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Hartogs' Theorem on Separate Holomorphicity for Projective Spaces

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Abstract. If a mapping of several complex variables into projective space is holomorphic in each pair of variables, then it is globally holomorphic.

1 Introduction

Hartogs' theorem [3] on separately holomorphic functions is not valid for holomorphic mappings into a general complex manifold. Some basic counterexamples involve the complex projective space $\mathbb{P}^1$. Let $g\colon \mathbb{C}^2 \to \mathbb{P}^1$ be the mapping given by $g(x, y) = [xy, x^2+y^2]$, with $g(0, 0) = [0, 1]$, in the homogeneous coordinates of $\mathbb{P}^1$. We have that g is a mapping holomorphic in each entry separately, but g is not even continuous at the origin. There are several works classifying those complex spaces for which the Hartogs Theorem on separately holomorphic mappings holds. A short list includes [4–6].

No complex projective space $\mathbb{P}^m$ satisfies the hypotheses presented in [4–6]. The main objective of this work is to prove that a weak version of Hartogs' theorem holds for mappings into complex projective spaces $\mathbb{P}^m$. Georges Dloussky has presented a paper [2] on separate analyticity with results similar to ours. We wish to present an alternative proof.

By a coordinate k-plane Π^k we understand any affine linear subspace of $\mathbb{C}^n$ with $n \geq k$ obtained by fixing $n-k$ of the coordinates. Hartogs' original theorem may be stated as follows: given an open subset $\Omega \subset \mathbb{C}^n$, and a function $f\colon \Omega \to \mathbb{C}$ whose restriction to each intersection $\Omega \cap \Pi^1$ is holomorphic, for every coordinate 1-plane Π^1, we have that f is holomorphic on Ω. We may now present the main result of this paper.

Main Theorem *Let Ω be an open domain in $\mathbb{C}^n$ with $n \geq 3$, and let $\mathbb{P}^m$ be a complex projective space. Given a mapping $f\colon \Omega \to \mathbb{P}^m$ whose restriction to each intersection $\Omega \cap \Pi^2$ is holomorphic for every coordinate 2-plane Π^2, we have that f is holomorphic on Ω.*

This theorem will be proved in the next section. The Main Theorem does not hold if we use continuity (or smoothness) instead of holomorphicity. For example, consider the mapping $h\colon \mathbb{C}^3 \to \mathbb{P}^1$ defined by $h(x, y, z) = [xyz, |x|^3+|y|^3+|z|^3]$, with

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$h(0, 0, 0) = [0, 1]$. The restriction of h to every coordinate 2-plane Π^2 is smooth, but h is not even continuous at the origin.

2 Proof of the Main Theorem

We begin by recalling the following theorem of Alexander, Taylor and Ullman.

Theorem 1 **([1, p. 340])** *Let Ω be an open subset of $\mathbb{C}^{n+1}$. Let $r \geq 1$ be a fixed integer, and $A \subset \Omega$ be a subset such that the intersection $A \cap \Pi^n$ with every coordinate n-plane Π^n is either empty or a complex subvariety of pure dimension r in Ω. Then the set A is closed in Ω.*

Remark. The theorem in [1, p. 340] does not say explicitly that the intersection $A \cap \Pi^n$ may be empty. However, in their proof, they use the hypothesis that each set W_j is an r-dimensional subvariety of Ω_1, recalling their own notation. Looking carefully at their proof, we have that no set W_j can be empty, for the point c_j is indeed contained in W_j. Hence, the proof in [1, p. 340] works perfectly under the hypothesis that each intersection $A \cap \Pi^n$ is either empty or an r-dimensional subvariety.

Lemma 2 *Let Ω be an open domain in $\mathbb{C}^{n+1}$ with $n \geq 2$, and let $\mathbb{P}^m$ be a complex projective space. Given a function $f \colon \Omega \to \mathbb{P}^m$ whose restriction to each intersection $\Omega \cap \Pi^n$ is holomorphic, for every coordinate n-plane Π^n, we have that f is holomorphic on Ω.*

Proof We shall prove that f is holomorphic on a neighbourhood of any fixed point $z_0 \in \Omega$. Thus, we shall suppose from now on that Ω is a polydisc, and that $f(z_0) = [1, 0, 0, \dots]$ in the homogeneous coordinates of $\mathbb{P}^m$. Consider the open set $U_0 \subset \mathbb{P}^m$ composed of the points $[1, \xi]$, for $\xi \in \mathbb{C}^m$. We only need to show that $f^{-1}(U_0)$ is an open neighbourhood of z_0 in Ω. Then a direct application of Hartogs' original theorem yields that f is holomorphic on $f^{-1}(U_0)$, because U_0 is obviously biholomorphic to $\mathbb{C}^m$; so we have that f is holomorphic on a neighbourhood of z_0.

Notice that the hyperplane at infinity $\mathbb{P}^m \backslash U_0$ is the set of all points $[0, \xi]$ with $\xi \neq 0$. Define the set E equal to $f^{-1}(\mathbb{P}^m \backslash U_0)$ in Ω, and consider any coordinate n-plane Π^n. The restriction of f to the complex manifold $\Omega \cap \Pi^n$ is a holomorphic mapping with image in $\mathbb{P}^m$, because of the given hypotheses. Hence, the set $E \cap \Pi^n$ is a subvariety of $\Omega \cap \Pi^m$ as the preimage of the complex submanifold $\mathbb{P}^m \backslash U_0$ by a holomorphic mapping.

It is easy to deduce that $E \cap \Pi^n$ has only three possibilities: it can either be empty, equal to $\Omega \cap \Pi^n$, or a subvariety of $\Omega \cap \Pi^n$ with pure dimension $n - 1$. Define J to be the union of all intersections $E \cap \Pi^n$ which are equal to $\Omega \cap \Pi^n$. We assert that J is a closed subset of Ω. First, we consider only the coordinate n-planes whose first coordinate is constant, that is, planes $\Pi^{n,0}$ of the form $\{x\} \times \mathbb{C}^n$. Recall that Ω is a polydisc, so it can be written as the product $D_0 \times \Delta_0$ with D_0 an open disc in $\mathbb{C}$. Define $\rho_0 \colon \Omega \to D_0$ to be the projection on the first coordinate, and J_0 to be the union of all sets $E \cap \Pi^{n,0}$ which are equal to $\Omega \cap \Pi^{n,0}$. It is easy to deduce that

$$J_0 = \bigcap_{y \in \Delta_0} \rho_0\big(E \cap (D_0 \times \{y\})\big) \times \Delta_0.$$

Let H^n be any coordinate n-plane which contains the line $\mathbb{C}\times\{y\}$. We know that $E\cap H^n$ is a closed subset of $\Omega\cap H^n$. Whence we also deduce that every $E\cap H^n$, each $E\cap(D_0\times\{y\})$, and J_0 are all closed subsets of Ω. We may analyse, in the same way, the coordinate n-planes $\Pi^{n,k}$ of the form $\mathbb{C}^k\times\{x\}\times\mathbb{C}^{n-k}$, for $0\le k\le n$, and define J_k to be union of all sets $E\cap\Pi^{n,k}$ which are equal to $\Omega\cap\Pi^{n,k}$. At the end, we deduce that each J_k is closed in Ω. Moreover, it is easy to deduce that $J=\bigcup_k J_k$ is also closed in Ω.

Finally, consider $E^* := E\setminus J$ and the open set $\Omega^* := \Omega\setminus J$ in $\mathbb{C}^{n+1}$. Every intersection $E^*\cap\Pi^n$ with a coordinate n-plane Π^n is either empty or an analytic set of pure dimension $n-1$ in $\Omega^*\cap\Pi^n$ and Ω^*. Then E^* is a closed subset of Ω^*, after applying Theorem 1. The sets $f^{-1}(U_0)$, $\Omega\setminus E$ and $\Omega^*\setminus E^*$ are all equal, so f is indeed holomorphic on the open neighbourhood $f^{-1}(U_0)$ of z_0, as we wanted to prove. ■

The proof of the Main Theorem is a direct application of Lemma 2.

Proof of the Main Theorem Considering Lemma 2, we may deduce that the restriction of f to each intersection $\Omega\cap\Pi^3$ is holomorphic for every coordinate 3-plane Π^3. Proceeding by induction, we only need to apply Lemma 2 a finite number of times in order to deduce that the restriction of f to each intersection $\Omega\cap\Pi^k$ is holomorphic for every coordinate k-plane Π^k with $k\ge 3$; and so f is holomorphic on Ω. ■

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