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ALMOST-ORTHONORMAL BASES FOR HILBERT SPACE

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Abstract. A basis $\{x_n\}$ for a Hilbert space H is called a *Riesz basis* if it has the property that $\sum a_n x_n$ converges in H if and only if $\sum |a_n|^2 < \infty$, and hence if and only if $\{x_n\}$ is the isomorphic image of some orthonormal basis for H. A consequence of a classical result of Bary [1] is that any basis for H that is quadratically near an orthonormal basis must be a Riesz basis. Motivated by this result, we study in this paper the class of normalized bases in a Hilbert space that are quadratically near some orthonormal basis, bases we call almost-orthonormal bases. In particular, we prove that any such basis must be quadratically near its Gram-Schmidt orthonormalization, and derive an internal characterization of these bases that indicates how restrictive the property of being almost-orthonormal is.

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1. Introduction. Let H denote a real, infinite-dimensional Hilbert space. A basis $\{x_n\}$ for H is called a *Riesz basis* when $\sum a_n x_n$ converges if and only if $\sum |a_n|^2 < \infty$. Equivalently, $\{x_n\}$ is a Riesz basis if and only if there is an isomorphism on H mapping some orthonormal basis $\{\phi_n\}$ onto $\{x_n\}$. It follows that Riesz bases have exactly the same linear-topological properties as orthonormal bases, lacking only the isometric properties of orthonormal bases. A classical result of Bary [**1**] (one of many in the literature of a similar nature) has as a corollary the fact that a basis $\{x_n\}$ for H that is *quadratically near* an orthonormal basis $\{\phi_n\}$, in the sense that $\sum_n \|x_n - \phi_n\|^2 < \infty$, must be a Riesz basis.

In view of this result, an interesting question which arises concerns when the converse of this result is true; that is, which (Riesz) bases for H are, in fact, quadratically near some orthonormal basis? We call a basis $\{x_n\}$ with this property *almost-orthonormal* in virtue of the "nearness" of $\{x_n\}$ in this geometric sense to being orthonormal, and consider here the problem of characterizing those normalized bases for H that are almost-orthonormal.

2. Almost-orthonormal bases. Recall that if $\{x_n\}$ is a basis for H, then the sequence of coefficient functionals associated with $\{x_n\}$ is the biorthogonal set $\{f_n\}$ in H for which $(f_m, x_n) = \delta_{m,n}$, for all m and n. This sequence of coefficient functionals is a basis for H and is equivalent to $\{x_n\}$ if and only if $\{x_n\}$ is a Riesz basis. (See, [**3**] for this and other results on bases.)

THEOREM 1. *Let $\{x_n\}$ be a normalized basis for* H *having coefficient functionals $\{f_n\}$. Then the following are equivalent.*

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(a) $\{x_n\}$ *is almost orthonormal.*

(b) $\{x_n\}$ *is quadratically near* $\{f_n\}$.

(c) $\{x_n\}$ *is quadratically near the orthonormal basis* $\{\phi_n\}$ *obtained from* $\{x_n\}$ *by the Gram-Schmidt procedure.*

Proof. (a) $\Rightarrow$ (b). Suppose that $\{x_n\}$ is a normalized basis for H that is almost-orthonormal and $\{\phi_n\}$ some orthonormal basis for which $\sum_n \|x_n - \phi_n\|^2 < \infty$. By Bary's result $\{x_n\}$ is then a Riesz basis, as is its sequence of coefficient functionals $\{f_n\}$. Therefore the operator T on H for which $T(\phi_n) = x_n$, for all n, is an isomorphism, as is T^*, and $T^*(f_n) = \phi_n$ for all n. If I denotes the identity operator on H, then since $\sum_n \|x_n - \phi_n\|^2 < \infty$ we see that the operator $U = T - I$ is a Hilbert Schmidt operator on H [**2**] that maps $\{\phi_n\}$ to $x_n - \phi_n$, for all n. It follows that $U^* = T^* - I$ is also a Hilbert-Schmidt operator and $U^*(f_n) = \phi_n - f_n$, for all n. But then $V = U^*(T^*)^{-1}$ is a Hilbert-Schmidt operator for which $V(\phi_n) = \phi_n - f_n$, implying $U^* + V$ is also a Hilbert-Schmidt operator that maps $\{\phi_n\}$ to $x_n - f_n$, for all n, and hence that $\sum_n \|x_n - f_n\|^2 < \infty$. That is, $\{x_n\}$ and $\{f_n\}$ are quadratically near.

(b) $\Rightarrow$ (c). Suppose that $\sum_n \|x_n - f_n\|^2 < \infty$. If $\{\phi_n\}$ denotes the orthonormal basis obtained from $\{x_n\}$ using the Gram-Schmidt (GS) procedure, then

$$\sum_n \|x_n - f_n\|^2 = \sum_n \|(x_n - \phi_n) + (\phi_n - f_n)\|^2$$
$$= \sum_n [\|x_n - \phi_n\|^2 + \|\phi_n - f_n\|^2 + ((x_n, \phi_n) + (f_n, \phi_n)) - 2] < +\infty.$$

By the definition of the GS procedure, $x_n = \sum_{i=1}^{n} (\phi_i, x_n)\phi_i$, for all n. Since $\{f_n\}$ is biorthogonal to $\{x_n\}$ it follows that the expansion of each f_n has the form $f_n = (x_n, \phi_n)^{-1}\phi_n + \sum_{i=n+1}^{\infty} (\phi_i, f_n)\phi_i$, so $(f_n, \phi_n) = (x_n, \phi_n)^{-1}$ and therefore $(x_n, \phi_n) + (f_n, \phi_n) = (x_n, \phi_n) + (x_n, \phi_n)^{-1}$, where $0 < (x_n, \phi_n) \leq 1$, for all n. If f is the function on (0,1] defined by $f(x) = x + x^{-1}$, then $f(1) = 2$ and $f'(x) < 0$ $(0 < x < 1)$ implying that $f(x) \geq 2$ for all x in (0,1]. Consequently, $(x_n, \phi_n) + (x_n, \phi_n)^{-1} - 2 \geq 0$, for all n, and it follows that $(x_n, \phi_n) + (f_n, \phi_n) - 2 \geq 0$. But then

$$\sum_n \|x_n - \phi_n\|^2 \leq \sum_n [\|x_n - \phi_n\|^2 + \|f_n - \phi_n\|^2 + [(x_n, \phi_n) + (+f_n, \phi) - 2]] < +\infty,$$

so that $\{x_n\}$ is quadratically near $\{\phi_n\}$.

(c) $\Rightarrow$ (a). This is clear from the definition.

Theorem 1 shows the rather surprising result that if a normalized basis $\{x_n\}$ in H is quadratically near *some* orthonormal basis it must then be quadratically near the orthonormal basis derived from it by the Gram-Schmidt procedure. This leads to the following interpretation of quadratic nearness.

THEOREM 2. *A normalized basis* $\{x_n\}$ *is almost-orthonormal if and only if* $\sum_n (1 - d_n) < +\infty$, *where* $d_n = \mathrm{dist}(x_n, [x_i]_{i=1}^{n-1})$, *for all* n.

Proof. Let $\{\phi_n\}$ be the orthonormal basis obtained from $\{x_n\}$ by the GS procedure, and suppose that $\{x_n\}$ is almost-orthonormal. Then, by Theorem 1, $\sum_n \|x_n - \phi_n\|^2 < +\infty$ and, for any n, $\|x_n - \phi_n\|^2 = 1 - 2(\phi_n, x_n) + \|x_n\|^2 = 2(1 - (x_n, \phi_n))$. From the

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GS construction,

$$\phi_n = \frac{x_n - \sum_{i=1}^{n-1}(\phi_i, x_n)\phi_i,}{\left\| x_n - \sum_{i=1}^{n-1}(\phi_i, x_n)\phi_i \right\|},$$

where $\sum_{i=1}^{n-1}(\phi_i, x_n)\phi_i$ is the best approximation to x_n from $[\phi_i]_{i=1}^{n-1} = [x_i]_{i=1}^{n-1}$. Therefore, $\|x_n - \sum_{i=1}^{n-1}(\phi_i, x_n)\phi_i\| = d_n$, the distance from x_n to $[x_i]_{i=1}^{n-1}$, and it follows that $(x_n, \phi_n) = d_n$. Hence, $\|x_n - \phi_n\|^2 = 2(1 - d_n)$, for all n, and $\sum_{n=1}^{\infty}(1 - d_n) = \frac{1}{2}\sum_{n=1}^{\infty}\|x_n - \phi_n\|^2 < +\infty$.

Conversely, if $\sum_n(1 - d_n) < +\infty$ then since, as we showed above, $\|x_n - \phi_n\|^2 = 2(1 - d_n)$ for all n, we see that $\sum_n \|x_n - \phi_n\|^2 < +\infty$ and $\{x_n\}$ is almost-orthonormal.

We can also use Theorem 1 to derive the following characterization of almost-orthonormal bases that is "internal" in the sense that the criterion involves the basis $\{x_n\}$ alone without mention of an orthonormal basis.

THEOREM 3. *A normalized basis* $\{x_n\}$ *for* H *is almost-orthonormal if and only if* $\{x_n\}$ *is a Riesz basis for which* $\sum_{n=1}^{\infty}\sum_{i\neq n}(x_i, x_n)^2 < +\infty$.

Proof. Suppose that $\{x_n\}$ is almost-orthonormal. Then by Theorem 1 $\{x_n\}$ is quadratically near its sequence of coefficient functionals $\{f_n\}$ and, being quadratically near an orthonormal basis, is a Riesz basis by Bary's result.

For all n, the expansion of x_n in terms of the basis $\{f_i\}$ for H is given by $x_n = \sum_{i=1}^{\infty}(x_1, x_n)f_i$, where $(x_n, x_n) = 1$ since $\|x_n\| = 1$. Therefore we have $x_n - f_n = \sum_{i\neq n}(x_i, x_n)f_i$, from which it follows that

$$\sum_{n=1}^{\infty}\left\|\sum_{i\neq n}(x_i, x_n)f_i\right\| = \sum_{n=1}^{\infty}\|x_n - f_n\|^2 < +\infty.$$

Since $\{f_i\}$ is also a Riesz basis for H there is some $m > 0$ such that $\|\sum_{i=1}^{\infty} a_i f_i\|^2 \geq m\sum_{i=1}^{\infty} a_i{}^2$, for all $\{a_i\}$, and hence , in particular,

$$\left\|\sum_{i\neq n}(x_i, x_n)f_i\right\|^2 \geq m\sum_{i\neq n}(x_i, x_n)^2$$

for all n, implying that $\sum_{n=1}^{\infty}\sum_{i\neq n}(x_i, x_n)^2 < +\infty$ by the above.

Conversely, suppose that $\{x_n\}$ is a Riesz basis for which $\sum_{n=1}^{\infty}\sum_{i\neq n}(x_i, x_n)^2 < +\infty$. As we showed in the first part of the proof, $\|x_n - f_n\|^2 = \|\sum_{i\neq n}(x_i, x_n)f_i\|^2$, for all n. Since $\{f_i\}$ is also a Riesz basis for H, there exists $M > 0$ such that

$$\left\|\sum_{i\neq n}(x_i, x_n)f_i\right\|^2 \leq M\sum_{i\neq n}(x_i, x_n)^2,$$

for all n. Therefore we have

$$\sum_{n=1}^{\infty}\|x_n - f_n\|^2 \leq M\sum_{n=1}^{\infty}\sum_{i\neq n}(x_i, x_n)^2 < +\infty,$$

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implying that $\{x_n\}$ is quadratically near $\{f_n\}$ and hence almost-orthonormal by Theorem 1.

Finally, we observe that Theorem 3 shows how restrictive the property of almost-normality is, since bases that would appear to be "almost-orthonormal" from certain perspectives may fail to be quadratically near any orthonormal basis. For example, let $\{\phi_n\}$ be an orthonormal basis for H. Define $x_{2n-1} = \phi_{2n-1}$ and $x_{2n} = a\phi_{2n-1} + b\phi_{2n}$, for $n = 1, 2 \cdots$, where a and b are any non-zero real numbers for which $a^2 + b^2 = 1$. Then $\{x_n\}$ is a normalized Riesz basis for H. However since $\sum_{i \neq n}(x_i, x_n)^2 = a^2 \neq 0$, for all n, by Theorem 3 $\{x_n\}$ fails to be almost orthonormal. Since $a > 0$ can be chosen arbitrarily small we see the sense in which the property of being quadratically near an orthonormal basis is quite a restrictive one.

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