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LATTICE DILATIONS OF POSITIVE CONTRACTIONS ON L^p-SPACES

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ABSTRACT. In the spirit of our previous paper (Math. Z. **156**, 265–277 (1977)) we present a functional analytic proof of the following result of M. A. Akcoglu: Every positive contraction on a reflexive L^p-space has a lattice dilation.

M. A. Akcoglu developed a useful but very complicated dilation theory for positive contractions on L^p-spaces (see [2], [3], [4]). For L^1-spaces we presented a simple dilation [5] which can be said to be canonical in a certain sense. In this note we show that the same ideas, adequately modified, yield a dilation for all L^p-spaces, $1 \le p < \infty$. This is of some importance for applications to individual ergodic theorems (see [1]).

THEOREM. *Every positive contraction $T \in \mathscr{L}(E)$, E a reflexive L^p-space with weak order unit, has a lattice dilation $\hat{T}$ on an L^p-space $\hat{E}$. More precisely*:

There exists a finite measure space $(\hat{X}, \hat{\mu})$, a Banach lattice isomorphism $\hat{T} \in \mathscr{L}(\hat{E})$, $\hat{E} = L^p(\hat{X}, \hat{\mu})$, an isometric lattice injection $\hat{I} : E \to \hat{E}$, and a positive contraction $\hat{Q} : \hat{E} \to E$, such that

$$\begin{array}{ccc} E & \xrightarrow{T^n} & E \\ \downarrow{\scriptstyle \hat{I}} & & \downarrow{\scriptstyle \hat{Q}} \\ \hat{E} & \xrightarrow[T^n]{} & \hat{E} \end{array}$$

commutes for every $n = 0, 1, 2, \ldots$.

Proof. By the ultra power technique as developed in [4] it suffices to prove the theorem for finite dimensional E. But for a finite dimensional L^p-space E, it is known that there exist weak order units $0 \ll u \in E$ and $0 \ll v \in E'$, such that $Tu \le v^{q-1}$ and $T'v \le u^{p-1}$ (by [4], Theorem 2.5 there is a $u \gg 0$, such that $T'(Tu)^{p-1} \le u^{p-1}$; take $v_1 := (Tu)^{p-1}$ and $v := v_1 + v_2 \gg 0$, where $v_2 \perp v_1$).

Therefore the lattice dilation can be obtained as a composition (see [5], 1, remark 1) of the positive dilation in Lemma 1 and the lattice dilation in Lemma 2 (for the terminology, see [5], 1.1).

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LEMMA 1. *Let T be a positive contraction on $E=L^p(X,\mu)$ for $\mu(X)<\infty$ and $1<p<\infty$. Assume that there exist weak order units $0\ll u\in E$ and $0\ll v\in E'$ such that $Tu\le v^{q-1}$ and $T'v\le u^{p-1}$. Then (E,T) has a positive dilation $(L^p(\hat{X},\hat{\mu}),\hat{T})$ satisfying $T'\hat{v}=\hat{u}^{p-1}$ and $\hat{T}'\hat{\imath}=\hat{v}^{p-1}$ for some weak order units $0\ll\hat{u}\in L^p(\hat{X},\hat{\mu})$ and $0\ll\hat{v}\in L^q(\hat{X},\hat{\mu})$.*

Proof. Choose $\hat{X}:=X\cup\{y,z\}$ and $\hat{\mu}:=\mu+\alpha\delta_y+\beta\delta_z$ where

$$\alpha:=\langle u, u^{p-1}-T'v\rangle \quad\text{and}\quad \beta:=\langle v^{q-1}-Tu, v\rangle.$$

We identify $E=L^p(X,\mu)$ canonically with a projection band in $\hat{E}=L^p(\hat{X},\hat{\mu})$ and take $\hat{I}$ and $\hat{Q}$ to be the corresponding injection resp. projection. Finally, if we define $\hat{T}$ by

$$\hat{T}:=T+(2^{1/p}\beta)^{-1}\mathbb{1}_z\otimes(v^{q-1}-Tu+\mathbb{1}_z)+(2^{1/q}\alpha)^{-1}(u^{p-1}-T'v+\mathbb{1}_y)\otimes\mathbb{1}_y$$

we obtain a positive dilation of T satisfying the assertion for

$$\hat{u}:=u+\mathbb{1}_y+2^{1/p}\mathbb{1}_z \quad\text{and}\quad \hat{v}:=v+2^{1/q}\mathbb{1}_y+\mathbb{1}_z.$$

LEMMA 2. *Let T be a positive contraction on $E=L^p(X,\mu)$ satisfying $Tu=v^{q-1}$ and $T'v=u^{p-1}$ for some weak order units $0\ll u\in E$ and $0\ll v\in E'$. Then (E,T) has a lattice dilation $(L^p(\hat{X},\hat{\mu}),\hat{T})$.*

Proof. We assume $u=\mathbb{1}$ and define operators

$$S_+, S_-: L^\infty(\mu)\to L^\infty(\mu)$$

by

$$S_+f:=(T\mathbb{1})^{-1}\cdot Tf \quad\text{and}\quad S_-f:=T'(vf).$$

As in [5] and particularly by lemma 2.1 we obtain a positive linear operator

$$\hat{Q}_0: C(X^{\mathbb{Z}})\to L^\infty(X,\mu)$$

by extending continuously the mapping

$$\bigotimes_{-n}^{n} f_i \mapsto S_-(f_{-1}S_-(f_{-2}\ldots S_-f_{-n})\ldots))f_0S_+(f_1S_+(f_2\ldots S_+f_n)\ldots))$$

for $f_i\in C(X)$. If we define

$$\hat{\mu}:=\mu\circ\hat{Q}_0.$$

we obtain a Radon measure on $\hat{X}:=X^{\mathbb{Z}}$ and may extend $\hat{Q}_0$ to a positive contraction from $L^1(\hat{X},\hat{\mu})$ into $L^1(X,\mu)$. By the Riesz convexity theorem, its restriction

$$\hat{Q}: L^p(\hat{X},\hat{\mu})\to L^p(X,\mu)$$

is also a positive contraction.

Applying similar arguments to the canonical injection into the 0th coordinate (see [5], 2.ii)

$$\hat{I}_0 : C(X) \to C(\hat{X}),$$

we obtain an isometric lattice injection

$$\hat{I} : L^p(X, \mu) \to L^p(\hat{X}, \hat{\mu})$$

satisfying $\hat{Q}' = \hat{I}$ (compare [5], 3.iii).

Next we consider the lattice isomorphism on $C(\hat{X})$ induced by the right shift τ on $\hat{X}$ (see [5], 2.iii). For this operator and for $\hat{f} := \otimes_{-n}^{n} f_i \in C(\hat{X})$ we have

$$\begin{aligned}\int \hat{f} \circ \tau \, d\hat{\mu} &= \int S_-(f_0 S_-(f_{-1} \ldots)) f_1 S_+(f_2 S_+(f_3 \ldots)) \, d\mu \\ &= \int v f_0 S_-(f_{-1} \ldots) T(f_1 S_+(f_2 \ldots)) \, d\mu \\ &= \int S_-(f_{-1} \ldots) f_0 S_+(f_1 S_+(f_2 \ldots)) v^q \, d\mu \\ &= \int \hat{Q}\hat{f} \cdot v^q \, d\mu = \int \hat{f} \cdot \hat{I} v^q \, d\hat{\mu},\end{aligned}$$

hence τ can be used to define a Banach lattice isomorphism

$$\hat{T} : L^p(\hat{X}, \hat{\mu}) \to L^p(\hat{X}, \hat{\mu}),$$

namely

$$\hat{T}\hat{f} := \hat{I} v^{q-1} \cdot f \circ \tau^{-1}.$$

We have

$$\int |\hat{T}\hat{f}|^p \, d\hat{\mu} = \int |\hat{I} v^{q-1} \cdot \hat{f} \circ \tau^{-1}|^p \, d\hat{\mu} = \int \hat{I} v^q \cdot |f|^p \circ \tau^{-1} \, d\hat{\mu} = \int |\hat{f}|^p \, d\hat{\mu}.$$

It remains to show that $\hat{T}$ is a dilation of T, i.e. that $\hat{Q}\hat{T}^n\hat{I} = T^n$ for $n = 0, 1, 2 \ldots$. Observe first that

$$\hat{T}^n \hat{I} f = v_{(0)}^{q-1} \otimes \cdots \otimes v_{(n-1)}^{q-1} \otimes f_{(n)} \text{ for } f \in L^p(X, \mu).$$

therefore we have

$$\begin{aligned}\langle \hat{Q}\hat{T}^n\hat{I} f, g \rangle &= \langle \hat{T}^n \hat{I} f, \hat{Q}' g \rangle \\ &= \int v_{(0)}^{q-1} \otimes \cdots \otimes v_{(n-1)}^{q-1} \otimes f_{(n)} \cdot \hat{I} g \, d\hat{\mu} \\ &= \int \hat{Q}(v_{(0)}^{q-1} \otimes \cdots \otimes v_{(n-1)}^{q-1} \otimes f_{(n)}) g \, d\mu \\ &= \langle T^n f, g \rangle \quad \text{for every} \quad n \geq 0 \quad \text{and} \quad f, g \in L^p(X, \mu).\end{aligned}$$

REMARK. For the proof of Akcoglu's individual ergodic theorem [1], the dilation for the finite dimensional case is sufficient.

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