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GAPS OF OPERATORS, II

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Abstract. In [**11**] the authors obtained an operator matrix with two variables that distinguishes the classes of p-hyponormal operators, w-hyponormal, absolute-p-paranormal, and normaloid operators on Hilbert spaces. We establish the general model for n variables, which provides many more examples to show that such classes are distinct.

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1. Introduction. Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. There are several classes of operators with weaker conditions than quasinormality; for example, p-hyponormal, w-hyponormal, p-quasihyponormal, absolute p-paranormal, p-paranormal, and normaloid operators, etc. Here is a brief review of those operators (see [**5**], [**6**], [**10**], and [**12**] for further discussion).

- An operator $T \in \mathcal{L}(\mathcal{H})$ is *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$ $(0 < p < \infty)$.
- T is *∞-hyponormal* if $(T^*T)^p \geq (TT^*)^p$, for all $p \in (0, \infty)$.
- T is *p-quasihyponormal* if $T^*((T^*T)^p - (TT^*)^p)T \geq 0$ $(0 < p < \infty)$.
- T is an *$A(p)$-operator* if $(T^*\,|T|^{2p}\,T)^{1/(p+1)} \geq |T|^2$ $(0 < p < \infty)$.
- T is *p-paranormal* if $\|\,|T|^p\,U\,|T|^p\,x\| \geq \|\,|T|^p\,x\|^2$ for all unit vectors $x \in \mathcal{H}$ $(0 < p < \infty)$, where U is the partially isometric part of the polar decomposition of T. In particular, 1-paranormality is referred to as paranormality.
- T is *absolute-p-paranormal* if $\||T|^pTx\| \geq \|Tx\|^{p+1}$ for all unit vectors $x \in \mathcal{H}$ $(0 < p < \infty)$. Note that absolute-1-paranormality and 1-paranormality are equivalent.
- T is *w-hyponormal* if $|\widetilde{T}| \geq |T|$, where $\widetilde{T} := |T|^{1/2}\,U\,|T|^{1/2}$ is the *Aluthge transform* of T ([**2**], [**3**], [**9**]).
- T is *normaloid* if $\|T\| = r(T)$, where $r(T)$ is the spectral radius of T, which is equivalent to $\|T^n\| = \|T\|^n$ for all natural numbers n. (See [**6**, p. 100].)

There are several well-known relationships among the classes of operators described above. The interesting implications in this note are as follows:

- quasinormal $\Rightarrow$ p-hyponormal $\Rightarrow$ p-quasihyponormal $\Rightarrow$ $A(p)$-operator $\Rightarrow$ absolute-p-paranormal $\Rightarrow$ normaloid.

Only a few examples of these operators, in particular p-hyponormal operators, have been known, and so it is worthwhile to develop examples to show that these classes are distinct. For this purpose, in [**11**] the authors considered a matrix of block operators with 2 variables and obtained a graph to classify those classes in 2-dimensional space. In this note we extend the study of the 2 variable version in [**11**] to yield the general version of n variables which establishes some examples to show that the classes of

p-hyponormal operators are distinct for each $n \geq 2$. Thus such examples are abundant! In addition, we consider the graphs for operators. These will be discussed in this note in 3-dimensional space.

2. p-hyponormality. Let $C = (c_{ij})$ be an $n \times n$ matrix with $c_{ij} = 1/n\,(1 \leq i, j \leq n)$ and let $D \equiv D(x_1, x_2, \ldots, x_n) := \mathrm{Diag}\{x_1, x_2, \ldots, x_n\}$ with $x_i \geq 0,\ i = 1, \ldots, n$. We define $T(x_1, x_2, \ldots, x_n)$ on $\mathcal{H} \equiv \mathbb{C}^n \otimes \ell_2(\mathbb{Z})$ by

$$T(x_1, x_2, \ldots, x_n) = \begin{pmatrix} \ddots & & & & & & \\ \ddots & O & & & & & \\ & C & O & & & & \\ & & C & \boxed{O} & & & \\ & & & D & O & & \\ & & & & D & O & \\ & & & & & \ddots & \ddots \end{pmatrix},$$

where $\boxed{\cdot}$ denotes the center of the two sided infinite matrix.

THEOREM 2.1. *Let $p \in (0, \infty)$. Then the following assertions are equivalent*:
(i) *$T(x_1, x_2, \ldots, x_n)$ is p-hyponormal*;
(ii) *$n \geq (1/x_1)^{2p} + \cdots + (1/x_n)^{2p}$ with $x_i > 0,\ 1 \leq i \leq n$.*

The following lemma proves Theorem 2.1.

LEMMA 2.2. *Let $p \in (0, \infty)$. Then the following assertions are equivalent*:
(i) *$T(x_1, x_2, \ldots, x_n)$ is p-hyponormal*;
(ii) *$\Delta_k > 0\ (1 \leq k \leq n-1)$ and $\Delta_n \geq 0$, where*

$$\Delta_1 := x_1^{2p} - \frac{1}{n} \quad \textit{and}$$

$$\Delta_k := -\frac{1}{n} \sum_{1 \leq i \leq k} \left(\prod_{1 \leq j \leq k, j \neq i} x_j^{2p} \right) + \prod_{1 \leq i \leq k} x_i^{2p} \quad (2 \leq k \leq n).$$

Proof. First note that $T(x_1, x_2, \ldots, x_n)$ is p-hyponormal if and only if $D^{2p} - C \geq 0$. Let

$$A := D^{2p} - C = \begin{pmatrix} x_1^{2p} - \frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & x_2^{2p} - \frac{1}{n} & \cdots & -\frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & x_n^{2p} - \frac{1}{n} \end{pmatrix} \tag{2.1}$$

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and let S_k be the $k \times k$ matrix in the upper left corner of A. Then the determinant of the matrix S_k, $2 \le k \le n$, is

$$\Delta_k := -\frac{1}{n}\sum_{1\le i\le k}\left(\prod_{1\le j\le k, j\ne i} x_j^{2p}\right) + \prod_{1\le i\le k} x_i^{2p}. \tag{2.2}$$

According to the Nested Determinants Test (cf. [**4**, p. 213]), obviously (ii) implies (i). We shall show that (i) implies (ii) here. If $D^{2p} - C \ge 0$, then obviously $S_k \ge 0$ and

$$x_i \ne 0 \quad \text{for } i = 1, 2, \ldots, n. \tag{2.3}$$

Suppose that $\Delta_k = 0$ for $1 \le k < n$. Then $\Delta_{k+1} = \cdots = \Delta_n = 0$ (cf. [**4**, Proposition 2.6]). Multiplying by x_{k+1}^{2p} and adding $-\frac{1}{n}\prod_{1\le i\le k} x_i^{2p}$ to (2.2), we have

$$\Delta_{k+1} = \Delta_k x_{k+1}^{2p} - \frac{1}{n}\prod_{1\le i\le k} x_i^{2p} = 0 \quad (2 \le k < n).$$

Hence $x_1^{2p}\cdots x_k^{2p} = 0$. Therefore $x_{k_0} = 0$ for some $2 \le k_0 \le k$, which contradicts (2.3). □

Proof of Theorem 2.1. The implication (i) $\Rightarrow$ (ii) follows easily from Lemma 2.2. For the reverse implication (ii) $\Rightarrow$ (i), we assume $n \ge (1/x_1)^{2p} + \cdots + (1/x_n)^{2p}$ with $x_i > 0$, $1 \le i \le n$. Then

$$n > \left(\frac{1}{x_1}\right)^{2p} + \cdots + \left(\frac{1}{x_k}\right)^{2p}, \quad (1 \le k \le n-1).$$

Hence we may obtain $\Delta_k > 0$ $(1 \le k \le n-1)$ by a simple computation. Since $\Delta_n \ge 0$ is equivalent to $n \ge (1/x_1)^{2p} + \cdots + (1/x_n)^{2p}$, we have proved this theorem. □

For $0 < p \le \infty$, we denote by $\mathcal{R}_p^{(n)}$ the set of $(x_1, \ldots, x_n)$ in $\mathbb{R}_+^{(n)}$ such that $T(x_1, x_2, \ldots, x_n)$ is p-hyponormal, where $\mathbb{R}_+$ is the set of nonnegative real numbers. The following proposition shows that the classes of p-hyponormal operators are distinct for $0 < p \le \infty$.

Proposition 2.3. *For* $0 < q < p < \infty$, $\mathcal{R}_\infty^{(n)} \subsetneqq \mathcal{R}_p^{(n)} \subsetneqq \mathcal{R}_q^{(n)}$.

Proof. Since the case of $n = 2$ was proved in [**11**], we may assume that $n \ge 3$. Let us consider $x_1 = x_2 = \cdots = x_{n-2} = 1$ and let

$$\mathcal{R}_p^{(n)}|_{(1,\ldots,1,x_{n-1},x_n)} := \left\{(\underbrace{1,\ldots,1}_{(n-2)}, x_{n-1}, x_n) : 2 \ge \left(\frac{1}{x_{n-1}}\right)^{2p} + \left(\frac{1}{x_n}\right)^{2p}\right\},$$

which is the projection of $\mathcal{R}_p^{(n)}$ w.r.t. $x_1 = x_2 = \cdots = x_{n-2} = 1$. Then a computation similar to that in the proof of [**11**, Lemma 2.2] shows that $\mathcal{R}_p^{(n)}|_{(1,\ldots,1,x_{n-1},x_n)} \subsetneqq \mathcal{R}_q^{(n)}|_{(1,\ldots,1,x_{n-1},x_n)}$. Obviously $\mathcal{R}_p^{(n)} \subsetneqq \mathcal{R}_q^{(n)}$. □

Remark 2.4. Let $\mathcal{R}_{\text{qn}}^{(n)}$ be the set of $(x_1, \ldots, x_n) \in \mathbb{R}_+^{(n)}$ for which the operator $T(x_1, x_2, \ldots, x_n)$ is quasinormal. Then by the definition of quasinormality, we may

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show that $\mathcal{R}_{\text{qn}}^{(n)}$ is the singleton $\{(1, \ldots, 1)\}$. This fact points out an error in [**11**, Theorem 2.3 (v)].

PROPOSITION 2.5. *For $p \in (0, \infty)$, we have the following equalities.*

(i) $\mathcal{R}_{\infty}^{(n)} = [1, \infty) \times \cdots \times [1, \infty)$ (n*−copies*),

(ii) $\bigcup_{p>0} \mathcal{R}_p^{(n)} = \{(x_1, \ldots, x_n) : x_1x_2\cdots x_n > 1\} \cup \{(1, \ldots, 1)\}$.

Proof. (i) Let $T(x_1, \ldots, x_n)$ be ∞-hyponormal. Then by (2.1), we have $x_i^{2p} \geq \frac{1}{n}$ for all $p > 0$ and $i = 1, \ldots, n$. Hence $x_i \geq \lim_{p\to\infty}(\frac{1}{n})^{\frac{1}{2p}} = 1$.

Conversely, suppose $x_i \geq 1$ $(1 \leq i \leq n)$. By (2.1) and (2.2), we may show easily that $I_n \geq C$, where I_n is the $n \times n$ identity matrix. Hence $D^{2p} \geq I_n \geq C$ for all $p \in (0, \infty)$, which proves that $T(x_1, \ldots, x_n)$ is p-hyponormal.

(ii) Let $(x_1, \ldots, x_n) \in \mathcal{R}_p^{(n)}$ for some $p > 0$ and suppose $x_i \neq 1$ for some $i = 1, \ldots, n$. Then

$$n \cdot x_1^{2p}x_2^{2p}\cdots x_n^{2p} \geq \sum_{i=1}^{n} \prod_{1\leq j\neq i\leq n} x_j^{2p} \geq n \cdot \sqrt[n]{(x_1x_2\cdots x_n)^{2p(n-1)}}, \tag{2.4}$$

which shows that $x_1x_2\cdots x_n \geq 1$. Now suppose that $x_1x_2\cdots x_n = 1$. According to (2.4), we have

$$n = \left(\frac{1}{x_1}\right)^{2p} + \left(\frac{1}{x_2}\right)^{2p} + \cdots + \left(\frac{1}{x_n}\right)^{2p}.$$

Hence $x_1 = \cdots = x_n = 1$ (use the relationship of arithmetic and geometric means). This contradiction shows that $x_1x_2\cdots x_n > 1$. Hence $(x_1, \ldots, x_n)$ belongs to the set on the right side in (ii).

On the other hand, since $(1, \ldots, 1) \in \mathcal{R}_p^{(n)}$ for every $p > 0$, we may suppose that $x_1x_2\cdots x_n > 1$ to prove the reverse inclusion. Let us define

$$\varphi(p) = \left(\frac{1}{x_1}\right)^{2p} + \left(\frac{1}{x_2}\right)^{2p} + \cdots + \left(\frac{1}{x_n}\right)^{2p}.$$

Then a simple computation shows that $\lim_{p\to 0^+} \varphi'(p) = -2\ln(x_1\cdots x_n) < 0$, and hence $\varphi'(p) < 0$ on $(0, p_0)$ for sufficiently small $p_0 > 0$. Since $\varphi(0) = n$, we have

$$n > \left(\frac{1}{x_1}\right)^{2p} + \left(\frac{1}{x_2}\right)^{2p} + \cdots + \left(\frac{1}{x_n}\right)^{2p},$$

for such $p \in (0, p_0)$. Thus by Theorem 2.1, $(x_1, \ldots, x_n) \in \cup_{0<p<p_0}\mathcal{R}_p^{(n)}$. □

We consider the graph of the set $\mathcal{R}_p^{(3)}$ for p-hyponormal operators.

EXAMPLE 2.6 (The case $n = 3$). It follows from Theorem 2.1 that $T(x, y, z)$ is p-hyponormal if and only if $3 \geq (1/x)^{2p} + (1/y)^{2p} + (1/z)^{2p}$ $(x, y, z > 0)$. The regions for the p-hyponormality of $T(x, y, z)$ are shown in Figure 2.1.

3. Absolute *p*-paranormality. By direct computation, we obtain the following lemma.

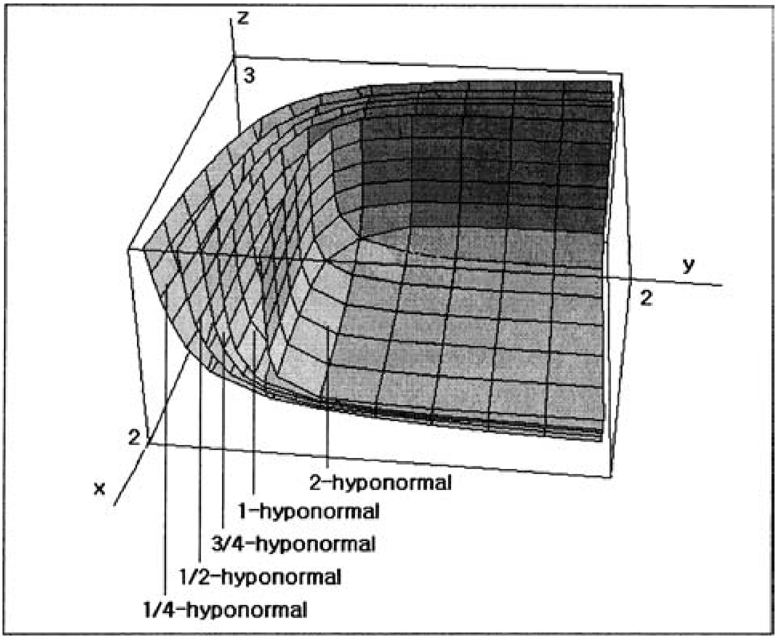

Figure 2.1

LEMMA 3.1. *Let* $B = (b_{ij})$ *be an* $n \times n$ *matrix satisfying* $b_{ij} = a + b$ *(if* $i = j$*) and* $b_{ij} = a$ *(if* $i \neq j$*), where* $a, b \in \mathbb{R}$. *Then* $\det B = b^{n-1}(na + b)$ *for all* $n \geq 1$.

THEOREM 3.2. *For any* $p > 0$, *the following assertions are equivalent*:

(i) $T(x_1, x_2, \ldots, x_n)$ *is absolute p-paranormal*;

(ii) $x_1^{2p} + x_2^{2p} + \cdots + x_n^{2p} \geq n$ *and* $x_i \geq 0$ $(1 \leq i \leq n)$.

Proof. For brevity, let us put $T := T(x_1, x_2, \ldots, x_n)$. It is well known ([**6**, Theorem 1, p. 174] and its proof) that T is absolute p-paranormal if and only if for all $\lambda > 0$, we have

$$\Theta(\lambda) := T^*(T^*T)^pT - (p+1)\lambda^p(T^*T) + p\lambda^{p+1} \geq 0.$$

Obviously we have

$$\Theta(\lambda) = \text{Diag}\,\{\ldots, C^{2p+2}, C^{2p+2}, CD^{2p}C, \boxed{D^{2p+2}}, D^{2p+2}, D^{2p+2}, \ldots\}$$
$$- (p+1)\lambda^p\text{Diag}\,\{\ldots, C^2, C^2, \boxed{D^2}, D^2, D^2, \ldots\} + p\lambda^{p+1}.$$

To show that $\Theta(\lambda) \geq 0$, for all $\lambda > 0$, we need to prove that

(i) $\Theta_1(\lambda) := C^{2p+2} - (p+1)\lambda^pC^2 + p\lambda^{p+1} \geq 0$ $(\lambda > 0)$,

(ii) $\Theta_2(\lambda) := CD^{2p}C - (p+1)\lambda^pC^2 + p\lambda^{p+1} \geq 0$ $(\lambda > 0)$,

(iii) $\Theta_3(\lambda) := D^{2p+2} - (p+1)\lambda^pD^2 + p\lambda^{p+1} \geq 0$ $(\lambda > 0)$.

Let $\Delta_k^{(i)}$ denote the determinant of the $k \times k$ submatrix of the upper left corner in $\Theta_i(\lambda)$ for each $i = 1, 2, 3$ and $k = 1, \ldots, n$. In order to obtain $\Theta_i(\lambda) \geq 0$ for all $\lambda > 0$, it is sufficient to show that $\Delta_n^{(i)} \geq 0$ and $\Delta_k^{(i)} > 0$ for $k = 1, \ldots, n-1$.

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First, by a simple calculation, we have the $n \times n$ matrix $\Theta_1(\lambda) = [\theta_{ij}^{(1)}]$, where

$$\theta_{ij}^{(1)} = \begin{cases} \dfrac{1-(p+1)\lambda^p}{n} + p\lambda^{p+1} & \text{for} \quad (i=j) \\ \dfrac{1-(p+1)\lambda^p}{n} & \text{for} \quad (i \neq j) \end{cases}$$

and $i, j = 1, 2, \ldots, n$. Using Lemma 3.1, we have for $k = 1, 2, \ldots, n$,

$$\Delta_k^{(1)} = (p\lambda^{p+1})^{k-1}\left(\frac{k}{n} - \frac{k}{n}(p+1)\lambda^p + p\lambda^{p+1}\right).$$

Put $f_1(\lambda) := p\lambda^{p+1} - \frac{k}{n}(p+1)\lambda^p + \frac{k}{n}$, for $k = 1, 2, \ldots, n$. Then, $f_1(\lambda)$ has its minimum value at $\lambda = \frac{k}{n}$ and $f_1(\frac{k}{n}) = \frac{k}{n}(1 - (\frac{k}{n})^p)$ for $k = 1, 2, \ldots, n$. Since $\Delta_k^{(1)} = (p\lambda^{p+1})^{k-1} \cdot f_1(\lambda)$, we have that $\Delta_k^{(1)} > 0$ for $k = 1, 2, \ldots, n-1$ and $\Delta_n^{(1)} \geq 0$, and so $\Theta_1(\lambda) \geq 0$ $(\lambda > 0)$.

Consider also the matrix $\Theta_3(\lambda) = [\theta_{ij}^{(3)}]$, where

$$\theta_{ij}^{(3)} = \begin{cases} x_i^{2p+2} - (p+1)\lambda^p x_i^2 + p\lambda^{p+1} & \text{for} \quad (i=j) \\ 0 & \text{for} \quad (i \neq j) \end{cases}$$

and $i, j = 1, 2, \ldots, n$. Put $f_3(\lambda) := x_i^{2p+2} - (p+1)\lambda^p x_i^2 + p\lambda^{p+1}$ for $i = 1, 2, \ldots, n$. Then, by the same method, $f_3(\lambda)$ has its minimum value at $\lambda = x_i^2$ and $f_3(x_i^2) = 0$ for $i = 1, 2, \ldots, n$. Therefore, $\Theta_3(\lambda) \geq 0$ for all $\lambda > 0$.

Next, by another calculation, we have the matrix $\Theta_2(\lambda) = [\theta_{ij}^{(2)}]$, where

$$\theta_{ij}^{(2)} = \begin{cases} \dfrac{x_1^{2p} + \cdots + x_n^{2p} - n(p+1)\lambda^p}{n^2} + p\lambda^{p+1} & \text{for} \quad (i=j) \\ \dfrac{x_1^{2p} + \cdots + x_n^{2p} - n(p+1)\lambda^p}{n^2} & \text{for} \quad (i \neq j) \end{cases}$$

and $i, j = 1, 2, \ldots, n$. Using Lemma 3.1, we have for $k = 1, 2, \ldots, n$,

$$\Delta_k^{(2)} = (p\lambda^{p+1})^{k-1}\left[\frac{k}{n^2}\left(\sum_{i=1}^n x_i^{2p} - n(p+1)\lambda^p\right) + p\lambda^{p+1}\right].$$

Put

$$f_2(\lambda) := p\lambda^{p+1} - \frac{k}{n}(p+1)\lambda^p + \frac{k}{n^2}\sum_{i=1}^n x_i^{2p},$$

for $k = 1, 2, \ldots, n$. Then, we can see that $f_2(\lambda)$ has its minimum value at $\lambda = \frac{k}{n}$. If $x_1^{2p} + \cdots + x_n^{2p} \geq n$, then

$$f_2\left(\frac{k}{n}\right) = \frac{k}{n^2} \cdot \sum_{i=1}^n x_i^{2p} - \left(\frac{k}{n}\right)^{p+1} \geq \frac{k}{n}\left[1 - \left(\frac{k}{n}\right)^p\right] > 0$$

for $k = 1, 2, \ldots, n-1$. Since $f_2\left(\frac{n}{n}\right) = \frac{1}{n}\sum_{i=1}^n x_i^{2p} - 1 \geq 0$, the inequality $x_1^{2p} + \cdots + x_n^{2p} \geq n$ implies that $\Theta_2(\lambda) \geq 0$ $(\lambda > 0)$. The reverse implication is obvious. □

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PROPOSITION 3.3. *Let*

$$\Omega_p = \{(x_1, \dots, x_n) \in \mathbb{R}_+^{(n)} : T(x_1, \dots, x_n) \text{ is absolute } p\text{-paranormal}\}.$$

Then

$$\bigcap_{p>0} \Omega_p = \{(x_1, \dots, x_n) : x_1 \cdots x_n \geq 1\}.$$

Proof. Let $(x_1, \dots, x_n) \in \bigcap_{p>0} \Omega_p$. Then it follows from Theorem 3.2 that

$$x_1^{2p} + \cdots + x_n^{2p} \geq n \tag{3.1}$$

for $p > 0$. Note that $x_i > 0$ $(1 \leq i \leq n)$ (because (3.1) holds for all $p > 0$). Since

$$x_k \geq \left(n - \sum_{1 \leq i \leq n, i \neq k} x_i^{2p} \right)^{\frac{1}{2p}}, \tag{3.2}$$

then, by letting $p \to 0$ in the inequality of (3.2), we have that

$$x_k \geq \frac{1}{x_1 \cdots x_{k-1} \cdot x_{k+1} \cdots x_n}.$$

Conversely, consider the vector $(x_1, \dots, x_n)$ in $\mathbb{R}_+^{(n)}$ satisfying $x_1 \cdots x_n \geq 1$. Then

$$\frac{x_1^{2p} + \cdots + x_n^{2p}}{n} \geq \left(x_1^{2p} \cdots x_n^{2p}\right)^{\frac{1}{n}} \geq 1 \quad \text{for all } p > 0,$$

and hence $x_1^{2p} + \cdots + x_n^{2p} \geq n$ for all $p > 0$. □

PROPOSITION 3.4. *For $p \in (0, \infty)$, the following assertions are equivalent*:

(i) $T(x_1, \dots, x_n)$ *is absolute p-paranormal*;
(ii) $T(x_1, \dots, x_n)$ *is p-quasihyponormal*;
(iii) $T(x_1, \dots, x_n)$ *is p-paranormal*;
(iv) $T(x_1, \dots, x_n)$ *is $A(p)$-operator*;
(v) $x_1^{2p} + \cdots + x_n^{2p} \geq n$ $(x_i \geq 0,\ 1 \leq i \leq n)$.

In particular, $T(x_1, \dots, x_n)$ is $1/2$-paranormal if and only if $T(x_1, x_2, \dots, x_n)$ is w-hyponormal.

Proof. For brevity, we denote $T = T(x_1, \dots, x_n)$. Then, according to the definition of p-quasihyponormality, we have

$$\begin{aligned} T \text{ is } p\text{-quasihyponormal} &\iff T^*\{(T^*T)^p - (TT^*)^p\}T \geq 0 \\ &\iff CD^{2p}C - C = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{2p} - 1 \right) C \geq 0 \\ &\iff x_1^{2p} + \cdots + x_n^{2p} \geq n. \end{aligned}$$

Observe that T is p-paranormal if and only if $U|T|^p = T(x_1^p, \dots, x_n^p)$ is paranormal (i.e., absolute-1-paranormal), which is equivalent to the condition $x_1^{2p} + \cdots + x_n^{2p} \geq n$

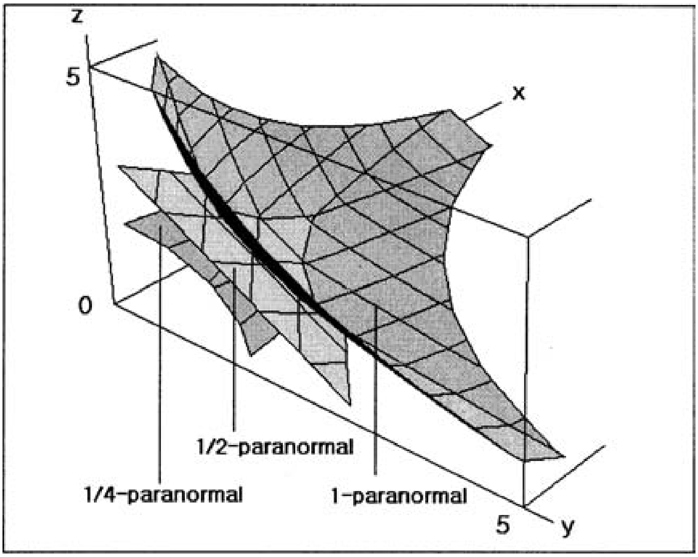

Figure 3.2

by Theorem 3.2. Finally, a proof similar to that of Theorem 3.1 shows that assertions (iv) and (v) are equivalent. □

COROLLARY 3.5. *Let $\mathcal{N}(n)$ be the set of $(x_1, \ldots, x_n)$ in $\mathbb{R}_+^{(n)}$ such that $T(x_1, x_2, \ldots, x_n)$ is normaloid. Then*

$$\mathcal{N}(n)\setminus\bigcup_{p>0}\Omega_p = \underbrace{[0,1]\times\cdots\times[0,1]}_{(n)}\setminus(\underbrace{1,\ldots,1}_{(n)}).$$

Proof. Use the proof of [**11**, Proposition 2.8] and Proposition 3.4. □

EXAMPLE 3.6. (Continued from Example 2.6). Recall that $T(x, y, z)$ is p-paranormal if and only if $x^{2p} + y^{2p} + z^{2p} \geq 3$ and $x, y, z \geq 0$. The following Figure 3.2 shows the regions for p-paranormality.

REMARK 3.7. Let

$$I \neq A = \begin{pmatrix} a & \overline{b} \\ b & c \end{pmatrix}$$

be an otherwise arbitrary 2×2 positive matrix with entries of complex numbers. If $A^2 = A$, then $A^p = A$ for all $p > 0$. Hence

$$A^p = A \text{ for all } p > 0 \Longleftrightarrow |b|^2 = a(1-a) \;\&\; a + c = 1. \tag{3.4}$$

(The case of $a = b = c = 1/2$ is special.) If we take $a > 0$, $c = 1 - a \geq 0$, and $b \in \{a(1-a)e^{i\theta} : 0 \leq \theta \leq 2\pi\}$, by (3.4) we have that $T(x, y)$ is p-hyponormal if and

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only if

$$y \geq \left(\frac{c(x^{2p} - a) + |b|^2}{x^{2p} - a} \right)^{\frac{1}{2p}} \text{ and } x > a^{\frac{1}{2p}}.$$

More generally, we may consider a $k \times k$ matrix $A = (a_{ij})$. If we take an arbitrary a_{ij} satisfying $A^2 = A$ (which implies $A^p = A$ for any $p > 0$), our technique introduced in this paper may provide a lot of examples to show that the classes of such operators discussed in this paper are distinct. We leave this work to the interested readers.

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