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CATEGORIFYING RATIONALIZATION

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Abstract

We construct, for any set of primes S, a triangulated category (in fact a stable ∞-category) whose Grothendieck group is $S^{-1}\mathbf{Z}$. More generally, for any exact ∞-category E, we construct an exact ∞-category $S^{-1}E$ of equivariant sheaves on the Cantor space with respect to an action of a dense subgroup of the circle. We show that this ∞-category is precisely the result of categorifying division by the primes in S. In particular, $K_n(S^{-1}E) \cong S^{-1}K_n(E)$.

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It is a peculiar fact that rationalized algebraic K-groups have largely remained out of reach of algebraic techniques. For example, the rationalized K-groups of a number field F were computed by Borel [5]: for $n \geqslant 2$,

$$\dim K_n(F)\otimes \mathbf{Q} = \begin{cases} 0 & \text{if } n \equiv 0 \mod 2; \\ r_1 + r_2 & \text{if } n \equiv 1 \mod 4; \\ r_2 & \text{if } n \equiv 3 \mod 4, \end{cases}$$

where r_1 is the number of real places and r_2 is the number of complex places of F. But Borel's proof depends upon a delicate analysis of invariant differential forms on the Borel–Serre compactification of a symmetric space. As far as we know, no algebraic approach to this computation has appeared in the literature.

For function fields, the situation is at least as dire. For example, we have the following.

CONJECTURE (Parshin). *If X is a smooth projective variety over a finite field, then $K_n(X) \otimes \mathbf{Q} = 0$ for any $n \geqslant 1$.*

But only when the dimension of X is 0 or 1 is this assertion known.

The task of this paper is to *categorify rationalization*, in order to get a more explicit grasp on rational K-theory classes. That is, we introduce explicit categories of *divisible objects* whose K-theory gives the rational K-theory directly.

More precisely, if S is a set of prime numbers, then for any exact ∞-category E (in particular, for any exact ordinary category or any stable ∞-category [2]), we construct here an exact ∞-category $S^{-1}E$ such that $K(S^{-1}E) \simeq S^{-1}K(E)$ as spectra, and, consequently,

$$K_*(S^{-1}E) \cong S^{-1}K_*(E)$$

as graded abelian groups.

When E is an idempotent-complete stable ∞-category, we can offer an explicit—though perhaps unwieldy—characterization of $S^{-1}E$: it is an ∞-category of what we call *S-divisible objects*. These are sequences $\{X_i\}$ of objects X_i of $\operatorname{Ind} E$, indexed over the various products i of the primes in S, along with suitably compatible identifications, when m divides n, between the object X_m and the n/m-fold direct sum $X_n \oplus X_n \oplus \cdots \oplus X_n$, all subject to a finiteness condition.

Our main theorem goes a step still further, and identifies $S^{-1}E$ as an ∞-category of sheaves of objects of $\operatorname{Ind} E$ on the Cantor space Ω that are equivariant with respect to a free action (Construction 4.2) of the *S-adic circle*

$$\mathbf{T}_S := S^{-1}\mathbf{Z}/\mathbf{Z}$$

on Ω. When E is the ∞-category of coherent complexes on a reasonable scheme X, we may think of Ω as an affine scheme with its S-adic circle action, and we prove:

THEOREM. *One has an equivalence of ∞-categories*

$$S^{-1}\mathbf{IndCoh}(X) \simeq \mathbf{IndCoh}^{\mathbf{T}_S}(X \times \Omega)$$

between the S-divisible ind-coherent complexes on X and the ∞-category of $\mathbf{T}_S$-equivariant ind-coherent complexes on $X \times \Omega$.

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Recall that the G-theory of a scheme X is by definition the K-theory of the stable ∞-category of coherent complexes on X. We therefore deduce that

$$S^{-1}G_n(X) \cong G_n^{\mathbf{T}_S}(X \times \Omega), \quad \text{and in particular} \quad G_n(X) \otimes \mathbf{Q} \cong G_n^{\mathbf{Q}/\mathbf{Z}}(X \times \Omega);$$

that is, the rationalized G-theory of X is the $\mathbf{Q}/\mathbf{Z}$-equivariant G-theory of $X \times \Omega$. Recall also that $G_n(X) \cong K_n(X)$ when the scheme X is regular.

This paper is motivated by a question of Khovanov [7, 2.3 and 2.4], who sought such a "categorification of division." In particular, he asked for a monoidal triangulated category whose Grothendieck group is $\mathbf{Q}$, and more generally, one whose Grothendieck group is $m^{-1}\mathbf{Z}$ for an integer m. In fact, for any field k, the stable ∞-category $\mathbf{QCoh}^{m^{-1}\mathbf{Z}/\mathbf{Z}}(\operatorname{Spec} k \times \Omega)$ of $m^{-1}\mathbf{Z}/\mathbf{Z}$-equivariant sheaves of complexes of k-vector spaces on Ω can be viewed as the localization of the derived category of k away from m. The compact objects therein have not only the desired Grothendieck group $m^{-1}\mathbf{Z}$, but one even has

$$m^{-1}K_n(k) \cong G_n^{m^{-1}\mathbf{Z}/\mathbf{Z}}(\operatorname{Spec} k \times \Omega).$$

The slogan is thus: *Vector spaces with rational dimension are circle-equivariant sheaves of complexes on the Cantor space.* However, our construction does not fully answer Khovanov's question, because the monoidal structure on the ∞-category $\mathbf{QCoh}^{m^{-1}\mathbf{Z}/\mathbf{Z}}(\operatorname{Spec} k \times \Omega)$ does not restrict to the subcategory of compact objects.

Finally, though our motivation was to contemplate rational algebraic K-theory, we must note that nowhere have we really used anything special about the functor K, save only that it preserves finite products and filtered colimits. Any functor with this property (for example, topological Hochschild homology) can replace K in the assertions above. This reflects the fact that our procedure really inverts the primes in S at the *categorical* level.

1. Localizations

RECOLLECTION 1.1. An abelian group E is *S-local* if and only if, for any product k of primes in S, the multiplication by k map $k: E \longrightarrow E$ is an isomorphism.

More generally, we have the following.

DEFINITION 1.2. Suppose C an ∞-category with direct sums. For any object E of C, and for any natural number k, write kE for the k-fold direct sum $E \oplus E \oplus \cdots \oplus E$. The composite

$$E \longrightarrow kE \longrightarrow E$$

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of the codiagonal followed by the diagonal deserves the name *multiplication by* k. We will say that E is S-*local* if and only if, for any product k of primes in S, the multiplication by k map $k\colon E \longrightarrow E$ is an equivalence.

This recovers, for example, the notion of S-locality for spectra.

NOTATION 1.3. Let Φ_S denote the ordinary category in which an object is a (positive) natural number that is a product of elements of S, and a morphism $m \longrightarrow n$ is a natural number k such that $n = mk$.

We will show in Section 2 that every object E of an ∞-category C with direct sums determines a functor

$$E[S]\colon \Phi_S \longrightarrow C$$

that carries every object to E and every morphism $k\colon m \longrightarrow n$ to the morphism $k\colon E \longrightarrow E$, as well as a dual diagram

$$E[S]^\vee\colon \Phi_S^{\mathrm{op}} \longrightarrow C$$

that carries every object to E and every morphism $k\colon m \longrightarrow n$ to the morphism $k\colon E \longrightarrow E$.

The proof of the following is easy.

PROPOSITION 1.4. *Suppose C an ∞-category that admits direct sums. Then the following are equivalent for an object E of C.*

- *The object E is S-local.*
- *The functor $E[S]$ is essentially constant.*

NOTATION 1.5. If C is an ∞-category that admits direct sums and filtered colimits, then we write $S^{-1}\colon C \longrightarrow C$ for the functor $E \rightsquigarrow \operatorname{colim} E[S]$.

If E is S-local, it follows from Proposition 1.4 (and the fact that the category Φ_S is filtered and hence weakly contractible) that the natural map $E \longrightarrow S^{-1}E$ is an equivalence.

WARNING 1.6. If C is compactly generated, it is tempting to believe that the functor $S^{-1}\colon C \longrightarrow C$ is a localization onto the full subcategory spanned by the S-local objects. This is true when C is **Ab** or **Sp**. However, it is not true in general: see Warning 3.11. In order for $S^{-1}E$ to be S-local, it is sufficient that for any $p \in S$, there exist $N \geqslant 2$ such that the cyclic permutation of $p^N\colon E \longrightarrow E$ is homotopic to the identity.

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2. The effective Burnside ∞-category and the functors $E[S]$ and $E[S]^\vee$

We give a precise construction of the functors $E[S]$ and $E[S]^\vee$ for any object E of any ∞-category C that admits direct sums.

To this end, let $A^{\mathrm{eff}}(\mathbf{Fin})$ denote the effective Burnside ∞-category of finite sets [3]. (This is in fact a 2-category.) We have shown that this is the Lawvere theory of E_∞ objects. That is, for any ∞-category D with all finite products, there is an equivalence

$$\mathbf{CAlg}(D^\times) \simeq \mathrm{Fun}^\times(A^{\mathrm{eff}}(\mathbf{Fin}), D),$$

where $\mathrm{Fun}^\times$ denotes the ∞-category of product-preserving functors. Equivalently, $A^{\mathrm{eff}}(\mathbf{Fin})$ can be identified with the ∞-category of free, finitely generated E_∞ spaces.

Now since C has direct sums, every object is an E_∞-algebra in a unique way. That is, the forgetful functor

$$\mathbf{CAlg}(C^\times) \longrightarrow C$$

is an equivalence. Consequently, the functor

$$\mathrm{Fun}^\times(A^{\mathrm{eff}}(\mathbf{Fin}), C) \longrightarrow C$$

given by evaluation at the one-point set $\langle 1\rangle := \{0\}$ is an equivalence. Selecting once and for all a homotopy inverse F to this equivalence, we obtain for each $E \in C$ a product-preserving functor $F(E)\colon A^{\mathrm{eff}}(\mathbf{Fin}) \longrightarrow C$. Now in order to construct $E[S]$ and $E[S]^\vee$ for any object E of C, we need only to define a functor

$$M_S\colon \Phi_S \longrightarrow A^{\mathrm{eff}}(\mathbf{Fin})$$

that carries each natural number in Φ_S to the singleton, and every map $m \longrightarrow n$ given by $n = mk$ to the span

$$\langle 1\rangle \longleftarrow \langle k\rangle \longrightarrow \langle 1\rangle,$$

where

$$\langle k\rangle := \{0, 1, \ldots, k-1\}.$$

We then obtain $E[S]$ as the composite $F(E) \circ M_S$, and we obtain $E[S]^\vee$ as the composite $F(E) \circ D \circ M_S^{\mathrm{op}}$, where $D\colon A^{\mathrm{eff}}(\mathbf{Fin})^{\mathrm{op}} \xrightarrow{\sim} A^{\mathrm{eff}}(\mathbf{Fin})$ is the duality functor.

In fact it will be useful to define a functor

$$\widetilde{M}_S\colon O(\Phi_S) \longrightarrow A^{\mathrm{eff}}(\mathbf{Fin}),$$

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where $O(\Phi_S) := \mathrm{Fun}(\Delta^1, \Phi_S)$ is the arrow category of Φ_S, such that the precomposition of $\tilde{M}_S$ with the inclusion $\Phi_S \subseteq O(\Phi_S)$ sending every object to the identity on it is the required functor M_S.

To define $\widetilde{M}_S$ carefully, if $n = mk$, then we define two maps

$$p_{m|n} : \langle n \rangle \longrightarrow \langle m \rangle \quad \text{and} \quad j_{m|n} : \langle n \rangle \longrightarrow \langle m \rangle$$

by the formulas

$$p_{m|n}(i) := \left\lfloor \frac{i}{k} \right\rfloor \quad \text{and} \quad j_{m|n}(i) := i \mod m.$$

Now for any p-simplex

$$(m_0|n_0)|(m_1|n_1)|\cdots|(m_p|n_p)$$

of $O(\Phi_S)$ (by which we mean that $m_s|m_{s+1}$ and $n_t|n_{t+1}$) in which $n_t = k_{s,t}m_s$, the p-simplex

$$\widetilde{M}_S((m_0|n_0)|(m_1|n_1)|\cdots|(m_p|n_p)) \in A^{\mathrm{eff}}(\mathbf{Fin})_p$$

will be the diagram

$$\begin{tikzcd}[column sep=tiny]
 & & & & & \langle k_{0,p}\rangle \arrow[dl] \arrow[dr] & & & & & \\
 & & & & \langle k_{0,p-1}\rangle \arrow[dl] \arrow[dr] & \Diamond & \langle k_{1,p}\rangle \arrow[dl] \arrow[dr] & & & & \\
 & & & \iddots \arrow[dl] \arrow[dr] & \Diamond & \ddots\iddots \arrow[dl] \arrow[dr] & \Diamond & \ddots \arrow[dl] \arrow[dr] & & & \\
 & & \langle k_{0,2}\rangle \arrow[dl] \arrow[dr] & \Diamond & \langle k_{1,3}\rangle \arrow[dl] \arrow[dr] & \Diamond & \langle k_{p-3,p-1}\rangle \arrow[dl] \arrow[dr] & \Diamond & \langle k_{p-2,p}\rangle \arrow[dl] \arrow[dr] & & \\
 & \langle k_{0,1}\rangle \arrow[dl] \arrow[dr] & \Diamond & \langle k_{1,2}\rangle \arrow[dl] \arrow[dr] & \Diamond & \ddots\iddots \arrow[dl] \arrow[dr] & \Diamond & \langle k_{p-1,p-2}\rangle \arrow[dl] \arrow[dr] & \Diamond & \langle k_{p-1,p}\rangle \arrow[dl] \arrow[dr] & \\
\langle k_{0,0}\rangle & & \langle k_{1,1}\rangle & & \langle k_{2,2}\rangle & & \langle k_{p-2,p-2}\rangle & & \langle k_{p-1,p-1}\rangle & & \langle k_{p,p}\rangle
\end{tikzcd}$$

in which the backward pointing maps are all of the form

$$p_{k_{i,j}|k_{i+1,j}},$$

and the forward pointing maps are all of the form

$$j_{k_{i,j}|k_{i,j+1}}.$$

It is a trivial matter to see that this assignment defines a simplicial map

$$\widetilde{M}_S : O(\Phi_S) \longrightarrow A^{\mathrm{eff}}(\mathbf{Fin}),$$

as desired.

3. Localizing exact ∞-categories

Now we apply the construction of Section 2 when $C = \mathbf{Exact}_\infty$, the ∞-category of exact ∞-categories [2, 3.1]. In particular, since exact ∞-categories form an ∞-category with direct sums, we may form, for any exact ∞-category E, the exact ∞-category $S^{-1}E$ via this filtered colimit. Since the multiplication by k functor $k\colon E \longrightarrow E$ induces the multiplication by k map $k\colon K(E) \longrightarrow K(E)$, and since algebraic K-theory preserves filtered colimits, we deduce that

$$K(S^{-1}E) \simeq S^{-1}K(E).$$

The question, now, is whether our exact ∞-category $S^{-1}E$ is at all understandable. Happily, the answer is yes: we can identify $S^{-1}E$ with an ∞-category of certain graded objects, not quite of E, but of a natural enlargement thereof, where we might find suitably infinite objects for our analysis.

DEFINITION 3.1. If E is an essentially small exact ∞-category, then a *large* object of E is a functor $E^{\mathrm{op}} \longrightarrow \mathbf{Top}$ that carries any zero object in E to a terminal object and any admissible bicartesian square

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d, two heads] & Y \arrow[d, two heads] \\ X' \arrow[r, tail] & Y' \end{tikzcd}$$

to a pullback square. We write $P_+(E)$ for the full subcategory of $\mathrm{Fun}(E^{\mathrm{op}}, \mathbf{Top})$ spanned by the large objects of E.

It is easy to check that $P_+(E)$ is a compactly generated, additive ∞-category, and that the Yoneda embedding of E into $P_+(E)$ carries admissible bicartesian squares to squares that are both pushout and pullback squares. We may declare a morphism of $P_+(E)$ to be ingressive or egressive if and only if it is a filtered colimit of ingressive or egressive morphisms of E, respectively. With this structure, $P_+(E)$ is an exact ∞-category, and $j_+\colon E \hookrightarrow P_+(E)$ is exact.

Furthermore, $P_+(E)$ has the following universal property: for any additive, presentable ∞-category D, precomposition with j_+ defines an equivalence

$$\mathrm{Fun}^L(P_+(E), D) \xrightarrow{\sim} \mathrm{Fun}_{\mathbf{Exact}_\infty}(E, D).$$

EXAMPLE 3.2. When E is the ordinary category of finitely generated projective modules over a commutative ring R, then $P_+(E)$ is equivalent to the ∞-category $\mathbf{Ch}^+(R)$ of non-negative chain complexes of R-modules.

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EXAMPLE 3.3. More generally, when E has its *minimal* exact structure, so that the only ingressive morphisms are summand inclusions, the ∞-category $P_+(E)$ is the non-abelian derived ∞-category of E.

EXAMPLE 3.4. When E is a stable ∞-category with its maximal exact structure, so that every morphism is ingressive, the ∞-category $P_+(E)$ is simply $\mathrm{Ind}(E)$.

DEFINITION 3.5. Suppose again E an exact ∞-category and S a set of primes. Then an *S-divisible large object* of E is an object of the (homotopy) limit of the functor

$$P_+(E)[S]^\vee \colon \Phi_S^{\mathrm{op}} \longrightarrow \mathbf{Cat}_\infty.$$

We write $\mathbf{Div}_S(P_+(E))$ for this homotopy limit.

More concretely, an S-divisible object is a sequence of large objects

$$\{X_i\}_{i\in\Phi_S}$$

along with equivalences

$$\rho_{i,j} \colon X_i \xrightarrow{\sim} jX_{ij}$$

for any $i, j \in \Phi_S$, which fit together to give, for every $i_0, i_1, \ldots, i_n \in \Phi_S$, an n-simplex

$$X_{i_0} \xrightarrow{\sim} i_1 X_{i_0 i_1} \xrightarrow{\sim} \cdots \xrightarrow{\sim} i_1 i_2 \cdots i_n X_{i_0 i_1 \cdots i_n}$$

of equivalences.

NOTATION 3.6. For any $m \in \Phi_S$, we have the projection

$$\omega_m \colon \mathbf{Div}_S(P_+(E)) \longrightarrow P_+(E),$$

given by evaluation at $m \in \Phi_S$, and we also have its left adjoint σ_m.

Given an object V of E and a natural number m, we may define an S-divisible large object

$$\frac{V}{m} := \sigma_m(j_+(V)).$$

We write $\mathbf{Div}_S(E)$ for the full subcategory of $\mathbf{Div}_S(P_+(E))$ spanned by the objects of the form V/m.

3.7. Note that if $n = mk$ in Φ_S, then

$$m\frac{V}{n} \simeq \frac{V}{k},$$

justifying our notation.

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THEOREM 3.8. *Suppose E an exact ∞-category and S a set of primes. Then the exact ∞-category $S^{-1}E$ is equivalent to $\mathbf{Div}_S E$.*

Proof. The ∞-category $S^{-1}E$ is the colimit of the diagram $E[S]\colon \Phi_S \longrightarrow \mathbf{Cat}_\infty$. We consider the embedding $E \hookrightarrow P_+(E)$, which is visibly functorial in E and lands in the subcategory of compact objects. Hence the induced functor $S^{-1}E \hookrightarrow S^{-1}P_+(E)$ is fully faithful and exact, where $S^{-1}P_+(E)$ is computed in the ∞-category $\mathbf{Pr}^L$.

Now $S^{-1}P_+(E)$ is by definition the filtered colimit of $P_+(E)[S]$ computed in $\mathbf{Pr}^L$, which is in turn equivalent to the filtered limit of the adjoint diagram in $\mathbf{Pr}^R$, which is in turn the limit in $\mathbf{Cat}_\infty$. The adjoint diagram is clearly $P_+(E)[S]^\vee$, whence we find that $S^{-1}P_+(E) \simeq \mathbf{Div}_S P_+(E)$.

Now the essential image of the functor is spanned by those objects that lie in the image of an object V of E lying in some degree $m \in \Phi_S$. These are exactly the objects V/m defined above. □

EXAMPLE 3.9. In the particular case in which E is an idempotent-complete stable ∞-category, the ∞-category $S^{-1}E \simeq \mathbf{Div}_S(E)$ is the full subcategory of $\mathbf{Div}_S(\mathrm{Ind}(E))$ spanned by the compact objects.

REMARK 3.10. If E is a symmetric monoidal exact ∞-category (that is, an exact ∞-category whose underlying Waldhausen ∞-category is symmetric monoidal in the sense of [1]), then one can show that $S^{-1}E$ is naturally an E-module, and the functors $\sigma_m \circ j_+\colon E \longrightarrow S^{-1}E$ are E-module functors.

WARNING 3.11. We stress that $S^{-1}E$ will not in general be an S-local exact ∞-category. In fact, it is not hard to see that the only S-local exact ∞-category is 0. Thus, even transfinite iterations of the construction $E \rightsquigarrow S^{-1}E$ will not be S-local.

4. Divisible objects as equivariant sheaves

In this section we will find a more geometric description of $S^{-1}C$ when C is a presentable exact category, such as $P_+(E)$, and then we will cut the resulting large ∞-category back down to size. To begin, let us describe an action of the *S-adic circle group* $\mathbf{T}_S = S^{-1}\mathbf{Z}/\mathbf{Z}$ on the Cantor space Ω.

NOTATION 4.1. For any prime number p, write

$$\Omega_p := \mathrm{Map}(\mathbf{N}, \langle p\rangle),$$

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equipped with the product topology. This is of course a Cantor space, as is the product

$$\Omega_S := \prod_{p \in S} \Omega_p.$$

(Of course Ω_p may be identified with the group $\mathbf{Z}_p$ of p-adic integers, but we will not use much of the abelian group structure.)

For any non-negative integer n, we obtain a continuous map

$$p^n \colon \Omega_p \longrightarrow \Omega_p,$$

which carries r to the map given by

$$(p^n r)_i = \begin{cases} 0 & \text{if } i \leqslant n; \\ r_{i-n} & \text{if } i > n. \end{cases}$$

(In other words, this is multiplication by p^n in $\mathbf{Z}_p$.) For any product $m = \prod_{p \in S} p^{v_p(m)}$ of primes in S, we therefore obtain a continuous map

$$m \colon \Omega_S \longrightarrow \Omega_S.$$

We write $m\Omega_S \subseteq \Omega_S$ for the image of this map, which is again a Cantor space. There is also a surjection $f_{p^n} \colon \Omega_p \longrightarrow p^n \Omega_p$ given by

$$f_{p^n}(r)_i = \begin{cases} 0 & \text{if } i \leqslant n; \\ r_i & \text{if } i > n; \end{cases}$$

this extends to a surjection $f_m \colon \Omega_S \longrightarrow m\Omega_S$ for any natural number m.

CONSTRUCTION 4.2. Of course we have the free action of the cyclic group C_{p^n} on $\langle p^n \rangle$. Identifying $\mathrm{Map}(\langle n \rangle, \langle p \rangle)$ with $\langle p^n \rangle$ by means of the lexicographic ordering, we obtain a free continuous action of $\mathbf{T}_p = \mathrm{colim}_n\, C_{p^n}$ on Ω_p. Moreover, two elements $x, y \in \Omega_p$ lie in the same orbit if and only if $f_{p^n}(x) = f_{p^n}(y)$ for some non-negative integer n.

These actions together provide an action of $\mathbf{T}_S \cong \bigoplus_{p \in S} \mathbf{T}_p$ on Ω_S, and two elements $x, y \in \Omega_S$ lie in the same orbit if and only if $f_m(x) = f_m(y)$ for some natural number m.

PROPOSITION 4.3. *Let C be an exact presentable ∞-category (for example, $P_+(E)$ for an exact ∞-category E). Then there is an equivalence*

$$S^{-1}C \simeq \mathbf{Sh}_C^{\mathbf{T}_S}(\Omega_S)$$

where the right hand side is the ∞-category of C-valued $\mathbf{T}_S$-equivariant sheaves on the space Ω_S, with the S-adic circle group $\mathbf{T}_S$ acting as above.

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Proof. The category $S^{-1}C$ is the colimit of a diagram $\Phi_S \longrightarrow \mathbf{Pr}^L$. We can interpret the arrows appearing in this diagram as formed via a push–pull construction

$$C \xrightarrow{\pi^*} \mathbf{Sh}_C(\langle n\rangle) \xrightarrow{\pi_*} C$$

, where $\pi : \langle n\rangle \longrightarrow \langle 1\rangle$ is the projection. But we can decouple the pullback and the pushforward by employing Section 2 to define a factorization of $\Phi_S \longrightarrow \mathbf{Pr}^L$ through a functor $O(\Phi_S) = \mathrm{Fun}(\Delta^1, \Phi_S) \longrightarrow \mathbf{Pr}^L$ that carries each object $(m|n)$ of $O(\Phi_S)$ to the ∞-category

$$\mathbf{Sh}_C\left(\left\langle \frac{n}{m}\right\rangle\right).$$

Precisely, we compose $\widetilde{M}_S$ with the unique functor $\mathbf{Sh}_C : A^{\mathrm{eff}}(\mathbf{Fin}) \longrightarrow \mathbf{Pr}^L$ that preserves finite products and carries $\langle 1\rangle$ to C (with the direct sum symmetric monoidal structure).

Since Φ_S is a filtered category, the inclusion $\Phi_S \longrightarrow O(\Phi_S)$ is cofinal and we can compute

$$S^{-1}C := \operatorname*{colim}_{m\in\Phi_S} C \simeq \operatorname*{colim}_{(m|n)\in O(\Phi_S)} \mathbf{Sh}_C\left(\left\langle \frac{n}{m}\right\rangle\right) \simeq \operatorname*{colim}_{m\in\Phi_S} \operatorname*{colim}_{n\in\Phi_S,\ m|n} \mathbf{Sh}_C\left(\left\langle \frac{n}{m}\right\rangle\right),$$

where in the last equality we have used that the projection $O(\Phi_S) \longrightarrow \Phi_S$ sending $(m|n)$ to m is a cocartesian fibration and so we can compute colimits fiberwise. But, since colimits in $\mathbf{Pr}^L$ can be computed as limits in $\mathbf{Pr}^R$, we have for any fixed $m \in \Phi_S$,

$$\operatorname*{colim}_{n\in\Phi_S,\ m|n} \mathbf{Sh}_C\left(\left\langle \frac{n}{m}\right\rangle\right) = \lim_{n\in\Phi_S,\ m|n} \mathbf{Sh}_C\left(\left\langle \frac{n}{m}\right\rangle\right) = \mathbf{Sh}_C(m\Omega_S).$$

Here, the final identification follows from the fact that the ∞-category of sheaves on the lattice of clopen sets $m\Omega_S$ (that is, the union of the lattices of subsets of $\langle n/m\rangle$ as n varies through Φ_S) is equivalent to the ∞-category of sheaves on the topological space $m\Omega_S$, because clopen sets form a basis that is closed under finite intersections.

So we have shown that

$$S^{-1}C \simeq \operatorname*{colim}_{m\in\Phi_S} \mathbf{Sh}_C(m\Omega_S),$$

where the maps in the diagram are given by the pushforward along the projection

$$j_{m|n} : m\Omega_S \longrightarrow n\Omega_S.$$

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But colimits in $\mathbf{Pr}^L$ can be computed as limits in $\mathbf{Pr}^R$ after replacing all the functors with their right adjoints. Since $j_{m|n}$ is étale and proper, the right adjoint of the pushforward is the pullback. Hence we can write

$$S^{-1}C \simeq \lim_{m\in\Phi_S^{\mathrm{op}}} \mathbf{Sh}_C(m\Omega_S).$$

Now we observe that the map $j_{1|m}\colon \Omega_S \longrightarrow m\Omega_S$ is the surjection f_m above. In particular we can write

$$S^{-1}C \simeq \lim_{m\in\Phi_S^{\mathrm{op}}} \mathbf{Sh}_C(\Omega_S/R_m) \simeq \lim_{m\in\Phi_S^{\mathrm{op}}} \lim_{\Delta^{\mathrm{op}}} \mathbf{Sh}_C(R_m \times_{\Omega_S} \cdots \times_{\Omega_S} R_m),$$

where R_m is the equivalence relation given by

$$R_m = \{(x, y) \in \Omega_S \times \Omega_S \mid f_m(x) = f_m(y)\},$$

and we conclude that

$$S^{-1}C \simeq \lim_{\Delta^{\mathrm{op}}} \mathbf{Sh}_C(R \times_{\Omega_S} \cdots \times_{\Omega_S} R),$$

where $R = \mathrm{colim}_{m\in\Phi_S} R_m$. Finally, by Construction 4.2, the equivalence relation R is exactly the equivalence relation induced on Ω_S by the action of $\mathbf{T}_S$. So

$$S^{-1}C \simeq \mathbf{Sh}_C(\Omega_S)^{h\mathbf{T}_S},$$

as desired. ☐

REMARK 4.4. A simple analysis of this proof shows that if C is a presentably symmetric monoidal exact ∞-category, then the equivalence $S^{-1}C \simeq \mathbf{Sh}_C^{\mathbf{T}_S}(\Omega_S)$ is an equivalence of C-modules.

4.5. Note that since Ω_S is a compact Hausdorff space of finite covering dimension, it follows that the corresponding ∞-topos is hypercomplete. This ensures that equivalences in $\mathbf{Sh}_C(\Omega_S)$ and $\mathbf{Sh}_C^{\mathbf{T}_S}(\Omega_S)$ can be detected on stalks.

4.6. Of course we wish to apply this to the case in which $C = P_+(E)$ for some exact ∞-category E. The full subcategory $S^{-1}E \subset S^{-1}P_+(E)$ can be identified with a full subcategory

$$\mathbf{Sh}^{\mathbf{T}_S}_{P_+(E)}(\Omega_S)^{\mathrm{small}} \subseteq \mathbf{Sh}^{\mathbf{T}_S}_{P_+(E)}(\Omega_S).$$

The objects V/m of $\mathbf{Sh}^{\mathbf{T}_S}_{P_+(E)}(\Omega_S)^{\mathrm{small}}$ can be described as follows. Form the constant sheaf V on $\langle m\rangle$ with the obvious C_m action; call the result V again. Now

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V/m is the induced $\mathbf{T}_S$-equivariant sheaf

$$\mathbf{T}_S \times_{C_m} V \cong \bigoplus_{g \in \mathbf{T}_S/C_m} g^{\star} V$$

on Ω_S.

Now if E is an idempotent-complete stable ∞-category, then $\mathbf{Sh}^{\mathbf{T}_S}_{P_+(E)}(\Omega_S)^{\text{small}}$ is the full subcategory of $\mathbf{Sh}^{\mathbf{T}_S}_{\operatorname{Ind} E}(\Omega_S)$ spanned by the compact objects.

If E is a symmetric monoidal exact ∞-category, then one can show that the $P_+(E)$-module equivalence $S^{-1}P_+(E) \simeq \mathbf{Sh}^{\mathbf{T}_S}_{P_+(E)}(\Omega_S)$ restricts to an E-module equivalence $S^{-1}E \simeq \mathbf{Sh}^{\mathbf{T}_S}_{P_+(E)}(\Omega_S)^{\text{small}}$.

We now turn our attention to the G-theory of a quasicompact quasiseparated scheme X. (Everything will also work in the derived or spectral settings with small modifications that are best left to the reader.) Following Illusie [4, Exposé I], one defines the ∞-category $\mathbf{Coh}(X) \subset \mathbf{QCoh}(X)$ of *coherent complexes* on X as follows:

(1) If $X = \operatorname{Spec} A$ is an affine scheme, then $\mathbf{Coh}(X)$ is defined as the full subcategory of the derived ∞-category $\mathbf{D}(A)$ spanned by those bounded complexes of A-modules M such that for any filtered diagram $\{N_\alpha\}_{\alpha \in \Lambda}$ of A-modules, and any integer n, the natural map

$$\operatorname*{colim}_{\alpha \in \Lambda} \operatorname{Map}(M, N_\alpha[n]) \longrightarrow \operatorname{Map}\left(M, \operatorname*{colim}_{\alpha \in \Lambda} N_\alpha[n]\right)$$

is an equivalence.

(2) In general, an object of $\mathbf{QCoh}(X)$ belongs to the subcategory $\mathbf{Coh}(X)$ if and only if its restriction to every affine open subscheme $U \subset X$ belongs to $\mathbf{Coh}(U)$. We set

$$\mathbf{IndCoh}(X) := \operatorname{Ind} \mathbf{Coh}(X).$$

Following Thomason [9, 3.3], one defines the G-theory of X by

$$G(X) := K(\mathbf{Coh}(X)).$$

Now recall that Ω_S can be seen as an affine scheme (precisely as the spectrum of the ring of locally constant $\mathbf{Z}$-valued functions on Ω_S). Since

$$\mathbf{Sh}_{\mathbf{IndCoh}(X)}(\Omega_S) \simeq \mathbf{IndCoh}(X \times \Omega_S)$$

we can express Proposition 4.3 in a different way:

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PROPOSITION 4.7. *Let X be a quasicompact quasiseparated scheme. There is an equivalence of stable presentable ∞-categories*

$$S^{-1}\mathbf{IndCoh}(X) \simeq \mathbf{IndCoh}(X \times \Omega_S)^{h\mathbf{T}_S}.$$

Following Gaitsgory [6], we may extend the definition of **IndCoh** to more general objects by stipulating that the functor $X \rightsquigarrow \mathbf{IndCoh}(X)$, $f \rightsquigarrow f^!$, transform colimits into limits. The quotient algebraic space

$$[(X \times \Omega_S)/\mathbf{T}_S] \simeq X \times [\Omega_S/\mathbf{T}_S]$$

can be expressed as a colimit of schemes

$$\operatorname*{colim}_{m \in \Phi_S} (X \times \Omega_S)/C_m$$

in which all maps are finite étale. Since $f^! = f^*$ for such maps f, we obtain

$$S^{-1}\mathbf{IndCoh}(X) \simeq \mathbf{IndCoh}([(X \times \Omega_S)/\mathbf{T}_S]).$$

As $S^{-1}\mathbf{IndCoh}(X)$ is furthermore compactly generated [8, Pr. 5.5.7.6], it is sensible to define $\mathbf{Coh}([(X \times \Omega_S)/\mathbf{T}_S])$ as the full stable subcategory of the ∞-category $\mathbf{IndCoh}([(X \times \Omega_S)/\mathbf{T}_S])$ spanned by the compact objects. Consequently, the proposition above induces an identification

$$S^{-1}\mathbf{Coh}(X) \simeq \mathbf{Coh}^{\mathbf{T}_S}(X \times \Omega_S) = \mathbf{Coh}([(X \times \Omega_S)/\mathbf{T}_S]).$$

We thus obtain the desired identification of spectra (and even $K(X)$-modules)

$$S^{-1}G(X) \simeq G^{\mathbf{T}_S}(X \times \Omega_S).$$

In particular, when $X = \operatorname{Spec} A$, then one has

$$S^{-1}G(A) \simeq G^{\mathbf{T}_S}(C(\Omega_S, A)),$$

where C denotes the ring of locally constant functions.

REMARK 4.8. We caution that the algebraic space $[(X \times \Omega_S)/\mathbf{T}_S]$ is not perfect: compact objects such as $O_X/1$ are not dualizable in the symmetric monoidal ∞-category $\mathbf{QCoh}([(X \times \Omega_S)/\mathbf{T}_S])$, and conversely the unit object is not compact. Hence, we cannot simply replace G-theory by K-theory in the above formulas.

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