type of function space generalizing the L^p spaces. On the other hand, for a locally compact abelian group G and an Orlicz function Φ, Hudzik *et al.* [4, Theorem 2] gave necessary and sufficient conditions for an Orlicz space $L^\Phi(G)$ to be a Banach algebra with respect to the convolution. Thus, the Orlicz space generalization is significant as it allows a wide area of applications without restricting G to be compact. Similar to L^p spaces, it is natural to consider weighted Orlicz spaces and study their properties.

In the present paper, for an arbitrary locally compact group G and a Young function Φ, we define weighted Orlicz spaces $L^\Phi_w(G)$ and view them as Banach algebras with respect to the convolution multiplication. We find sufficient conditions for which the corresponding space becomes an algebra and study some properties such as the existence of an approximate identity, characterization of closed left ideals and the spectrum of the algebra when the underlying group is abelian. Let us note that in comparison, the weighted Orlicz algebra $L^\Phi_w(G)$ has no bounded left approximate identity for nondiscrete G, while the Banach algebra $L^1_w(G)$ always has a bounded left approximate identity. It should be noted that for $w = 1$, the existence of a bounded approximate identity and the semisimplicity of the Orlicz algebra $L^\Phi(G)$ were studied independently by using different techniques in [1] for an N-function Φ. Note also that every N-function is a Young function.

The structure of the paper is as follows. In Section 2, we provide necessary definitions and some technical results. In Section 3, we give the conditions under which an Orlicz space related to a locally compact group is a Banach algebra under the convolution multiplication, which we call the weighted Orlicz algebra. In Section 4, we show that the weighted Orlicz algebra $L^\Phi_w(G)$ has a left approximate identity, but has no bounded left approximate identity when G is nondiscrete. We give necessary and sufficient conditions for a weighted Orlicz algebra to have an identity. Further, similar to that of the Banach algebra $L^1_w(G)$, the closed left ideals of the weighted Orlicz algebra $L^\Phi_w(G)$ turn out to be nothing but the closed left translation invariant subspaces of $L^\Phi_w(G)$. In the last section, we study the weighted Orlicz algebra in the commutative case. We describe the spectrum (the maximal ideal space) of the weighted Orlicz algebra and show that these algebras are semisimple. Some results are also new for the unweighted Orlicz space.

2. Preliminaries

In this section, we give some definitions and state some technical results that will be crucial in the rest of this paper. In the paper, G denotes a locally compact group with a fixed left Haar measure μ.

A nonzero function $\Phi : \mathbb{R} \to [0, \infty]$ is called a Young function if Φ is convex, even and $\Phi(0) = 0$. By this definition, a Young function can have the value ∞ at a point and hence be discontinuous at such a point. However, unless otherwise specified, we consider only real-valued Young functions. Hence, Φ is necessarily continuous and $\lim_{x\to\infty} \Phi(x) = \infty$.

For a Young function Φ, the complementary function Ψ of Φ is given by

$$\Psi(y) = \sup\{x|y| - \Phi(x) : x \geq 0\} \quad (y \in \mathbb{R}).$$

If Ψ is the complementary function of Φ, then Φ is the complementary function of Ψ and (Φ, Ψ) is called a complementary pair. We have the Young inequality

$$xy \leq \Phi(x) + \Psi(y) \quad (x, y \geq 0)$$

for complementary functions Φ and Ψ. Note that even if a Young function Φ is continuous, it may happen that Ψ is not a continuous function, but is still a Young function as in the above definition.

A Young function Φ satisfies the Δ_2 condition if there exist a constant $K > 0$ and $x_0 > 0$ such that $\Phi(2x) \leq K\Phi(x)$ for all $x \geq x_0$ when G is compact and $\Phi(2x) \leq K\Phi(x)$ for all $x \geq 0$ when G is noncompact. In this case, we write $\Phi \in \Delta_2$.

Let us recall some facts concerning Orlicz spaces.

Given a Young function Φ, the Orlicz space $L^\Phi(G)$ is defined by

$$L^\Phi(G) = \left\{f : G \to \mathbb{C} : \int_G \Phi(\alpha|f|)\, d\mu < \infty \text{ for some } \alpha > 0\right\},$$

where f indicates the μ-equivalence classes of measurable functions. Then the Orlicz space is a Banach space under the (Orlicz) norm $\|\cdot\|_\Phi$ defined for $f \in L^\Phi(G)$ by

$$\|f\|_\Phi = \sup\left\{\int_G |fv|\, d\mu : \int_G \Psi(|v|)\, d\mu \leq 1\right\},$$

where Ψ is the complementary function to Φ. One can also define the (Luxemburg) norm $\|\cdot\|_\Phi^\circ$ on $L^\Phi(G)$ by

$$\|f\|_\Phi^\circ = \inf\left\{k > 0 : \int_G \Phi\left(\frac{|f|}{k}\right) d\mu \leq 1\right\}.$$

It is known that these two norms are equivalent; that is, $\|\cdot\|_\Phi^\circ \leq \|\cdot\|_\Phi \leq 2\|\cdot\|_\Phi^\circ$) and $\|f\|_\Phi^\circ \leq 1$ if and only if $\int_G \Phi(|f|)\, d\mu \leq 1$. If (Φ, Ψ) is a complementary pair of Young functions and $\Phi \in \Delta_2$, then the dual space of $(L^\Phi(G), \|\cdot\|_\Phi)$ is $(L^\Psi(G), \|\cdot\|_\Phi^\circ)$ and, if also $\Psi \in \Delta_2$, then the Orlicz space $L^\Phi(G)$ is reflexive. Note that a measurable function f is in $L^\Phi(G)$ if and only if $\|f\|_\Phi < \infty$. Denote by $C_c(G)$ the space of all continuous functions on G with compact support and let $M^\Phi(G)$ be the closure of $C_c(G)$ in $L^\Phi(G)$. If $\Phi \in \Delta_2$, then $L^\Phi(G) = M^\Phi(G)$. Hence, $C_c(G)$ is dense in $L^\Phi(G)$. For more detail, see [6, 9].

Let G be a locally compact group. A weight on G is a Borel measurable function $w : G \to (0, \infty)$ such that

$$w(xy) \leq w(x)w(y) \quad (x, y \in G).$$

There is no loss of generality in assuming that w is continuous (see [5, Section 1.3] and [10, Section 3.7]).

For a weight w and a Young function Φ, we define the weighted Orlicz space as

$$L^\Phi_w(G) := \{f : fw \in L^\Phi(G)\}.$$

We also set

$$\|f\|_{\Phi,w} := \|fw\|_\Phi \quad (f \in L^\Phi_w(G));$$

then the function $\|\cdot\|_{\Phi,w}$ defines a norm on $L^\Phi_w(G)$ called a weighted Orlicz norm. With this norm, all weighted Orlicz spaces $L^\Phi_w(G)$ are Banach spaces. Moreover, if $\Phi \in \Delta_2$, then the dual space of $(L^\Phi_w(G), \|\cdot\|_{\Phi,w})$ is

$$L^\Psi_{w^{-1}}(G) = \left\{f : \frac{f}{w} \in L^\Psi(G)\right\},$$

where Ψ is the complementary function of Φ and the space $L^\Psi_{w^{-1}}(G)$ is endowed with the norm $\|f\|^\circ_{\Psi,w^{-1}} = \|f/w\|^\circ_\Psi$. So, if $\Phi, \Psi \in \Delta_2$, then the weighted Orlicz space $L^\Phi_w(G)$ is a reflexive Banach space.

Let us note that when $w = 1$, the weighted Orlicz space $(L^\Phi_w(G), \|\cdot\|_{\Phi,w})$ reduces to the usual Orlicz space $(L^\Phi(G), \|\cdot\|_\Phi)$.

For $1 < p < \infty$ and the Young function $\Phi(x) = |x|^p/p$, the space $L^\Phi_w(G)$ becomes the weighted Lebesgue space $L^p_w(G)$ and the norm $\|\cdot\|_{\Phi,w}$ is equivalent to the classical norm $\|\cdot\|_{p,w}$. In particular, if $p = 1$, then the complementary Young function of $\Phi(x) = |x|$ is

$$\Psi(x) = \begin{cases} 0 & \text{if } 0 \leq |x| \leq 1, \\ \infty & \text{otherwise} \end{cases} \tag{2.1}$$

and $\|f\|_{\Phi,w} = \|f\|_{1,w}$ for all $f \in L^1_w(G)$, since $\int_G \Psi(|v|)\,d\mu \leq 1$ if and only if $|v(x)| \leq 1$ almost everywhere on G. Note that the function Ψ defined in (2.1) is still a Young function as in the first definition of the Young function. If $p = \infty$, then for the Young function Φ given in (2.1), the space $L^\Phi_w(G)$ is equal to the space $L^\infty_w(G) = \{f : G \to \mathbb{C} : fw \in L^\infty(G)\}$ and we have $\|f\|_{\Phi,w} = \|f\|_{\infty,w}$ for all $f \in L^\infty_w(G)$.

Recall that a net $(e_\alpha)_{\alpha\in\Lambda}$ in the Banach algebra $(A, \|\cdot\|_A)$ is called a left approximate identity for A if $\lim_\alpha \|e_\alpha a - a\|_A = 0$ for all $a \in A$, and a right approximate identity is defined similarly. If there exists a $K > 0$ such that $\|e_\alpha\|_A \leq K$ for all $\alpha \in \Lambda$, then $(e_\alpha)_{\alpha\in\Lambda}$ is called a bounded approximate identity for A.

Now we give some technical lemmas.

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Lemma 2.1. *Let G be a locally compact group. If $\Phi \in \Delta_2$, then $C_c(G)$ is dense in $L^\Phi_w(G)$.*

Proof. Let $f \in L^\Phi_w(G)$. Then $fw \in L^\Phi(G)$. Since $C_c(G)$ is dense in $L^\Phi(G)$ given $\varepsilon > 0$, there exists $f_c \in C_c(G)$ such that $\|f_c - fw\|_\Phi < \varepsilon$. Set $g_c = f_c/w$. Then $g_c \in C_c(G)$ and $\|g_c - f\|_{\Phi,w} < \varepsilon$. □

Lemma 2.2. *Let G be a locally compact group, w be a weight and Φ be a Young function. Then the inclusion $L^\Phi_w(G) \subseteq L^1_w(G)$ is true if and only if there exists a $c > 0$ such that $\|f\|_{1,w} \le c\|f\|_{\Phi,w}$ for all $f \in L^\Phi_w(G)$.*

Proof. The 'if' part is trivial. Conversely, assume that $L^\Phi_w(G) \subseteq L^1_w(G)$. It is easy to show that $L^\Phi_w(G)$ is a Banach space with respect to the norm $\|f\| = \|f\|_{1,w} + \|f\|_{\Phi,w}$. Let I be the identity map from $(L^\Phi_w(G), \|\cdot\|)$ to $(L^\Phi_w(G), \|\cdot\|_{\Phi,w})$. Then I is continuous and one-to-one. By a consequence of the open mapping theorem, there exists a $c > 0$ such that

$$\|f\|_{1,w} \le c\|f\|_{\Phi,w}$$

for all $f \in L^\Phi_w(G)$. □

Lemma 2.3. *Let G be a locally compact group, w be a weight, Φ be a Young function and $f \in L^\Phi_w(G)$ be given.*

(i) *For all $x \in G$, $L_x f \in L^\Phi_w(G)$ and $\|L_x f\|_{\Phi,w} \le w(x)\|f\|_{\Phi,w}$.*
(ii) *If $\Phi \in \Delta_2$, then the mapping $x \mapsto L_x f$ from G into $L^\Phi_w(G)$ is continuous.*

Here $L_x f(y) = f(x^{-1}y)$ for all $x, y \in G$.

Proof. The proof of (i) follows from the definition of weight w easily. For the proof of (ii), it is sufficient to show that the map $x \mapsto L_x f$ is continuous at the identity element of the group G. Suppose first that $f \in C_c(G)$ with support S, and choose a compact neighbourhood K of $e \in G$. Then $\mathrm{supp}(L_x f) = xS$ for all $x \in K$. If we define $K' = K \cup S \cup (KS)$, then we have the inclusion $L^\Psi(K') \subseteq L^1(K')$ for a compact set $K' \subseteq G$, where Ψ is the complementary function of Φ. By Lemma 2.2 with $w = 1$, there exists a $c > 0$ such that

$$\|v\chi_K\|_1 \le c\|v\chi_K\|_\Psi \le 2c\|v\chi_K\|^\circ_\Psi \le 2c\|v\|^\circ_\Psi \le 2c \tag{2.2}$$

for all functions $v \in L^\Psi(G)$ for which $\int_G \Psi(|v(y)|)\,d\mu(y) \le 1$. On the other hand, there exists a $B > 0$ such that $w(y) \le B$ for all $y \in K'$, since a weight is bounded on compact sets. By (2.2), for all $x \in K$,

$$\begin{aligned}\|L_x f - f\|_{\Phi,w} &= \sup\Big\{\int_G |(L_x f - f)wv|\,d\mu : \int_G \Psi(|v|)\,d\mu \le 1\Big\}\\ &\le B\|L_x f - f\|_\infty \sup\Big\{\int_{K'} |v|\,d\mu : \int_G \Psi(|v|)\,d\mu \le 1\Big\}\\ &\le 2cB\|L_x f - f\|_\infty,\end{aligned}$$

since $\mathrm{supp}(L_x f - f) \subseteq K'$. Then the last norm is less than ε for any x in a sufficiently small neighbourhood $V \subseteq K$ of the identity.

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Now let the function $f \in L^\Phi_w(G)$ be arbitrary and let C be a compact neighbourhood of the identity. Then there exists an $A > 0$ such that $w(x) \le A$ for all $x \in C$. Since the set $C_c(G)$ is dense in $L^\Phi_w(G)$ for $\Phi \in \Delta_2$, for any $\varepsilon > 0$, there exists a function $g \in C_c(G)$ such that $\|f - g\|_{\Phi,w} < \varepsilon/2(A + 1)$. Also, from the first part of the proof, there exists a neighbourhood $V \subseteq C$ such that $\|L_x g - g\|_{\Phi,w} < \varepsilon/2$ for all $x \in V$ and

$$\begin{aligned}\|L_x f - f\|_{\Phi,w} &\le \|L_x f - L_x g\|_{\Phi,w} + \|L_x g - g\|_{\Phi,w} + \|g - f\|_{\Phi,w}\\ &\le (A + 1)\|f - g\|_{\Phi,w} + \|L_x g - g\|_{\Phi,w} < \varepsilon. \qquad \square\end{aligned}$$

Remark 2.4. If $w = 1$, then $\|L_x f\|_\Phi = \|f\|_\Phi$ for all $f \in L^\Phi(G)$ and for all $x \in G$. So, the translation operator L_x is an isometry on $L^\Phi(G)$.

3. Weighted Orlicz algebras

Hudzik *et al.* in [4, Theorem 2] proved that the Orlicz space $L^\Phi(G)$ for an abelian group G is a Banach algebra with respect to convolution if and only if $L^\Phi(G) \subseteq L^1(G)$. We give a sufficient condition on a Young function Φ for the weighted Orlicz space $L^\Phi_w(G)$ to become a Banach algebra with respect to convolution, and we show that this condition is not necessary. We also study some properties of this Banach algebra, which we call the weighted Orlicz algebra.

Theorem 3.1. *Let G be a locally compact group, w be a weight and Φ be a Young function. If $L^\Phi_w(G) \subseteq L^1_w(G)$, then the weighted Orlicz space $L^\Phi_w(G)$ is a Banach algebra with respect to the convolution. If also G is abelian, then this algebra is commutative.*

Proof. Assume that $L^\Phi_w(G) \subseteq L^1_w(G)$ is true. Then, by Lemma 2.2, there exists a $c > 0$ such that

$$\|f\|_{1,w} \le c\|f\|_{\Phi,w} \tag{3.1}$$

for all $f \in L^\Phi_w(G)$. Given any $f, g \in L^\Phi_w(G)$, Fubini's theorem implies that

$$\begin{aligned}\|f * g\|_{\Phi,w} &\le \sup\Big\{\int_G |f(y)w(y)| \int_G |w(y^{-1}x)g(y^{-1}x)v(x)|\,d\mu(x)\,d\mu(y)\\ &\qquad : \int_G \Psi(|v(x)|)\,d\mu(x) \le 1\Big\}\\ &\le \|fw\|_1 \|L_{y^{-1}}(gw)\|_\Phi\\ &= \|f\|_{1,w}\|g\|_{\Phi,w}.\end{aligned}$$

By (3.1), we obtain $\|f * g\|_{\Phi,w} \le c\|f\|_{\Phi,w}\|g\|_{\Phi,w}$ for all $f, g \in L^\Phi_w(G)$. $\square$

Without loss of generality, we may assume that $c = 1$ by renorming $L^\Phi_w(G)$.

Observation 3.2. If Φ is a Young function with $\Phi'_+(0) > 0$ (where Φ'_+ is the right derivative of Φ), then the inclusion $L^\Phi_w(G) \subseteq L^1_w(G)$ is true. So, by Theorem 3.1, $(L^\Phi_w(G), \|\cdot\|_{\Phi,w})$ becomes a weighted Orlicz algebra. If also G is noncompact and abelian, then $\Phi'_+(0) > 0$ is equivalent to $L^\Phi_w(G) \subseteq L^1_w(G)$. So, we can give the following corollary.

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Corollary 3.3. *If Φ is a Young function such that $\Phi'_+(0) > 0$, then $L^\Phi_w(G)$ is Banach algebra with respect to convolution.*

To obtain a Banach algebra structure on $L^\Phi_w(G)$, here we put a condition on Φ. On the other hand, without any assumption on the Young function Φ, we can have that the weighted Orlicz space $L^\Phi_w(G)$ is a left Banach $L^1_w(G)$-module with respect to convolution. So, we obtain on $L^p_w(G)$ an $L^1_w(G)$-module structure.

Remark 3.4. Note that the converse of Theorem 3.1 is not true in general. For example, if $1 < p < \infty$ and $\Phi(x) = |x|^p/p$, then, for some weights w, $L^p_w(G)$ is a Banach algebra with respect to convolution [7, Theorem 1]. But $L^p_w(G) \not\subseteq L^1_w(G)$ for a noncompact G (for example, when $G = \mathbb{R}$), since $\Phi'_+(0) = 0$.

4. Approximate identities

It is known that the weighted Lebesgue algebra $L^p_w(G)$, $1 < p < \infty$, always has a left approximate identity and the algebra $L^p_w(G)$ on a nondiscrete group G has no bounded left approximate identity [8, Theorems 4.1 and 4.2]. In this section, we investigate the existence of an approximate identity for the weighted Orlicz algebra $L^\Phi_w(G)$. We show that it has a left approximate identity if the Young function Φ satisfies the Δ_2 condition. If also the complementary function Ψ of Φ satisfies the Δ_2 condition, then, for a nondiscrete G, the weighted Orlicz algebra $L^\Phi_w(G)$ admits no bounded left approximate identity (Theorem 4.4). Similar assertions hold for right approximate identities.

Remark 4.1. Note that, for $p = 1$, the Banach algebra $L^1_w(G)$ always has a bounded left approximate identity. But here we get $L^1_w(G)$ from the $L^\Phi_w(G)$ for the Young function $\Phi(x) = |x|$, and the complementary function of $\Phi(x) = |x|$ does not satisfy the Δ_2 condition. So, there is no contradiction with Theorem 4.4. In particular, for $w = 1$ and nondiscrete G, the Orlicz algebra $L^\Phi(G)$ has no bounded left approximate identity if the complementary function Ψ of the Young function Φ satisfies the Δ_2 condition. This result does not contradict the assertion mentioned in [2] that $L^\Phi(G)$ has no bounded left approximate identity for a Young function Φ.

From now on, we assume that Φ is a Young function with $\Phi \in \Delta_2$. Firstly, by using classical techniques in harmonic analysis, we show the existence of the left approximate identity of the weighted Orlicz algebra $L^\Phi_w(G)$.

Theorem 4.2. *Let $L^\Phi_w(G)$ be a weighted Orlicz algebra. Then $L^\Phi_w(G)$ has a left approximate identity consisting of compactly supported functions.*

Proof. Let K be a compact neighbourhood of the identity element e of G and denote by $\mathcal{V}$ the family of all neighbourhoods $V \subseteq K$ of the identity element ordered by inclusion so that $\mathcal{V}$ is a directed set. Defining $e_V = \chi_V/\mu(V)$ for all $V \in \mathcal{V}$, by the continuity of the weight w, there exists a $B > 0$ such that $\|e_V\|_{1,w} \le B$ for all $V \in \mathcal{V}$. So, $(e_V)_{V\in\mathcal{V}}$ becomes a left approximate identity of the weighted Orlicz algebra $L^\Phi_w(G)$.

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Indeed, given $f \in L^{\Phi}_w(G)$ and $\varepsilon > 0$, by Lemma 2.3, we can find a neighbourhood $W \in \mathcal{V}$ such that $\|L_x f - f\|_{\Phi,w} < \varepsilon$ for all $x \in W$. Then, for all $V \geq W$ with $V \in \mathcal{V}$, by using Fubini's theorem,

$$\int_G |(e_V * f - f)wv|\, d\mu \leq \frac{1}{\mu(V)} \int_V \|L_x f - f\|_{\Phi,w}\, d\mu(x) < \varepsilon$$

for all $v \in L^{\Psi}(G)$ satisfying $\int_G \Psi(|v(y)|)\, d\mu(y) \leq 1$. If we take the supremum over such functions v,

$$\|e_V * f - f\|_{\Phi,w} \leq \varepsilon.$$

Consequently, $(e_V)_{\mathcal{V}}$ is a left approximate identity for $L^{\Phi}_w(G)$. □

Now we prove that the left approximate identity of the weighted Orlicz algebra $L^{\Phi}_w(G)$ is not bounded if G is nondiscrete. For this purpose, we need the following lemma.

Lemma 4.3. *Let G be a locally compact group, w be a weight and Φ be a Young function. Then $L^{\Phi}_w(G) \subseteq L^{\infty}_w(G)$ if and only if G is discrete.*

Proof. Let $L^{\Phi}_w(G) \subseteq L^{\infty}_w(G)$. Suppose that G is not discrete. Then there exist a relatively compact subset K of G and a sequence (U_n) of pairwise disjoint open subsets of K such that $0 < \mu(U_n) < \mu(K)\Phi(1)/2^n\Phi(2^n)$ for all $n \in \mathbb{N}$. If we define the function f on G by

$$f = \sum_{n=1}^{\infty} 2^n \chi_{U_n},$$

then $f \in L^{\Phi}_w(G)$, since $\mathrm{supp}(f) \subseteq \overline{K}$, and the weight w is continuous on compact sets. Moreover, there exists an $A > 0$ such that $|f(x)w(x)| \geq A2^n$ for all $x \in U_n$. This says that $fw \notin L^{\infty}(G)$ or equivalently $f \notin L^{\infty}_w(G)$. But this contradicts the assumption.

Conversely, assume that G is a discrete group. Let $f \in L^{\Phi}_w(G)$; then there exists an $\alpha > 0$ such that

$$\int_G \Phi(\alpha|f(x)w(x)|)\, d\mu(x) = \sum_{x\in G} \Phi(\alpha|f(x)w(x)|) \leq 1.$$

Therefore, $\Phi(\alpha|f(x)w(x)|) \leq 1$ for all $x \in G$. So, $\alpha f w$ must be bounded on G, since $\lim_{x\to\infty} \Phi(x) = \infty$ for a Young function Φ. Thus, $fw \in L^{\infty}(G)$ and we have $f \in L^{\infty}_w(G)$. □

Theorem 4.4. *Let G be a nondiscrete group and $L^{\Phi}_w(G)$ be a weighted Orlicz algebra with $w \geq 1$. If the complementary function Ψ of Φ is continuous and satisfies the Δ_2 condition, then the weighted Orlicz algebra $L^{\Phi}_w(G)$ has no bounded left approximate identity.*

Proof. Suppose that $L^{\Phi}_w(G)$ is a weighted Orlicz algebra and has a bounded left approximate identity. If Ψ is the complementary function of Φ, then the space $L^{\Psi}_{w^{-1}}(G)$ is a left $L^{\Phi}_w(G)$-module: if $h \in L^{\Psi}_{w^{-1}}(G)$ and $f \in L^{\Phi}_w(G)$, then their product

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$f \circ h \in L^{\Psi}_{w^{-1}}(G)$ is defined by the formula $\langle g, f \circ h\rangle = \langle g * f, h\rangle$ for all $g \in L^{\Phi}_{w}(G)$ and can be explicitly expressed via convolution as

$$f \circ h = h * \check{f}, \quad \text{where } \check{f}(x) = f(-x).$$

Moreover, it is known that $L^{\Phi}_{w}(G) \circ L^{\Psi}_{w^{-1}}(G)$ is a closed set in the space $L^{\Psi}_{w^{-1}}(G)$ by [3, Theorem 32.22].

On the other hand, the space $L^{\Phi}_{w}(G) \circ L^{\Psi}_{w^{-1}}(G)$ is dense in the space $L^{\Psi}_{w^{-1}}(G)$. For this, it is sufficient to show that the closure of $L^{\Phi}_{w}(G) \circ L^{\Psi}_{w^{-1}}(G)$ contains the characteristic functions of all compact sets, since the linear span of all characteristic functions of compact sets is dense in the space $L^{\Psi}_{w^{-1}}(G)$. Let $K \subseteq G$ be a compact set. Take a relatively compact symmetric neighbourhood V of the identity element e of the group G. Since V and KV are relatively compact sets,

$$\check{\chi}_V = \chi_{V^{-1}} \in L^{\Phi}_{w}(G) \quad \text{and} \quad \chi_{KV} \in L^{\Psi}_{w^{-1}}(G).$$

Hence, if we define

$$f_V = \frac{1}{\mu(V)}(\chi_{KV} * \chi_V) = \frac{1}{\mu(V)}(\check{\chi}_V \circ \chi_{KV}),$$

then $f_V \in L^{\Phi}_{w}(G) \circ L^{\Psi}_{w^{-1}}(G)$. It is easy to see that

$$\chi_K \le f_V \le \chi_{KVV}$$

and, from it, we obtain $\|\chi_K - f_V\|^{\circ}_{\Psi,w^{-1}} \le \|\chi_K - \chi_{KVV}\|^{\circ}_{\Psi,w^{-1}}$. By assumption, $w \ge 1$, so

$$\|\chi_K - \chi_{KVV}\|^{\circ}_{\Psi,w^{-1}} \le \|\chi_K - \chi_{KVV}\|^{\circ}_{\Psi}.$$

Since Ψ is continuous, there exists an $x_0 > 0$ such that Ψ is strictly increasing on $[x_0, \infty)$. This shows that

$$\|\chi_K - f_V\|^{\circ}_{\Psi,w^{-1}} \le \|\chi_K - \chi_{KVV}\|^{\circ}_{\Psi} = \frac{1}{\Psi^{-1}\left(\frac{1}{\mu(KVV\backslash K)}\right)},$$

which tends to zero as $\mu(V) \to 0$. Thus, χ_K is contained in the closure of $L^{\Phi}_{w}(G) \circ L^{\Psi}_{w^{-1}}(G)$. Therefore, $L^{\Phi}_{w}(G) \circ L^{\Psi}_{w^{-1}}(G) = L^{\Psi}_{w^{-1}}(G)$, since $L^{\Phi}_{w}(G) \circ L^{\Psi}_{w^{-1}}(G)$ is closed.

Thus, for every $h \in L^{\Psi}_{w^{-1}}(G)$, there exist $f \in L^{\Phi}_{w}(G)$ and $g \in L^{\Psi}_{w^{-1}}(G)$ such that $h = f \circ g = g * \check{f}$. Using the Hölder inequality,

$$\begin{aligned}
|h(x)| = |(g * \check{f})(x)| &= \left|\int_G g(y)\check{f}(y^{-1}x)\,d\mu(y)\right| \\
&= \left|\int_G g(y) f(x^{-1}y)\,d\mu(y)\right| \le \int_G |g(y) L_x f(y)|\,d\mu(y) \\
&\le 2\|L_x f\|_{\Phi,w}\|g\|_{\Psi,w^{-1}} \le 2w(x)\|f\|_{\Phi,w}\|g\|_{\Psi,w^{-1}}
\end{aligned}$$

for all $x \in G$. Thus, $h/w \in L^{\infty}(G)$. This implies that $L^{\Psi}_{w^{-1}}(G) \subseteq L^{\infty}_{w^{-1}}(G)$ or equivalently $L^{\Psi}(G) \subseteq L^{\infty}(G)$, which is possible only for a discrete group G by Lemma 4.3 for $w = 1$. This completes the proof. □

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Theorem 4.5. *A weighted Orlicz algebra $L_w^\Phi(G)$ has an identity if and only if G is discrete.*

Proof. Assume that $L_w^\Phi(G)$ is a weighted Orlicz algebra with an identity and that $g \in L_w^\Phi(G)$ is an identity. So, $f * g = g * f = f$ for all $f \in L_w^\Phi(G)$. Let a neighbourhood U of $e \in G$ and any $\varepsilon > 0$ be given. Since $C_c(G)$ is dense in $L_w^\Phi(G)$, there exists a function $f \in C_c(G)$ such that $\operatorname{supp}(f) \subseteq U$ and $\|f * g - g\|_{\Phi,w} < \varepsilon$. Since $f * g = f$, we have $\|f - g\|_{\Phi,w} < \varepsilon$. Also, from the definition of $\|\cdot\|_{\Phi,w}$,

$$\begin{aligned}\varepsilon > \|f - g\|_{\Phi,w} &\geq \int_U w|f-g|\,|v|\,d\mu + \int_{G\setminus U} w|f-g|\,|v|\,d\mu \\ &\geq \int_{G\setminus U} w|g|\,|v|\,d\mu\end{aligned}$$

for all $v \in L^\Psi(G)$ satisfying $\int_G \Psi(|v|)\,d\mu \leq 1$. Then $w(x)|g(x)|\,|v(x)|$ should be zero for all $x \in G\setminus U$. If we select $v \in L^\Psi(G)$ such that $v(x) \neq 0$ for all $x \notin U$, then $g(x) = 0$ for all $x \in G\setminus U$, since the weight w is positive. Since U is any neighbourhood of the identity e, we have $\operatorname{supp}(g) \subseteq \{e\}$ and the measure of the singleton set $\{e\}$ should be positive. Because, if $\mu(\{e\}) = 0$, then g must be zero almost everywhere on G, but this contradicts that g is an identity of the algebra $L_w^\Phi(G)$. So, G must be discrete.

Conversely, assume that G is a discrete group. In this case, μ is the counting measure on G. If we denote the characteristic function of the singleton set $\{e\}$ by χ_e, then $\chi_e \in L_w^\Phi(G)$ and

$$(\chi_e * f)(x) = \int_G \chi_e(x^{-1}y)f(y)\,d\mu(y) = \sum_{y\in G} \chi_e(x^{-1}y)f(y) = f(x)$$

for all $f \in L_w^\Phi(G)$ and all $x \in G$. Thus, the function χ_e is an identity of the algebra $L_w^\Phi(G)$. □

Corollary 4.6 ([7, Theorem 4(iii)] and [8, Theorems 4.1 and 4.2]). *Let $1 < p < \infty$ and $L_w^p(G)$ be a Banach algebra. If we take $\Phi(x) = |x|^p/p$ in the above theorems, then the following hold.*

(i) *$L_w^p(G)$ has an approximate identity consisting of compactly supported functions.*
(ii) *If G is nondiscrete, then $L_w^p(G)$ admits no bounded approximate identity with respect to $\|\cdot\|_{p,w}$, since $\Psi(x) = |x|^q/q \in \Delta_2$.*
(iii) *$L_w^p(G)$ has an identity if and only if G is discrete.*

Corollary 4.7 [1, Proposition 4.2 and Theorem 4.3]. *Let Φ be an N-function and $L^\Phi(G)$ be an Orlicz algebra. If we take $w = 1$ in the above theorems, then the following hold.*

(i) *$L^\Phi(G)$ has an approximate identity consisting of compactly supported functions.*
(ii) *If the complementary function Ψ of Φ satisfies the Δ_2 condition, then, for nondiscrete G, the Orlicz algebra $L^\Phi(G)$ admits no bounded left approximate identity with respect $\|\cdot\|_\Phi$.*

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(iii) *$L^{\Phi}(G)$ has an identity if and only if G is discrete.*

Note that Corollary 4.7 is still true if we take Φ as a Young function.

Let $L^{\Phi}_w(G)$ be a weighted Orlicz algebra. Similar to the Banach algebra $L^1_w(G)$, by using the existence of a left approximate identity, we observe that the closed left ideals of the weighted Orlicz algebra $L^{\Phi}_w(G)$ turn out to be nothing but the closed left translation invariant subspaces of $L^{\Phi}_w(G)$.

Theorem 4.8. *Let $L^{\Phi}_w(G)$ be a weighted Orlicz algebra and I be a closed linear subspace of $L^{\Phi}_w(G)$. Then I is a left ideal in $L^{\Phi}_w(G)$ if and only if I is a left translation invariant subspace, that is, $L_x(I) \subseteq I$ for all $x \in G$.*

Proof. Suppose that $I \subseteq L^{\Phi}_w(G)$ is a left ideal. Given any $x \in G$ and $f \in I$, for any $\varepsilon > 0$, by Theorem 4.2, there exists a neighbourhood V of the identity element of G such that $\|e_V * f - f\|_{\Phi_w} < \varepsilon/w(x)$. On the other hand, $(L_x e_V) * f \in I$, since $L^{\Phi}_w(G)$ is left translation invariant and I is a left ideal in $L^{\Phi}_w(G)$. So,

$$\|L_x e_V * f - L_x f\|_{\Phi_w} \le w(x)\|e_V * f - f\|_{\Phi_w} < \varepsilon.$$

Thus, $L_x f \in I$, since I is closed.

Conversely, let I be a left translation invariant subspace of $L^{\Phi}_w(G)$. We have to show that $g * f \in I$ for each $f \in I$ and $g \in L^{\Phi}_w(G)$. To show this, take $f \in I$ and $g \in L^{\Phi}_w(G)$ and suppose that $g * f \notin I$. Then, from the consequences of the Hahn–Banach theorem, there exists a continuous linear functional F on $L^{\Phi}_w(G)$ such that $F|_I = \{0\}$ and $F(g * f) \neq 0$. On the other hand, since $\Phi \in \Delta_2$, the dual space of $L^{\Phi}_w(G)$ is $L^{\Psi}_{w^{-1}}(G)$, where Ψ is the complementary function of Φ. So, the continuous linear functional $F \in (L^{\Phi}_w(G))^*$ is uniquely determined by $\varphi \in L^{\Psi}_{w^{-1}}(G)$, such as

$$F(h) = \int_G h\varphi \, d\mu(x) \quad (h \in L^{\Phi}_w(G)).$$

Hence,

$$\begin{aligned} F(g * f) &= \int_G \varphi(x)\Big(\int_G g(y) f(y^{-1}x)\, d\mu(y)\Big)\, d\mu(x) \\ &= \int_G g(y) F(L_{y^{-1}} f)\, d\mu(y) = 0, \end{aligned}$$

since $L_{y^{-1}} f \in I$ and $F|_I = \{0\}$. This contradicts the assumption that $F(f * g) \neq 0$. □

From Theorem 4.8, we obtain the following new results related to the weighted Lebesgue algebra $L^p_w(G)$ for $\Phi(x) = |x|^p/p$, $1 < p < \infty$, and the Orlicz algebra $L^{\Phi}(G)$ for $w = 1$.

Corollary 4.9.

(i) *The closed left ideals of the unweighted Orlicz algebra $L^{\Phi}(G)$ coincide with its closed left translation invariant subspaces.*

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(ii) *If $L^p_w(G)$, $1 < p < \infty$, is a Banach algebra, then the closed left ideals of the algebra $L^p_w(G)$ coincide with the closed left translation invariant subspaces of $L^p_w(G)$.*

Proposition 4.10. *Let Φ be a Young function such that $\Phi'_+(0) > 0$. Then the weighted Orlicz algebra $L^\Phi_w(G)$ is a left ideal in $L^1_w(G)$.*

5. Weighted Orlicz algebras in the commutative case

Let G be a locally compact abelian group, w be a weight and Φ be a Young function. We now describe the maximal ideal space (spectrum) $\Delta(L^\Phi_w(G))$ of the commutative weighted Orlicz algebra $L^\Phi_w(G)$ in terms of so-called generalized characters of G determined by the complementary function Ψ of Φ and the weight w (Theorem 5.2). We also show that the algebras $L^\Phi_w(G)$ are semisimple (Theorem 5.8).

First we make the following definition.

Definition 5.1. Let G be a locally compact abelian group, w be a weight function and (Φ, Ψ) be a complementary pair of Young functions. A generalized character determined by the Young function Ψ and a weight function w on G is a continuous function $\gamma : G \to \mathbb{C}\backslash\{0\}$ satisfying the conditions:

(i) $\gamma(x + y) = \gamma(x)\gamma(y)$ for all $x, y \in G$;
(ii) $\gamma/w \in L^\Psi(G)$.

Let $\widehat{G_\Psi}(w)$ denote the set of all generalized characters on G equipped with the topology of uniform convergence on compact subsets of G.

Now we describe the maximal ideal space (spectrum) of the weighted Orlicz algebra $L^\Phi_w(G)$.

Theorem 5.2. *Let G be a locally compact abelian group and $L^\Phi_w(G)$ be a weighted Orlicz algebra. If $\gamma \in \widehat{G_\Psi}(w)$, then the map φ_γ defined by*

$$\varphi_\gamma(f) = \int_G f\gamma\, d\mu \quad (f \in L^\Phi_w(G))$$

is a complex homomorphism of $L^\Phi_w(G)$. Conversely, every complex homomorphism of $L^\Phi_w(G)$ is obtained in this way, and distinct generalized characters induce distinct homomorphisms.

Proof. Let $\gamma \in \widehat{G_\Psi}(w)$. It is straightforward to show that $\varphi_\gamma \in (L^\Phi_w(G))^*$, since $\gamma/w \in L^\Psi(G)$. Also, φ_γ is a homomorphism, since γ satisfies $\gamma(x + y) = \gamma(x)\gamma(y)$ for all $x, y \in G$.

Conversely, suppose that φ is a complex homomorphism of $L^\Phi_w(G)$; then φ is a bounded linear functional on $L^\Phi_w(G)$ with $\|\varphi\| \le 1$. Since $\varphi \in (L^\Phi_w(G))^*$, there exists a $\gamma \in L^\Psi_{w^{-1}}(G)$ such that

$$\varphi(g) = \int_G g\gamma\, d\mu \quad (g \in L^\Phi_w(G)).$$

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Choose $f \in L_w^\Phi(G)$ such that $\varphi(f) = 1$. We have

$$\varphi(g) = \varphi(f * g) = \int_G (f * g)\gamma \, d\mu = \int_G g(y)\varphi(L_y f) \, d\mu(y)$$

for all $g \in L_w^\Phi(G)$; or, take $\gamma(y) = \varphi(L_y f)$ for $y \in G$. Then γ is continuous because the map $y \mapsto L_y f$ from G into $L_w^\Phi(G)$ is continuous by Lemma 2.3 and

$$|\gamma(x) - \gamma(y)| = |\varphi(L_x f - L_y f)| \leq \|L_x f - L_y f\|_{\Phi,w}.$$

Moreover, since φ is linear and $\varphi(L_y f/\varphi(L_y f)) = 1$,

$$\gamma(x + y) = \varphi(L_{x+y}(f)) = \varphi\Big(L_x\Big(\frac{L_y f}{\varphi(L_y f)}\Big)\Big)\varphi(L_y f) = \gamma(x)\gamma(y)$$

for all $x, y \in G$. This shows that $\gamma \in \widehat{G_\Psi}(w)$ and thus $\varphi = \varphi_\gamma$.

Finally, if $\varphi_{\gamma_1} = \varphi_{\gamma_2}$, then $\gamma_1 = \gamma_2$, since $(L_w^\Phi(G))^* = L_{w^{-1}}^\Psi(G)$. □

As a consequence of this theorem, the maximal ideal space $\Delta(L_w^\Phi(G))$ of the weighted Orlicz algebra $L_w^\Phi(G)$ can be identified with the space of all generalized characters γ determined by the Young function Ψ and the weight w.

As another consequence, the maximal ideal spaces of the Orlicz algebra $L^\Phi(G)$ and the weighted Lebesgue algebra $L_w^p(G)$, $1 < p < \infty$, can be similarly identified.

Corollary 5.3. *If $w = 1$, then, for the Orlicz algebra $L^\Phi(G)$, $\Delta(L^\Phi(G))$ can be identified with $\widehat{G_\Psi}$, where Ψ is the complementary function of Φ.*

Corollary 5.4 [7, Theorem 2]. *Let $1 < p < \infty$, $1/p + 1/q = 1$ and $L_w^p(G)$ be a Banach algebra. Then, for $\Phi(x) = |x|^p/p$, $\Delta(L_w^p(G))$ can be identified with $\widehat{G_\Psi}(w)$, where $\Psi(x) = |x|^q/q$ for all $x \in \mathbb{R}$.*

Note that in the case $p = 1$, for $\Phi(x) = |x|$, we obtain that the maximal ideal space of the weighted Banach algebra $L_w^1(G)$ is $\widehat{G}(w)$. Also, if $w \geq 1$, then $L_w^1(G) \subseteq L^1(G)$ and we know that $\widehat{G} \subseteq \widehat{G}(w)$ [5, Section 2.8]. So, we have a similar inclusion for the weighted Orlicz algebra case.

Observation 5.5. If $w(x) \geq 1$ for all $x \in G$, between the weighted Orlicz algebra $L_w^\Phi(G)$ and the Orlicz algebra $L^\Phi(G)$, we have the containment $L_w^\Phi(G) \subseteq L^\Phi(G)$. So, we obtain $\widehat{G_\Psi} \subseteq \widehat{G_\Psi}(w)$.

In the case $G = \mathbb{R}$, we are able to characterize a part of $\Delta(L_w^\Phi(\mathbb{R}))$. To do this, let us define the subset $\mathcal{A}$ of complex numbers by requiring that $z \in \mathcal{A}$ if and only if there exists a $g \in L^1(\mathbb{R})$ such that

$$|g(x)| \leq 1 \quad \text{and} \quad -\ln(w(x)|g(x)|)^{1/x} \leq \operatorname{Re} z \leq -\ln(w(-x)|g(-x)|)^{-1/x} \tag{5.1}$$

for all $x \in \mathbb{R}$. We observe that there is an embedding of $\mathcal{A}$ into $\widehat{\mathbb{R}_\Psi}(w)$ which describes certain elements of $\widehat{\mathbb{R}_\Psi}(w)$.

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Proposition 5.6. *Let w be a weight on $\mathbb{R}$ and let (Φ, Ψ) be a complementary pair of Young functions with Ψ continuous. Then every $z \in \mathcal{A}$ defines an element φ_z of $\Delta(L^{\Phi}_w(\mathbb{R}))$ by*

$$\varphi_z(f) = \int_{\mathbb{R}} f(x)e^{-zx}\, d\mu(x) \quad (f \in L^{\Phi}_w(\mathbb{R})).$$

Proof. Let $z \in \mathcal{A}$ and obtain the g satisfying (5.1) for $x \in \mathbb{R}$. It is clear that an arbitrary $z \in \mathbb{C}$ gives a nonzero continuous homomorphism $\gamma_z(x) = e^{-zx}$ on $\mathbb{R}$. We show that if $z \in \mathcal{A}$, then $\gamma_z/w \in L^{\Psi}(\mathbb{R})$. Firstly, by the continuity and the convexity of Ψ,

$$\Psi\Big(\frac{e^{-\mathrm{Re}(zx)}}{w(x)}\Big) \le \Psi(1)\frac{e^{-\mathrm{Re}(zx)}}{w(x)} \quad (x \in \mathbb{R}),$$

since $0 \le e^{-\mathrm{Re}(zx)}/w(x) \le 1$ for all $x \in \mathbb{R}$. Consequently,

$$\int_{\mathbb{R}} \Psi\Big(\frac{|\gamma_z(x)|}{w(x)}\Big)\, d\mu(x) \le \Psi(1)\int_{\mathbb{R}} \frac{e^{-\mathrm{Re}(zx)}}{w(x)}\, d\mu(x) \le \Psi(1)\int_{\mathbb{R}} |g|\, d\mu < \infty.$$

Thus, $\gamma_z/w \in L^{\Psi}(\mathbb{R})$, that is, γ_z belongs to $\widehat{\mathbb{R}}_{\Psi}(w)$. By Theorem 5.2, this γ_z yields the $\varphi_z \in \Delta(L^{\Phi}_w(\mathbb{R}))$ mentioned in the statement of the proposition. □

By using the characterization of the maximal ideal space $\Delta(L^{\Phi}_w(G))$, we now show that the weighted Orlicz algebras $L^{\Phi}_w(G)$ are semisimple, but first we must prove that they are not radical algebras.

Proposition 5.7. *The weighted Orlicz algebra $L^{\Phi}_w(G)$ is not a radical algebra.*

Proof. Suppose that $L^{\Phi}_w(G)$ is a weighted Orlicz algebra. It suffices to show that there exists a function $f \in L^{\Phi}_w(G)$ such that

$$\lim_{n\to\infty} \|f^n\|^{1/n}_{\Phi,w} > 0,$$

where f^n denotes the n times convolution product of f. To show this, choose a relatively compact symmetric neighbourhood U of the identity e of G and let $f = \chi_U$. Then

$$0 < M = \sup\{w(x) : x \in U\} < \infty,$$

since w is continuous. Moreover, if we define $U^n = UU\cdots U$ (n times) for all $n \in \mathbb{N}$, then, for all $x \in U^n$, there exist $x_1, x_2, \ldots, x_n \in U$ such that $x = x_1x_2\cdots x_n$ and

$$w(x) = w(x_1x_2\cdots x_n) \le w(x_1)w(x_2)\cdots w(x_n) \le MM\cdots M = M^n$$

for all $x \in U^n$. Also, it is clear that $w(x) \ge w(e)/w(x^{-1})$ for all $x \in G$. For $f = \chi_U$ and for all $v \in L^{\Psi}(G)$ with $\int_G \Psi(|v|)\, d\mu \le 1$,

$$\|f^n\|_{\Phi,w} \ge \int_G |wf^nv|\, d\mu \ge \frac{w(e)}{M^n}\int_{U^n} (\chi_U)^n|v|\, d\mu,$$

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since $\operatorname{supp}(f^n) = \operatorname{supp}((\chi_U)^n) \subseteq U^n$. If we take the supremum over all $v \in L^\Psi(G)$ satisfying $\int_G \Psi(|v|)\, d\mu \le 1$,

$$\|f^n\|_{\Phi,w} \ge \frac{w(e)}{M^n}\|(\chi_U)^n\|_\Phi.$$

Also, there exists a $c > 0$ such that $\|(\chi_U)^n\|_\Phi \ge c\|(\chi_U)^n\|_1$. This inequality implies that

$$\lim_{n\to\infty} \|f^n\|_{\Phi,w}^{1/n} \ge \frac{1}{M} \lim_{n\to\infty} \|\chi_U^n\|_1^{1/n} > 0,$$

since $f = \chi_U \in L^1(G)$. Thus, $f \notin \operatorname{rad}(L_w^\Phi(G))$. □

Theorem 5.8. *If $L_w^\Phi(G)$ is a weighted Orlicz algebra, then it is semisimple.*

Proof. Since $L_w^\Phi(G)$ is not radical, there exists a $\varphi \in \Delta(L_w^\Phi(G))$ and this φ is determined by a $\gamma \in \widehat{G_\Psi}(w)$ uniquely. For each $\alpha \in \widehat{G}$, define $\varphi_\alpha \in (L_w^\Phi(G))^*$ by

$$\varphi_\alpha(f) = \int_G f\overline{\alpha\gamma}\, d\mu.$$

Then $\varphi_\alpha \in \Delta(L_w^\Phi(G))$, since $\alpha\gamma \in \widehat{G_\Psi}(w)$ for all $\alpha \in \widehat{G}$. Let f be an element of the radical of $L_w^\Phi(G)$. Then $f\overline{\gamma} \in L^1(G)$ and

$$\widehat{f\overline{\gamma}}(\alpha) = \varphi_\alpha(f) = 0$$

for all $\alpha \in \widehat{G}$. From the uniqueness of the Fourier transform in $L^1(G)$, we obtain $f\overline{\gamma} = 0$ and so $f = 0$. □

Corollary 5.9.

(i) *If $L_w^p(G)$, $1 \le p < \infty$, is a Banach algebra, then, for $\Phi(x) = x^p/p$, we conclude that $L_w^p(G)$ is semisimple.*

(ii) *In particular, if $w = 1$, then the Orlicz algebra $L^\Phi(G)$ is semisimple.*

We use different techniques from those of [7] and [1] to obtain the semisimplicity.

On the other hand, if $L_w^\Phi(G) \subseteq L_w^1(G)$, then we know from Theorem 3.1 that $L_w^\Phi(G)$ is a Banach algebra with respect to convolution and the converse is not true in general. But the next theorem shows that if $L_w^\Phi(G)$ is a Banach algebra with respect to convolution, then one can always assume that the inclusion $L_w^\Phi(G) \subseteq L^1(G)$ is true. Hence, we can use the similar method used in [7, Theorem 4] to show the semisimplicity of the weighted Orlicz algebra $L_w^\Phi(G)$. Before this, we give the following easy lemma by considering the dual spaces.

Lemma 5.10. *Let w be a weight function and (Φ, Ψ) be a pair of complementary Young functions such that $\Phi \in \Delta_2$. The following conditions are equivalent:*

(i) $1/w \in L^\Psi(G)$;

(ii) $L_w^\Phi(G) \subseteq L^1(G)$;

(iii) $L^\infty(G) \subseteq L_{w^{-1}}^\Psi(G)$.

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THEOREM 5.11. *Each weighted Orlicz algebra $L^{\Phi}_{w}(G)$ is isometrically isomorphic to an algebra $L^{\Phi}_{\widetilde{w}}(G)$ with weight $\widetilde{w}$ satisfying $1/\widetilde{w} \in L^{\Psi}(G)$.*

PROOF. If we fix $\gamma \in \widehat{G_{\Psi}}(w)$ and set $\widetilde{w} = w/|\gamma|$, then the mapping $f \mapsto f|\gamma|$ from $L^{\Phi}_{w}(G)$ to $L^{\Phi}_{\widetilde{w}}(G)$ is obviously an isometric isomorphism. □

REMARK 5.12. Let $L^{\Phi}_{w}(G)$ be a weighted Orlicz algebra. Then we can assume that $L^{\Phi}_{w}(G) \subseteq L^{1}(G)$ and this again implies that $L^{\Phi}_{w}(G)$ is semisimple, following the argument in [7].

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References

[1] I. Akbarbaglu and S. Maghsoudi, 'Banach–Orlicz algebras on a locally compact group', *Mediterr. J. Math.* **10** (2013), 1937–1947.
[2] P. Bharucha, 'Amenability properties and their consequences in Banach algebras', PhD Thesis, Australian National University, 2008.
[3] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis II* (Springer, New York, 1997).
[4] H. Hudzik, A. Kamińska and J. Musielak, 'On some Banach algebras given by a modular', in: *Alfred Haar Memorial Conference,* Budapest, Colloquia Mathematica Societatis János Bolyai, 49 (North-Holland, Amsterdam, 1987), 445–463.
[5] E. Kaniuth, *A Course in Commutative Banach Algebras* (Springer, New York, 2009).
[6] M. A. Krasnosel'skii and Ja. B. Rutickii, *Convex Functions and Orlicz Spaces* (Noordhoff, Groningen, 1961).
[7] Yu. N. Kuznetsova, 'Weighted L_p-Algebras on Groups', *Funct. Anal. Appl.* **40**(3) (2006), 234–236; (English transl.).
[8] Yu. N. Kuznetsova, 'Invariant weighted algebras', *Mat. Zametki* **84**(4) (2008), 567–576.
[9] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces* (Marcel Dekker, New York, 1991).
[10] H. Reiter and J. D. Stegeman, *Classical Harmonic Analysis and Locally Compact Groups* (Clarendon Press, Oxford, 2000).
[11] S. Saeki, 'The L^p-conjecture and Young's inequality', *Illinois J. Math.* **34**(3) (1990), 614–627.

ALEN OSANÇLIOL, Department of Mathematics, Faculty of Science,
İstanbul University, İstanbul, Turkey
e-mail: aleno@istanbul.edu.tr

SERAP ÖZTOP, Department of Mathematics, Faculty of Science,
İstanbul University, İstanbul, Turkey
e-mail: oztops@istanbul.edu.tr

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