A GENERALIZATION OF COMMUTATIVE AND ALTERNATIVE RINGS

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Introduction. In [**3**] Schafer has defined generalized standard rings as rings satisfying the identities

$$(1) \qquad (x, y, x) = 0,$$

$$(2) \qquad (x, y, z)x + (y, z, x)x + (z, x, y)x = (x, y, xz) + (y, xz, x) + (xz, x, y),$$

$$(3) \qquad (x, y, wz) + (w, y, xz) + (z, y, xw) = (x, (w, z, y)) + (x, w, (y, z)),$$

and observed that these identities imply $(y, y, (x, z)) = 0$ and if the characteristic is not three, $(x, y, x^2) = 0$. Schafer determined the structure of simple, finite-dimensional generalized standard algebras of characteristic not two or three by showing that they must be either commutative, Jordan, or alternative.

Previously one of us [**2**] had studied accessible rings, which are defined by the identities $(x, y, z) + (z, x, y) - (x, z, y) = 0$ and $((w, x), y, z) = 0$. The structure of accessible rings is determined in that paper as it turns out that an accessible ring without trivial ideals must be a subdirect sum of an associative ring and a commutative ring.

Both of the above results generalize some results on standard algebras by Albert [**1**].

In this paper we define an even more general class of rings called generalized accessible, as consisting of those rings which satisfy the identities

$$(1) \qquad (x, y, x) = 0,$$

$$(4) \qquad (x, (z, y, y)) = 0,$$

$$(5) \qquad 3(x, y, (w, z)) = -(w, (x, y, z)) - 2(x, (y, z, w)) + 2(y, (z, w, x)) + (z, (w, x, y)).$$

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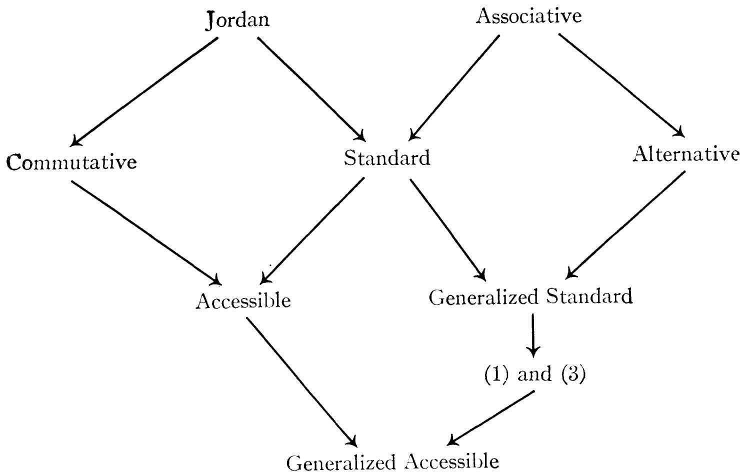
We will prove that identities (1) and (3) imply generalized accessible and also that accessible implies generalized accessible. The following schematic diagram summarizes the relationships which hold between the various types of rings under discussion. An arrow indicates implication.



The implications not proved in this paper are either trivial or can be found in [**1**; **2**; **3**]. Thus "generalized accessible" is implied by "commutative" and also by "alternative" so that it is a generalization of both.

The main theorems† of this paper are as follows.

THEOREM 1. *If R is a prime generalized accessible ring of characteristic not two or three, then R is either commutative or alternative.*

This then generalizes the results for prime rings in both [**3**] and [**2**]. Since primitive implies prime, we have the result also for primitive rings and with the usual extension of the Jacobson radical (as in the alternative case) we have the result that a semi-simple generalized accessible ring is isomorphic to a subdirect sum of primitive, commutative, and alternative rings.

THEOREM 2. *Let R be a generalized accessible ring of characteristic not two or three and without trivial ideals. Then R is isomorphic to a subdirect sum of a commutative ring and an alternative ring.*

†These results have been announced under the title: *The structure of generalized accessible rings*, Bull. Amer. Math. Soc. *75* (1969), 415–417.

1. Preliminaries. We make the following definitions.

$$H(x, y, z) \equiv (x, y, z) + (y, z, x) + (z, x, y),$$
$$T(w, x, y, z) \equiv (wx, y, z) - (w, xy, z) + (w, x, yz) - w(x, y, z) - (w, x, y)z,$$
$$B(x, y, z) \equiv (xy, z) - x(y, z) - (x, z)y - (x, y, z) - (z, x, y) + (x, z, y),$$
$$G(x, y, w, z) \equiv 3(x, y, (w, z)) + (w, (x, y, z)) + 2(x, (y, z, w)) - 2(y, (z, w, x)) - (z, (w, x, y)).$$

It can be checked by expanding the associators and commutators that in an arbitrary ring we have $T(w, x, y, z) = 0$ and $B(x, y, z) = 0$. In a ring satisfying the flexible law (1) we have

(6) $$0 = B(x, y, z) = (xy, z) - x(y, z) - (x, z)y - H(x, y, z).$$

Also in a ring satisfying (1) we may form $0 = T(w, x, y, z) + T(z, y, x, w)$ and verify that

(7) $$(w, x, (y, z)) - (w, (x, y), z) + ((w, x), y, z) = (w, (x, y, z)) - (z, (w, x, y)).$$

In the following we assume characteristic not two or three.

LEMMA 0. *Equations* (1) *and* (3) *imply*
(i) $(x, (z, y, y)) = 0$,
(ii) $(x, y, (w, z)) + (x, w, (z, y)) + (x, z, (y, w)) = 2(x, (w, z, y))$,
(iii) $G(x, y, w, z) = 0$,
(iv) *generalized accessibility.*
Also accessible implies generalized accessible.

Proof. Setting $w = y$ in (3) we see that

$$(x, y, yz) + (y, y, xz) + (z, y, xy) = (x, y, (y, z))$$

because of (1). By cancellation and subtraction, this last equation becomes $(x, y, zy) + (y, y, xz) + (z, y, xy) = 0$. In that equation, interchange x and z and subtract, so that

(8) $$(y, y, (x, z)) = 0.$$

By linearizing (8) and using (1), we obtain:

(9) $$(w, y, (x, z)) = -(y, w, (x, z)) = ((x, z), w, y) = -((x, z), y, w).$$

Setting $w = z = x$ in (3) and using (8) we see that

(10) $$3(x, y, x^2) = (x, (x, x, y)).$$

Now if we let $z = y = x$ in (7), we obtain $((w, x), x, x) = -(x, (w, x, x))$.

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Using (8) and the linearization of (1) we have $((w, x), x, x) = 0$, hence we have shown that $(x, (w, x, x)) = 0$. If we let $w = y$ in this last equation and use the linearization of (1), then $(x, (x, x, y)) = 0$. Comparing this with (10) yields $3(x, y, x^2) = 0$, and since we are assuming characteristic not three, it follows that

$$(x, y, x^2) = 0 \tag{11}$$

Schafer has already proved identities (8) and (11). We give proofs only to make the paper self-contained.

Now put $w = x$ in (3) and use (8) to obtain

$$2(x, y, xz) + (z, y, x^2) = (x, (x, z, y)).$$

But a linearization of (11) yields $(x, y, xz + zx) + (z, y, x^2) = 0$. Comparing the last two equations shows that

$$(x, y, (x, z)) = (x, (x, z, y)). \tag{12}$$

Now in (3) interchange w and z and subtract the resulting equation from (3). This results in

$$(x, y, (w, z)) = (x, (w, z, y) - (z, w, y)) + (x, w, (y, z)) - (x, z, (y, w)). \tag{13}$$

In (13) set $w = y$ and use (1). Then

$$(x, y, (y, z)) = -(x, (z, y, y)) + (x, y, (y, z))$$

which implies that

$$(x, (z, y, y)) = 0. \tag{14}$$

Now by linearization of (14) and use of (1) we see that

$$(x, (w\sigma, y\sigma, z\sigma)) = \operatorname{sgn} \sigma (x, (w, y, z)), \tag{15}$$

where σ is any permutation of the three letters w, y, and z. In the light of (15) we may rewrite (13) as

$$(x, y, (w, z)) + (x, w, (z, y)) + (x, z, (y, w)) = 2(x, (w, z, y)). \tag{16}$$

By linearizing (12) we see that

$$(w, y, (x, z)) + (x, y, (w, z)) = (w, (x, z, y)) + (x, (w, z, y)).$$

In this last equation interchange x and y to obtain

$$(w, x, (y, z)) + (y, x, (w, z)) = (w, (y, z, x)) + (y, (w, z, x)). \tag{17}$$

In (17) perform the permutation (wyz) to obtain

$$(y, x, (z, w)) + (z, x, (y, w)) = (y, (z, w, x)) + (z, (y, w, x)). \tag{18}$$

Now subtracting (17) from (18) and using (15), we have

$$(19)\quad 2(x, y, (w, z)) + (x, w, (y, z)) + (z, x, (y, w)) = -(w, (x, y, z)) + 2(y, (z, w, x)) + (z, (w, x, y)).$$

Adding (16) and (19) and using (15) and (9), it follows that

$$(20)\quad 3(x, y, (w, z)) = -(w, (x, y, z)) - 2(x, (y, z, w)) + 2(y, (z, w, x)) + (z, (w, x, y)).$$

In other words, $G(x, y, w, z) = 0$. Since (1), (14), and (20) constitute a generalized accessible ring, we have shown that (1) and (3) imply generalized accessibility.

In an arbitrary accessible ring R, it has been shown that R must be flexible [**3**, bottom of p. 335], that all commutators are in the nucleus, and that all associators satisfy $((x, y, z), w) = 0$ [**2**, (7)]. From this information it follows directly that R must be generalized accessible.

2. Main section. In the following we assume that R is a generalized accessible ring (as defined above) of characteristic not 2 or 3. Note that in a generalized accessible ring we have $(x, y, x) = 0$, $(x, (z, y, y)) = 0$ and $G(x, y, w, z) = 0$ by definition.

LEMMA 1. $((w\sigma, x\sigma, y\sigma), z) = \operatorname{sgn} \sigma((w, x, y), z)$, *where w, x, y, and z are arbitrary in R and σ is any permutation of the three letters w, x, and y.*

Proof. Immediate from the linearizations of (1) and (4).

LEMMA 2. *Let $V = \sum (R, R)$. Let $v \in V$ and $x, y \in R$. Then* $(v\sigma, x\sigma, y\sigma) = \operatorname{sgn} \sigma(v, x, y)$, *where σ is any permutation of the three elements v, x, and y.*

Proof. From $G(y, y, w, z) = 0$ we have:

$$(21)\quad 3(y, y, (w, z)) = -(w, (y, y, z)) - 2(y, (y, z, w)) + 2(y, (z, w, y)) + (z, (w, y, y)).$$

It follows from Lemma 1 that the right-hand side of equation (21) is zero. Thus we have $(y, y, (w, z)) = 0$. Linearization of this identity yields $(x, y, (w, z)) = -(y, x, (w, z))$. Because of the flexible law, $(y, y, (w, z)) = 0$ implies $((w, z), y, y) = 0$, and hence we have $((w, z), x, y) = -((w, z), y, x)$. Thus we have the following:

$$(x, y, (w, z)) = -(y, x, (w, z)) = ((w, z), x, y) = -((w, z), y, x).$$

We also know from the flexible law that $(x, (w, z), y) = -(y, (w, z), x)$.

We now need only show that $(x, (w, z), y) = -(x, y, (w, z))$. To do this, define

$$K(w, x, y, z) \equiv (w, x, (y, z)) - (w, (x, y), z) + ((w, x), y, z) - (w, (x, y, z)) + (z, (w, x, y)).$$

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Now equation (7) states precisely that $K(w, x, y, z) = 0$.

From $3K(x, z, w, y) + G(y, w, x, z) - G(x, z, w, y) = 0$ we obtain:

$$(22) \quad 3(x, (z, w), y) = 3((x, z), w, y) + 3(x, z, (w, y)) - 3(x, (z, w, y)) - 3((x, z, w), y) + 3(y, w, (x, z)) + (x, (y, w, z)) + 2(y, (w, z, x)) - 2(w, (z, x, y)) - (z, (x, y, w)) - 3(x, z, (w, y)) - (w, (x, z, y)) - 2(x, (z, y, w)) + 2(z, (y, w, x)) + (y, (w, x, z)).$$

Using Lemma 1 and collecting terms on the right-hand side of (22) yields:

$$(23) \quad -3(x, (w, z), y) = -(w, (x, y, z)) - 2(x, (y, z, w)) + 2(y, (z, w, x)) + (z, (w, x, y)).$$

But from $G(x, y, w, z) = 0$ we have

$$(24) \quad 3(x, y, (w, z)) = -(w, (x, y, z)) - 2(x, (y, z, w)) + 2(y, (z, w, x)) + (z, (w, x, y)).$$

Comparison of equations (23) and (24) yields $(x, y, (w, z)) = -(x, (w, z), y)$, as desired.

LEMMA 3. *Let $S = \{s \in R \mid (s, R) = 0\}$, let V be as in Lemma 2, and let C be the centre of R. If $s \in S$, $v \in V$, and $x, y \in R$, then*

(i) $H(x, y, s) = 0$,

(ii) $(s\sigma, x\sigma, y\sigma) \in S$,

(iii) *S is a subring of R,*

(iv) $(s\sigma, v\sigma, x\sigma) = 0$,

(v) $S \cap (R, R) \subset C$.

Proof. From (6) we have

$$0 = B(x, y, s) = (xy, s) - x(y, s) - (x, s)y - H(x, y, s) = -H(x, y, s).$$

Thus $H(x, y, s) = 0$. Since $H(x, y, z)$ is invariant under cyclic permutations of the variables x, y, and z, we have also $0 = H(y, s, x) = H(s, x, y)$. Now $0 = (z, H(x, y, s))$ and because of Lemma 1, $(z, H(x, y, s)) = 3(z, (x, y, s))$. Thus $(z, (x, y, s)) = 0$. Using Lemma 1 again this implies that

$$(z, (x\sigma, y\sigma, s\sigma)) = 0,$$

hence $(x\sigma, y\sigma, s\sigma) \in S$.

To see that S is a subring of R, let s_1 and s_2 be elements of S. Then by (6) we have

$$0 = B(s_1, s_2, z) = (s_1s_2, z) - s_1(s_2, z) - (s_1, z)s_2 - H(s_1, s_2, z) = (s_1s_2, z).$$

Hence $s_1s_2 \in S$.

By Lemma 2, $H(v, x, s) = 3(v, x, s)$ but by (i), $H(v, x, s) = 0$, thus $(v, x, s) = 0$ and using Lemma 2 again implies $(v\sigma, x\sigma, s\sigma) = 0$.

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To prove (v), let $z \in S \cap (R, R)$; then $(z, R) = 0$, since $z \in S$, $H(x, y, z) = 0$ by (i) and since $z \in (R, R)$, $H(x, y, z) = 3(x, y, z)$ by Lemma 2. Thus $(x, y, z) = 0$ and using Lemma 2 again yields $(x\sigma, y\sigma, z\sigma) = 0$. Thus $z \in C$.

LEMMA 4. *$Q = V + VR$ is an ideal of R.*

Proof. Equation (5) implies that $3(R, R, (R, R)) \subset (R, R)$, and since we have characteristic not three (the map $x \to 3x$ is one-to-one and onto) this implies that $(R, R, (R, R)) \subset (R, R)$. Linearization of (1) implies that $((R, R), R, R) \subset (R, R)$. By definition, Q is closed under addition, and $(x, y)z \in Q$ by definition. Now $z(x, y) = (x, y)z + (z, (x, y))$, and so $z(x, y) \in Q$. Now $(x, y)z \cdot w = ((x, y), z, w) + (x, y) \cdot zw$, hence $(x, y)z \cdot w \in Q$ while $w \cdot (x, y)z = (x, y)z \cdot w + (w, (x, y)z)$ implies that $w \cdot (x, y)z \in Q$.

LEMMA 5. *If $U = \{u \in S | uV = 0\}$, then U is an ideal of R and $UQ = 0$.*

Proof. First we show that $u \in U$ implies that $ux \in U$. To see that $ux \in S$, form $0 = B(u, x, z) = (ux, z) - (u, z)x - u(x, z) - H(u, x, z)$. Since $u \in S$ and u annihilates commutators, this yields $(ux, z) = H(u, x, z) = 0$ by Lemma 3 (i). Thus $ux \in S$. Now $ux \cdot (y, z) = xu \cdot (y, z) = x \cdot u(y, z) = 0$ because of Lemma 3 (iv) and the definition of U. Thus $ux \in U$, and hence U is a right ideal, since $U \subset S$; this is sufficient.

To see that $UQ = 0$, let $u \in U$. Then $uV = 0$ by definition of U and $u \cdot (x, y)r = u(x, y) \cdot r = 0$. It follows that $UQ = 0$.

LEMMA 6. *If $s \in S$, $y \in R$, and $v \in V$, then $(s, y, y)v \in S$.*

Proof. We have

$$0 = T(v, y, s, y) = (vy, s, y) - (v, ys, y) + (v, y, sy) - v(y, s, y) - (v, y, s)y.$$

Using (1) and Lemmas 2 and 3, this implies that $2(v, y, sy) = -(vy, s, y)$, and hence

$$(v, y, sy) \in S. \tag{25}$$

Now

$$0 = T(s, y, y, v) = (sy, y, v) - (s, y^2, v) + (s, y, yv) - s(y, y, v) - (s, y, y)v.$$

Using Lemmas 2, 3, and equation (25) we have

$$(s, y, y)v = [-(v, y, sy) - (s, y^2, v) + (s, y, yv)] \in S.$$

LEMMA 7. *$(S, R, R)V \subset S$.*

Proof. Let $s \in S$, $v \in V$, and $x, y \in R$. Linearizing (25) yields:

$$[(v, x, sy) + (v, y, sx)] \in S. \tag{26}$$

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From $0 = T(v, x, s, y)$ we obtain

$$(vx, s, y) - (v, xs, y) + (v, x, sy) = v(x, s, y) + (v, x, s)y.$$

Using Lemmas 2 and 3, this yields

$$v(x, s, y) = (vx, s, y) + [(v, x, sy) + (v, y, sx)].$$

It follows from (26) and Lemma 3 that $v(x, s, y) \in S$. By Lemma 3, $(x, s, y) \in S$, and so we have $(x, s, y)v = v(x, s, y) \in S$. Again by Lemma 3, $H(x, s, y) = 0$, hence $H(x, s, y)v = 0$ which implies that $(y, x, s)v + (s, y, x)v$ is an element of S, and using (1) yields

(27) $\qquad (s, x, y)v - (s, y, x)v$ is an element of S.

From Lemma 6 we have $(s, y, y)v \in S$. Linearizing yields

$$[(s, x, y)v + (s, y, x)v] \in S.$$

From this and (27) it follows that $(s, x, y)v \in S$.

LEMMA 8. *If $s \in S$ and $sV \subset S$ and if R has no trivial ideals, then $s \in U$.*

Proof. We have $s(a, b) \in S$ and $s(a, b^2) \in S$; thus $B(s, a, b) = 0$ implies $s(a, b) = (sa, b)$, and $B(s, a, b^2) = 0$ implies $s(a, b^2) = (sa, b^2)$ so that both (sa, b) and (sa, b^2) are elements of $S \cap (R, R)$ and $S \cap (R, R) \subset C$ by Lemma 3; thus we have $(sa, b) \in C$ and $(sa, b^2) \in C$. Now $B(b^2, s, a) = 0$ implies $(b^2, sa) = b(b, sa) + (b, sa)b = 2b(b, sa)$ since $(b, sa) \in C$; thus we have

(28) $$(b^2, sa) = 2b(b, sa).$$

Now $((b^2, sa), sa) = 0$ since $(b^2, sa) \in C$. This and (28) imply

(29) $$2(b(b, sa), sa) = 0,$$

while $B(b, (b, sa), sa) = 0$ implies $(b(b, sa), sa) = (b, sa)^2$. This and (29) imply $(b, sa)^2 = 0$. Now $(b, sa) \in C$ and the fact that R is without trivial ideals imply $(b, sa) = 0$. We have seen that $s(a, b) = (sa, b) = -(b, sa)$. It follows that $s(a, b) = 0$. Hence $sV = 0$ and $s \in U$.

COROLLARY. *If R has no trivial ideals, then $(S, R, R)V = 0$.*

Proof. By Lemma 3(ii) we have $(S, R, R) \subset S$. By Lemma 7 we have $(S, R, R)V \subset S$. We can now apply Lemma 8 to conclude that $(S, R, R) \subset U$. But $UV = 0$ by definition of U.

LEMMA 9. *If $(S, R, R) = 0$ and R has no trivial ideals, then R is alternative.*

Proof. Because of (1), $(S, R, R) = 0$ implies $(R, R, S) = 0$ and these together with $H(x, y, S) = 0$ imply $(R, S, R) = 0$. Thus $S = C$. By Lemma 1

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we have $[(a\sigma, b\sigma, c\sigma) - \operatorname{sgn} \sigma(a, b, c)] \in S$. It then follows from $(S, R, R) = 0$ that

$$((a\sigma, b\sigma, c\sigma), w, z) = \operatorname{sgn} \sigma((a, b, c), w, z). \tag{30}$$

If $c \in C$, it follows from $T(c, x, y, z) = 0$ that

$$(cx, y, z) = c(x, y, z). \tag{31}$$

Now

$$0 = T(x, x, y, y) = (x^2, y, y) - (x, xy, y) + (x, x, y^2) - x(x, y, y) - (x, x, y)y.$$

Form $(T(x, x, y, y), y, y) = 0$. Since $(x^2, y, y) \in S$ and $(x, x, y^2) \in S$, these terms contribute nothing and we obtain

$$-((x, xy, y), y, y) - (x(x, y, y), y, y) - ((x, x, y)y, y, y) = 0. \tag{32}$$

But $(x, x, y) \in S = C$, thus by (31) we have

$$((x, x, y)y, y, y) = (x, x, y)(y, y, y) = 0$$

and $((x, x, y)x, y, y) = (x, x, y)^2$. Thus (32) becomes

$$-((x, xy, y), y, y) = (x, y, y)^2. \tag{33}$$

Now

$$0 = B(a, b, c) - B(b, a, c) = ((a, b), c) + ((b, c), a) + ((c, a), b) - H(a, b, c) + H(b, a, c).$$

It follows from linearization of (1) that $H(b, a, c) = -H(a, b, c)$, and so the above yields $2H(a, b, c) = ((a, b), c) + ((b, c), a) + ((c, a), b)$. Thus $H(a, b, c) \in V$, and so it follows by Lemma 2 that $(H(xy, x, y), y, y) = 0$, while (30) implies that $(H(xy, x, y), y, y) = -3((x, xy, y), y, y)$. Thus $((x, xy, y), y, y) = 0$ and comparison with (33) yields $(x, y, y)^2 = 0$. Since $(x, y, y) \in S = C$ and R has no trivial ideals, it follows that $(x, y, y) = 0$. This and (1) imply R is alternative.

THEOREM 1. *If R is a prime generalized accessible ring of characteristic not two or three, then R is either commutative or alternative.*

Proof. If R is not commutative, the ideal Q of Lemma 4 is not zero. Since $UQ = 0$ by Lemma 5, we must have $U = 0$. Then Lemmas 3(ii) and 7 show that (S, R, R) satisfy the hypothesis of Lemma 8 so that $(S, R, R) \subset U = 0$. But then Lemma 9 implies that R must be alternative.

COROLLARY 1. *If R is a primitive generalized accessible ring of characteristic not two or three, then R is either commutative or alternative.*

Proof. Since R is primitive, there is a maximal regular right ideal P contained in R which contains no two-sided ideal other than 0. Let A and B be ideals

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such that $AB = 0$ and assume that $A \neq 0$; then $A + P = R$ and $RB = AB + PB = PB$. Now P is a right ideal, and so PB is contained in P. Thus we have RB contained in P. Since P is regular, there exists an element $e \in R$ such that $ex - x \in P$ for all $x \in R$. If $x \in B$, we have $ex \in RB \subset P$, and since $ex - x \in P$, this implies that $x \in P$. Therefore $B \subset P$ and we must have $B = 0$. Thus R is prime and we can apply the preceding theorem.

To pursue the study of semi-simple rings we define $\mathfrak{A}$ to be the intersection of the regular maximal right ideals of a generalized accessible ring T, and $\mathfrak{D}$ to be the intersection of the primitive ideals of T. If there are no regular maximal right ideals, then define $\mathfrak{A} = T$. In this case there are also no primitive ideals and we define $\mathfrak{D} = T$ also. Now if $\mathfrak{A} \neq T$, then let A be a regular maximal right ideal. The sum of all the ideals of T contained in A forms a maximal ideal of T in A, call it I. Then T/I is primitive since A/I is a regular maximal right ideal containing no ideal of T/I other than zero. Thus I is a primitive ideal contained in A. Since A is an arbitrary maximal regular right ideal, we have proved that $\mathfrak{D} \subseteq \mathfrak{A}$. On the other hand, if Q is a primitive ideal of T, then T/Q is a primitive ring. If T/Q is alternative, hence either a Cayley-Dickson algebra or associative, it is known that the intersection of the regular maximal right ideals of T/Q is 0. If T/Q is commutative, then any maximal right ideal that contains no two-sided ideal other than zero must be zero. Thus again the intersection of regular maximal right ideals is zero. Thus Q equals the intersection of those regular maximal right ideals which contain it and so $\mathfrak{A} \subseteq \mathfrak{D}$. But then $\mathfrak{A} = \mathfrak{D}$ if $\mathfrak{A} \neq T$. Now it is obvious that if $J(T) = \mathfrak{A} = \mathfrak{D}$, then $J(T/J(T)) = 0$, using $J(T) = \mathfrak{A}$. Also, using $J(T) = \mathfrak{D}$, there is a natural homomorphism from T into $\sum \oplus(T/Q_\alpha)$, where $\{Q_\alpha\}$ equals the set of primitive ideals of T and kernel of this homomorphism is $\bigcap_\alpha Q_\alpha$. If T is semisimple, i.e., if $J(T) = 0$, then this homomorphism is an isomorphism into the direct sum of primitive generalized accessible rings $\sum \oplus(T/Q_\alpha)$. By the corollary to Theorem 1, each of these summands is either commutative or alternative. Thus if we define $J(T) = \mathfrak{A}$ and T is semisimple if $J(T) = 0$, we have proved the following result.

COROLLARY 2. *If T is a semisimple generalized accessible ring, then T is isomorphic to a subdirect sum of primitive, commutative, and alternative rings.*

We now proceed to the more general case of generalized accessible rings without trivial ideals. Since we used primeness only in the proof of Theorem 1 and not in any of the preceding lemmas, these results are still applicable.

Definition. An element $a \in R$ which is of the form $a = (x, x, y)$ or $a = (y, x, x)$ for $x, y \in R$ will be called an *alternator.*

LEMMA 10. *If $s \in S$ as defined in Lemma 3, then* $(xs, y) = (sx, y) = s(x, y)$.

Proof. $0 = B(s, x, y) = (sx, y) - s(x, y) - (s, y)x - H(s, x, y)$ from equation (6). We have $H(s, x, y) = H(x, y, s) = 0$, because of the definition

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of H and Lemma 3(i). Moreover, $(s, y) = 0$ by definition of S. Thus we have proved that $(sx, y) = s(x, y)$. That $(sx, y) = (xs, y)$ is immediate from the definition of S.

Definition. $t = (a, b, b)$ for fixed a and b in R.

Remark. Note that $t \in S$ follows from equation (4). Thus by Lemma 10, t moves freely in and out of commutators. Note also that because of Lemma 3(iv), the triplet t, v, and r, where $v \in V$ as defined in Lemma 2 and $r \in R$, associate freely, so that we may write the product tvr without ambiguity. We also will be frequently using the fact that $t \in S$ implies $(t, R, R) \in S$ by Lemma 3(ii) and that if R has no trivial ideals, then $(S, R, R)V = 0$ by the corollary to Lemma 8.

LEMMA 11. *If R has no trivial ideals and if $A = tV + \sum tVR$, then A is a two-sided ideal of R.*

Proof. We have $t(x, y)z \in A$ by definition while we may write

$$zt(x, y) = (z, t(x, y)) + t(x, y)z = t(z, (x, y)) + t(x, y)z \in A.$$

Also $w \cdot t(x, y)z = t(x, y)z \cdot w + (w, t(x, y)z)$. Since

$$(w, t(x, y)z) = t(w, (x, y)z) \in A,$$

it now suffices to show that $t(x, y)z \cdot w \in A$. By the definition of the associator we have $t(x, y)z \cdot w = (t(x, y), z, w) + t(x, y) \cdot zw$. Since

$$t(x, y) \cdot zw \in tVR \subset A,$$

we need only show that the associator $(t(x, y), z, w) \in A$. To establish this, note that $3(t(x, y), z, w) = 3((tx, y), z, w)$ and by Lemma 2,

$$3((tx, y), z, w) = 3(z, w, (tx, y)).$$

Thus we have

$$3(t(x, y), z, w) = 3(z, w, (tx, y)). \tag{34}$$

To establish that $3(z, w, (tx, y)) \in A$, form $G(z, w, tx, y) = 0$. This yields

$$3(z, w, (tx, y)) = -(tx, (z, w, y)) - 2(z, (w, y, tx)) + 2(w, (y, tx, z)) + (y, (tx, z, w)). \tag{35}$$

Now $(tx, (z, w, y)) = t(x, (z, w, y)) \in A$. Form $(y, T(t, x, z, w))$. This yields

$$(y, (tx, z, w) - (t, xz, w) + (t, x, zw) - t(x, z, w) - (t, x, z)w) = 0. \tag{36}$$

Now $(y, (t, xz, w)) = 0 = (y, (t, x, zw))$ since $(S, R, R) \subset S$ and $t \in S$. Thus (36) becomes

$$(y, (tx, z, w) - t(x, z, w) - (t, x, z)w) = 0. \tag{37}$$

But now $(y, -t(x, z, w)) = -t(y, (x, z, w)) \in A$ and since $(t, x, z) \in S$, we have $(y, -(t, x, z)w) = -(t, x, z)(y, w) = 0$ using Lemma 10 and the fact that $(S, R, R)V = 0$, by the corollary to Lemma 8. It now follows from (37) that $(y, (tx, z, w)) \in A$. This is the fourth term on the right-hand side of equation (35). Because of Lemma 1, the second and third terms can be brought into this form also, and hence are in A. Since we have already shown that the first term on the right-hand side of (35) is in A, we conclude that $3(z, w, (tx, y)) \in A$ and hence by equation (34), $3(t(x, y), z, w) \in A$. Since we are assuming characteristic not 3, this completes the proof.

LEMMA 12. *If R has no trivial ideals, $v = (x, y)$ and t is defined as above, then $t^2v = 0$ and in fact $t^2Q = 0$, where Q is the commutator ideal and $tA = 0$, where A is as above.*

Proof. We form $(T(a, a, b, b), b, b)v = 0$. This yields

$$(38)\quad ((a^2, b, b), b, b)v - ((a, ab, b), b, b)v + ((a, a, b^2), b, b)v - (a(a, b, b), b, b)v - ((a, a, b)b, b, b)v = 0.$$

Since alternators are in S by (4) and since $(S, R, R)V = 0$, the first and third terms of (38) vanish. Now $H(x, y, z) \in V$ follows from

$$B(x, y, z) - B(y, x, z) = 0$$

and equation (6) as in the proof of Lemma 9, thus by Lemma 2,

$$(H(a, ab, b)b, b) = 0.$$

Since alternators are in S and $(S, R, R)V = 0$, we have $(H(a, ab, b), b, b)v = 3((a, ab, b), b, b)v$. It follows that $((a, ab, b), b, b) = 0$ so that the second term of (38) vanishes. To see that the fifth term of (38) vanishes, form $T((a, a, b), b, b, b)v = 0$. This yields

$$(39)\quad (tb, b, b)v - (t, b^2, b)v + (t, b, b^2)v - t(b, b, b)v - (t, b, b)b \cdot v = 0.$$

The second and third terms are zero since $(S, R, R)V = 0$. The fourth term is zero by (1). Now $(tb, b) \in S$ so that

$$-(t, b, b)b \cdot v = -b(t, b, b) \cdot v = -b \cdot (t, b, b)v = 0$$

using the definition of S, Lemma 3(iv) and $(S, R, R)V = 0$ by the corollary to Lemma 8. Putting these results in equation (39) we conclude that $(tb, b, b)v = 0$. Now equation (38) becomes

$$(40)\quad (a(a, b, b), b, b)v = 0.$$

Since $t = (a, b, b)$ and $t \in S$, (40) may be rewritten as

$$(41)\quad (ta, b, b)v = 0.$$

But now form $T(t, a, b, b)v = 0$ to obtain

$$(42)\quad (ta, b, b)v - (t, ab, b)v + (t, a, b^2)v - t^2v - (t, a, b)b \cdot v = 0.$$

The second and third terms vanish since $(S, R, R)V = 0$ and as above $(t, a, b)b \cdot v = b(t, a, b) \cdot v = b \cdot (t, a, b)v = 0$, using Lemma 3(ii) and (iv). Thus (42) becomes

$$(43) \qquad t^2 v = (ta, b, b)v.$$

Combining (41) and (43) yields $t^2 v = 0$. Now referring back to Lemma 5, $t^2 v = 0$ for an arbitrary commutator means $t^2 V = 0$. Since $t^2 \in S$ by Lemma 3(iii), we have $t^2 \in U$, but it now follows from Lemma 5 that $t^2 Q = 0$. Now $tA = t(tV + \sum tVR)$. We have $t \cdot tV = t^2 V = 0$ while

$$t \cdot tvr = (t, t, vr) = -(vr, t, t).$$

Form $T(v, r, t, t) = 0$. We obtain

$$(vr, t, t) - (v, rt, t) + (v, r, t^2) - v(r, t, t) - (v, r, t)t = 0.$$

Now $(v, rt, t) = 0$, $(v, r, t^2) = 0$, and $(v, r, t)t = 0$ follow from Lemma 3(iv) and $t \in S$, while $v(r, t, t) = -(t, t, r)v = 0$ using (1), the fact that $t \in S$, and $(S, R, R)V = 0$. It follows that $(vr, t, t) = 0$, and hence that $t \cdot tvr = 0$. Thus we have proved that $tA = 0$.

LEMMA 13. *If R has no trivial ideals, then $A = 0$.*

Proof. We first prove that $A^2 = 0$. Let v be a commutator; then

$$tv \cdot A = vt \cdot A = v \cdot tA = 0$$

by Lemma 12, thus we have

$$(44) \qquad tv \cdot A = 0.$$

Now using the definition of the associator, $tvz \cdot A = (tv, z, A) + tv \cdot zA$, but since A is an ideal, $tv \cdot zA \subset tv \cdot A = 0$. Hence $tvz \cdot A = (tv, z, A)$. Using equation (1) we have $(tv, z, A) = -(A, z, tv)$. Combining this with the preceding equation yields

$$(45) \qquad tvz \cdot A = -(A, z, tv).$$

Let $v = (x, y)$, then by Lemma 10, $tv = t(x, y) = (tx, y)$, so that (45) becomes

$$(46) \qquad tvz \cdot A = -(A, z, (tx, y)).$$

Now form $G(A, z, tx, y) = 0$. This yields

$$(47) \quad -3(A, z, (tx, y)) = (tx, (A, z, y)) + 2(A, (z, y, tx)) - 2(z, (y, tx, A)) - (y, (tx, A, z)).$$

Now $(tx, (A, z, y)) = t(x, (A, z, y)) \in tA = 0$ by Lemma 12. Thus we have

$$(48) \qquad (tx, (A, z, y)) = 0.$$

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Moreover, if we form $(y, T(t, x, A, z)) = 0$ we obtain

$$(y, (tx, A, z)) = (y, t(x, A, z)) + (y, (t, x, A) z). \tag{49}$$

Now $t(x, A, z) \in tA = 0$ while since $(t, x, A) \in S$ we use Lemma 10 to get $(y, (t, x, A)z) = (t, x, A)(y, z) \in (S, R, R)V = 0$. It now follows from (49) that

$$(y, (tx, A, z)) = 0. \tag{50}$$

Since alternators are in S, we have $(z, (y, tx, A)) = (z, (tx, A, y))$, and since y and z are both arbitrary, we conclude from this and (50) that

$$(z, (y, tx, A)) = 0. \tag{51}$$

We now consider $(A, (z, y, tx))$. Since alternators are in S, we have

$$(A, (z, y, tx)) = (A, (tx, z, y)). \tag{52}$$

We form $(A, T(t, x, z, y)) = 0$ to obtain

$$(A, (tx, z, y)) = (A, t(x, z, y) + (t, x, z)y). \tag{53}$$

But now $(A, t(x, z, y)) = t(A, (x, z, y)) \in tA = 0$ and

$$(A, (t, x, z)y) = (t, x, z)(A, y) \in (S, R, R)V = 0.$$

Substituting this information in (53) yields

$$(A, (tx, z, y)) = 0. \tag{54}$$

Combining (54) and (52) yields

$$(A, (z, y, tx)) = 0. \tag{55}$$

We now substitute (48), (50), (51), and (55) into equation (47) to get that $-3(A, z, (tx, y)) = 0$, and hence $-(A, z, (tx, y)) = 0$. Comparison with (46) now yields $tvz \cdot A = 0$. We have proved that $A^2 = 0$. Since A is an ideal and we are assuming no trivial ideals, this implies that $A = 0$.

THEOREM 2. *Let R be a generalized accessible ring of characteristic not two or three and without trivial ideals. Then R is isomorphic to a subdirect sum of a commutative ring and an alternative ring.*

Proof. From Lemma 13 we have $A = 0$, thus $tV = 0$; by Lemma 5 we have $t \in U$. Now this is true for every alternator t, thus all alternators are in U. Since U is an ideal, this means that the ideal generated by all alternators, call it B, is contained in U. However, we have proved in Lemma 5 that $UQ = 0$, where Q is the commutator ideal. Therefore $BQ \subset UQ = 0$. If we let $D = B \cap Q$ we see that $D^2 = 0$, hence $D = 0$. Now the natural homomorphism from R into $R/B \oplus R/Q$ has kernel $B \cap Q = 0$, and so is an isomorphism. Moreover, R/B is alternative while R/Q is commutative.

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References

1. A. A. Albert, *Power associative rings*, Trans. Amer. Math. Soc. *64* (1948), 552–593.
2. Erwin Kleinfeld, *Standard and accessible rings*, Can. J. Math. *8* (1956), 335–340.
3. R. D. Schafer, *On generalized standard algebras*, Proc. Nat. Acad. Sci. U.S.A. *60* (1968) 73–74.

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