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A NOTE ON TENSOR PRODUCTS OF REFLEXIVE ALGEBRAS

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In this short note, we obtain a concrete description of rank-one operators in $\operatorname{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$. Based on this characterisation, we give a simple proof of the tensor product formula:

$$\operatorname{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n) = \operatorname{Alg}\mathcal{L}_1 \otimes_w \cdots \otimes_w \operatorname{Alg}\mathcal{L}_n$$

if $\operatorname{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$ is weakly generated by rank-one operators in itself and $\mathcal{L}_i (i = 1, \ldots, n)$ are subspace lattices.

1. INTRODUCTION

One of the central results in the theory of tensor products of von Neumann algebras is Tomita's commutation formula:

$$\mathcal{M}' \otimes_w \mathcal{N}' = (\mathcal{M} \otimes_w \mathcal{N})', \tag{1}$$

where $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras. It was observed in [**1**] that if we let $\mathcal{L}_1$ and $\mathcal{L}_2$ denote the projection lattices of $\mathcal{M}$ and $\mathcal{N}$ respectively, then (1) can be rewritten as

$$\operatorname{Alg}\mathcal{L}_1 \otimes_w \operatorname{Alg}\mathcal{L}_2 = \operatorname{Alg}(\mathcal{L}_1 \otimes \mathcal{L}_2). \tag{2}$$

This version of Tomita's theorem makes sense for any pair of reflexive algebras $\operatorname{Alg}\mathcal{L}_1$ and $\operatorname{Alg}\mathcal{L}_2$. It remains a deep open question whether the tensor product formula (2) is valid for general reflexive algebras, or even general commutative subspace lattice algebras. However, (2) has been proved in a number of special cases ([**1, 2, 3, 4**]). In particular, it is known that if $\mathcal{L}_1$ is a commutative subspace lattice that is either completely distributive ([**4**]) or finite width ([**2**]), then (2) is valid for $\mathcal{L}_1$ and any subspace lattice $\mathcal{L}_2$. The main purpose of this paper is to study the n-fold tensor product formula of reflexive algebras. The technique employed in this note is simple and different from the other papers about tensor products. We use rank-one operators to investigate tensor products and the technique shows its power in this note.

Let us introduce some notation and terminology. Throughout, $\mathcal{H}$ represents a complex separable Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$. A sublattice

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$\mathcal{L}$ of the projection lattice of $\mathcal{B}(\mathcal{H})$ is said to be a subspace lattice if it contains 0 and I and is strongly closed, where we identify projections with their ranges. If the elements of $\mathcal{L}$ pairwise commute, $\mathcal{L}$ is a commutative subspace lattice. A nest is a totally ordered subspace lattice. If $\mathcal{L}$ is a subspace lattice, $\operatorname{Alg}\mathcal{L}$ denotes the set of operators in $\mathcal{B}(\mathcal{H})$ that leave the elements of $\mathcal{L}$ invariant. If $\mathcal{L}$ is a commutative subspace lattice, $\operatorname{Alg}\mathcal{L}$ is said to be a commutative subspace lattice algebra. If $\mathcal{L}$ is a nest, $\operatorname{Alg}\mathcal{L}$ is said to be a nest algebra.

If $\mathcal{A}$ is a subset of $\mathcal{B}(\mathcal{H})$ then $\operatorname{Lat}\mathcal{A}$, the set of projections left invariant by each element of $\mathcal{A}$, is a subspace lattice. A subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is reflexive if $\mathcal{A} = \operatorname{Alg}\operatorname{Lat}\mathcal{A}$. The reflexive algebras are precisely the algebras of the form $\operatorname{Alg}\mathcal{L}$, where $\mathcal{L}$ is a subspace lattice. If $\mathcal{L}_i \subseteq \mathcal{B}(\mathcal{H}_i)(i = 1, \dots, n)$ are subspace lattices, $\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n$ is the subspace lattice in $\mathcal{B}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n)$ generated by $\{L_1 \otimes \cdots \otimes L_n : L_i \in \mathcal{L}_i, i = 1, \dots, n\}$. If $\mathcal{S}_i \subseteq \mathcal{B}(\mathcal{H}_i)(i = 1, \dots, n)$ are subspaces, then $\mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n$ denotes the linear span of $\{S_1 \otimes \cdots \otimes S_n : S_i \in \mathcal{S}_i\}$; $\mathcal{S}_1 \otimes_w \cdots \otimes_w \mathcal{S}_n$ denotes the weak closure of $\mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n$ in $\mathcal{B}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n)$.

2. Tensor products of reflexive algebras

For $x, y \in \mathcal{H}$, the operator xy^* is defined by the equation

$$(xy^*)(z) = \langle z, y\rangle x, \quad \text{for all} \quad z \in \mathcal{H}.$$

If $\mathcal{L}$ is a subspace lattice and $L \in \mathcal{L}$, we write L_- for the projection $\vee\{E \in \mathcal{L} : L \nleqslant E\}$. The following result of Longstaff [**6**] is essential.

Lemma 1. *Let $\mathcal{L}$ be a subspace lattice. Then $xy^* \in \operatorname{Alg}\mathcal{L}$ if and only if there is an element $L \in \mathcal{L}$ such that $x \in L$ and $y \in L_-^\perp$.*

Let $\mathcal{L}_i \subseteq \mathcal{B}(\mathcal{H}_i)(i = 1, \dots, n)$ be subspace lattices. For $1 \leqslant j \leqslant n$, let $I_1 \otimes \cdots \otimes \mathcal{L}_j \otimes \cdots \otimes I_n = \{I_1 \otimes \cdots \otimes L_j \otimes \cdots \otimes I_n : L_j \in \mathcal{L}_j\}$; certainly, it is a subspace lattice.

Lemma 2. *Let $\mathcal{L}_j \subseteq \mathcal{B}(\mathcal{H}_i)(1 \leqslant j \leqslant n)$ be subspace lattices. Suppose that $N_j \in \mathcal{L}_j$, then*

$$(I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n)_- = I_1 \otimes \cdots \otimes N_{j-} \otimes \cdots \otimes I_n$$

and

$$(I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n)_-^\perp = I_1 \otimes \cdots \otimes N_{j-}^\perp \otimes \cdots \otimes I_n$$

in $I_1 \otimes \cdots \otimes \mathcal{L}_j \otimes \cdots \otimes I_n$.

Proof: We first show that $I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n \leqslant I_1 \otimes \cdots \otimes L_j \otimes \cdots \otimes I_n$ if and only if $N_j \leqslant L_j$. For the forward implication choose unit vectors $x_i \in \mathcal{H}_i (i \neq j)$. For any $x_j \in \mathcal{H}_j$,

$$\begin{aligned} 0 &\leqslant \Big\langle \big(I_1 \otimes \cdots \otimes (L_j - N_j) \otimes \cdots \otimes I_n\big)(x_1 \otimes \cdots \otimes x_n), x_1 \otimes \cdots \otimes x_n \Big\rangle \\ &= \big\langle x_1 \otimes \cdots \otimes (L_j - N_j)x_j \otimes \cdots \otimes x_n, x_1 \otimes \cdots \otimes x_n \big\rangle \\ &= \big\langle (L_j - N_j)x_j, x_j \big\rangle. \end{aligned}$$

So $N_j \leqslant L_j$. The converse implication is also natural. Thus $I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n \not\leqslant I_1 \otimes \cdots \otimes L_j \otimes \cdots \otimes I_n$ if and only if $N_j \not\leqslant L_j$. Hence

$$\begin{aligned}(I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n)_- \\ &= \vee\{I_1 \otimes \cdots \otimes L_j \otimes \cdots \otimes I_n : I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n \not\leqslant I_1 \otimes \cdots \otimes L_j \otimes \cdots \otimes I_n\} \\ &= \vee\{I_1 \otimes \cdots \otimes L_j \otimes \cdots \otimes I_n : N_j \not\leqslant L_j\} \\ &= I_1 \otimes \cdots \otimes (\vee\{L_j : N_j \not\leqslant L_j\}) \otimes \cdots \otimes I_n \\ &= I_1 \otimes \cdots \otimes N_{j-} \otimes \cdots \otimes I_n,\end{aligned}$$

(The proof of the third equality is routine). Since

$$(I_1 \otimes \cdots \otimes N_{j-}^{\perp} \otimes \cdots \otimes I_n)(I_1 \otimes \cdots \otimes N_{j-} \otimes \cdots \otimes I_n) = 0,$$

we have

$$I_1 \otimes \cdots \otimes N_{j-}^{\perp} \otimes \cdots \otimes I_n \leqslant (I_1 \otimes \cdots \otimes N_{j-} \otimes \cdots \otimes I_n)^{\perp}.$$

If $(I_1 \otimes \cdots \otimes N_{j-} \otimes \cdots \otimes I_n)^{\perp} \neq I_1 \otimes \cdots \otimes N_{j-}^{\perp} \otimes \cdots \otimes I_n$, we can choose a non-zero vector $z \in (I_1 \otimes \cdots \otimes N_{j-} \otimes \cdots \otimes I_n)^{\perp} \ominus (I_1 \otimes \cdots \otimes N_{j-}^{\perp} \otimes \cdots \otimes I_n)$. Thus

$$\begin{aligned} z &= (I_1 \otimes \cdots \otimes I_j \otimes \cdots \otimes I_n)z \\ &= (I_1 \otimes \cdots \otimes N_{j-} \otimes \cdots \otimes I_n)z + (I_1 \otimes \cdots \otimes N_{j-}^{\perp} \otimes \cdots \otimes I_n)z \\ &= 0. \end{aligned}$$

This is a contradiction. So

$$\begin{aligned} (I_1 \otimes \cdots \otimes N_j \otimes \cdots \otimes I_n)_-^{\perp} &= (I_1 \otimes \cdots \otimes N_{j-} \otimes \cdots \otimes I_n)^{\perp} \\ &= I_1 \otimes \cdots \otimes N_{j-}^{\perp} \otimes \cdots \otimes I_n. \end{aligned}$$

∎

Lemma 3. *Let $\mathcal{L}_i \subseteq \mathcal{B}(\mathcal{H}_i)(i = 1, \ldots, n)$ be subspace lattices. Then a rank-one operator $xy^* \in \operatorname{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$ if and only if there exist $N_i \in \mathcal{L}_i$ such that $x \in N_1 \otimes \cdots \otimes N_n$ and $y \in N_{1-}^{\perp} \otimes \cdots \otimes N_{n-}^{\perp}$.*

Proof: Set $\mathcal{F}_i = I_1 \otimes \cdots \otimes \mathcal{L}_i \otimes \cdots \otimes I_n$. Thus

$$\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n = \mathcal{F}_1 \vee \cdots \vee \mathcal{F}_n,$$

and

$$\operatorname{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n) = (\operatorname{Alg} \mathcal{F}_1) \cap \cdots \cap (\operatorname{Alg} \mathcal{F}_n).$$

Now suppose that $xy^* \in \operatorname{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$. Since $\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n \supseteq \mathcal{F}_i$, $\operatorname{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n) \subseteq \operatorname{Alg} \mathcal{F}_i$. Thus $xy^* \in \operatorname{Alg} \mathcal{F}_i$ and, by the definition of $\mathcal{F}_i$ and Lemma 1 and Lemma 2, there is an element $N_i \in \mathcal{L}_i$ such that $x \in I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n$ and $y \in (I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n)_-^{\perp} = I_1 \otimes \cdots \otimes N_{i-}^{\perp} \otimes \cdots \otimes I_n$. This is valid for each $i = 1, \ldots, n$, whence

$$x \in N_1 \otimes \cdots \otimes N_n \quad \text{and} \quad y \in N_{1-}^{\perp} \otimes \cdots \otimes N_{n-}^{\perp}.$$

For the converse, if $x \in N_1 \otimes \cdots \otimes N_n$ and $y \in N_{1-}^{\perp} \otimes \cdots \otimes N_{n-}^{\perp}$ then, in particular, $x \in I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n$ and $y \in I_1 \otimes \cdots \otimes N_{i-}^{\perp} \otimes \cdots \otimes I_n$. Lemma 1 and Lemma 2 imply that $xy^* \in \operatorname{Alg} \mathcal{F}_i$, for each i. Hence

$$xy^* \in \bigcap_{i=1}^{n} \operatorname{Alg} \mathcal{F}_i = \operatorname{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n).$$

∎

Proposition 4. *Let $\mathcal{L}_i \subseteq \mathcal{B}(\mathcal{H}_i)(i = 1, \ldots, n)$ be subspace lattices. If $L \in \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n$ and $L \nleqslant L_-$, then*

$$L = \vee\{N_1 \otimes \cdots \otimes N_n : N_1 \otimes \cdots \otimes N_n \leqslant L\}.$$

Proof: Suppose that $0 \neq x \in L$. Since $L \nleqslant L_-$, $L_- \neq I_1 \otimes \cdots \otimes I_n$. For any $0 \neq y \in L_-^{\perp}$, Lemma 1 shows that the rank-one operator $xy^* \in \operatorname{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$. By Lemma 3, there exist $N_i \in \mathcal{L}_i (i = 1, \ldots, n)$ such that $x \in N_1 \otimes \cdots \otimes N_n$ and $y \in N_{1-}^{\perp} \otimes \cdots \otimes N_{n-}^{\perp}$. If $N_1 \otimes \cdots \otimes N_n \nleqslant L$, it follows from the definition of $(N_1 \otimes \cdots \otimes N_n)_-$ that $L \leqslant (N_1 \otimes \cdots \otimes N_n)_-$. By virtue of Lemma 2, we then have

$$\begin{aligned} L &\leqslant (N_1 \otimes \cdots \otimes N_n)_- \\ &\leqslant (I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n)_- \\ &= I_1 \otimes \cdots \otimes N_{i-} \otimes \cdots \otimes I_n \end{aligned}$$

and

$$L^{\perp} \geqslant I_1 \otimes \cdots \otimes N_{i-}^{\perp} \otimes \cdots \otimes I_n, \quad \text{for each } i.$$

So $L^{\perp} \geqslant N_{1-}^{\perp} \otimes \cdots \otimes N_{n-}^{\perp}$ and we have shown that $y \in L^{\perp}$. Thus for any $y \in L_-^{\perp}$, we show that $y \in L^{\perp}$. This implies that $L_-^{\perp} \leqslant L^{\perp}$ and $L \leqslant L_-$. This contradicts our hypothesis. Hence $N_1 \otimes \cdots \otimes N_n \leqslant L$ and for any $x \in L$,

$$x \in \vee\{N_1 \otimes \cdots \otimes N_n : N_1 \otimes \cdots \otimes N_n \leqslant L\}.$$

Thus

$$L \leqslant \vee\{N_1 \otimes \cdots \otimes N_n : N_1 \otimes \cdots \otimes N_n \leqslant L\}.$$

The converse inequality is obvious and this completes the proof. ∎

Lemma 5. *Let $x_i, y_i \in \mathcal{H}_i (i = 1, \ldots, n)$. Then*

$$(x_1 \otimes \cdots \otimes x_n)(y_1 \otimes \cdots \otimes y_n)^* = (x_1 y_1^*) \otimes \cdots \otimes (x_n y_n^*).$$

Proof: For any $z_i \in \mathcal{H}_i$, it follows from the definition that

$$\begin{aligned} &\big[(x_1 \otimes \cdots \otimes x_n)(y_1 \otimes \cdots \otimes y_n)^*\big](z_1 \otimes \cdots \otimes z_n) \\ &\qquad = \langle z_1 \otimes \cdots \otimes z_n, y_1 \otimes \cdots \otimes y_n \rangle (x_1 \otimes \cdots \otimes x_n) \\ &\qquad = \langle z_1, y_1 \rangle \cdots \langle z_n, y_n \rangle (x_1 \otimes \cdots \otimes x_n) \\ &\qquad = \big(\langle z_1, y_1 \rangle x_1\big) \otimes \cdots \otimes \big(\langle z_n, y_n \rangle x_n\big) \\ &\qquad = \big[(x_1 y_1^*) z_1\big] \otimes \cdots \otimes \big[(x_n y_n^*) z_n\big] \\ &\qquad = \big[(x_1 y_1^*) \otimes \cdots \otimes (x_n y_n^*)\big](z_1 \otimes \cdots \otimes z_n). \end{aligned}$$

Since the linear span of simple tensors is everywhere dense in $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$, so

$$(x_1 \otimes \cdots \otimes x_n)(y_1 \otimes \cdots \otimes y_n)^* = (x_1 y_1^*) \otimes \cdots \otimes (x_n y_n^*).$$

□

THEOREM 6. *Suppose that $\mathcal{L}_i \subseteq \mathcal{B}(\mathcal{H}_i)(i = 1, \ldots, n)$ are subspace lattices and $\mathrm{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$ is weakly generated by rank-one operators in itself. Then*

$$\mathrm{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n) = \mathrm{Alg}\,\mathcal{L}_1 \otimes_w \cdots \otimes_w \mathrm{Alg}\,\mathcal{L}_n.$$

PROOF: Each of the operators which generate $\mathrm{Alg}\,\mathcal{L}_1 \otimes_w \cdots \otimes_w \mathrm{Alg}\,\mathcal{L}_n$ leaves invariant each of the projections which generate $\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n$; therefore

$$\mathrm{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n) \supseteq \mathrm{Alg}\,\mathcal{L}_1 \otimes_w \cdots \otimes_w \mathrm{Alg}\,\mathcal{L}_n.$$

It remains to show that $\mathrm{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n) \subseteq \mathrm{Alg}\,\mathcal{L}_1 \otimes_w \cdots \otimes_w \mathrm{Alg}\,\mathcal{L}_n$. Since $\mathrm{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$ is weakly generated by rank-one operators in itself, it suffices to show that each rank-one operator in $\mathrm{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$ belongs to $\mathrm{Alg}\,\mathcal{L}_1 \otimes_w \cdots \otimes_w \mathrm{Alg}\,\mathcal{L}_n$. Now for any $N_i \in \mathcal{L}_i$ and $x_i, y_i \in \mathcal{H}_i$, we have that

$$\begin{aligned}
&(N_1 \otimes \cdots \otimes N_n)\big[(x_1 \otimes \cdots \otimes x_n)(y_1 \otimes \cdots \otimes y_n)^*\big](N_{1-}^{\perp} \otimes \cdots \otimes N_{n-}^{\perp}) \\
&\qquad = (N_1 \otimes \cdots \otimes N_n)\big[(x_1 y_1^*) \otimes \cdots \otimes (x_n y_n^*)\big](N_{1-}^{\perp} \otimes \cdots \otimes N_{n-}^{\perp}) \\
&\qquad = N_1(x_1 y_1^*)N_{1-}^{\perp} \otimes \cdots \otimes N_n(x_n y_n^*)N_{n-}^{\perp} \\
&\qquad \in \mathrm{Alg}\,\mathcal{L}_1 \otimes_w \cdots \otimes_w \mathrm{Alg}\,\mathcal{L}_n.
\end{aligned}$$

For any rank-one operator $zw^* \in \mathrm{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$, it follows from Lemma 3 that there exist $N_i \in \mathcal{L}_i(i = 1, \ldots, n)$ such that $z \in N_1 \otimes \cdots \otimes N_n$ and $w \in N_{1-}^{\perp} \otimes \cdots \otimes N_{n-}^{\perp}$. Since $z, w \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$, there exist sequences $\{z_m\}$ and $\{w_m\}$ such that

$$z_m \overset{\|\cdot\|}{\to} z \quad \text{and} \quad w_m \overset{\|\cdot\|}{\to} w,$$

where z_m, w_m are finite linear combinations of simple tensors. It is routine to show that

$$(N_1 \otimes \cdots \otimes N_n)(z_m w_m^*)(N_{1-}^{\perp} \otimes \cdots \otimes N_{n-}^{\perp}) \overset{\|\cdot\|}{\to} (N_1 \otimes \cdots \otimes N_n)(zw^*)(N_{1-}^{\perp} \otimes \cdots \otimes N_{n-}^{\perp}) = zw^*.$$

The preceding pragragh shows that

$$(N_1 \otimes \cdots \otimes N_n)(z_m w_m^*)(N_{1-}^{\perp} \otimes \cdots \otimes N_{n-}^{\perp}) \in \mathrm{Alg}\,\mathcal{L}_1 \otimes_w \cdots \otimes_w \mathrm{Alg}\,\mathcal{L}_n,$$

so $zw^* \in \mathrm{Alg}\,\mathcal{L}_1 \otimes_w \cdots \otimes_w \mathrm{Alg}\,\mathcal{L}_n$. This completes the proof. □

COROLLARY 7. *Let $\mathcal{L}_i \subseteq \mathcal{B}(\mathcal{H}_i)(i = 1, \ldots, n)$ be subspace lattices. If $\mathrm{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_{n-1})$ is weakly generated by rank-one operators in itself, then*

$$\mathrm{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n) = \mathrm{Alg}\,\mathcal{L}_1 \otimes_w \cdots \otimes_w \mathrm{Alg}\,\mathcal{L}_n.$$

PROOF: It follows from [**4**, Proposition 1.1 and Theorem 2.1] that

$$\operatorname{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n) = \operatorname{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_{n-1}) \otimes_w \operatorname{Alg}\mathcal{L}_n.$$

By virtue of Theorem 6, we obtain that

$$\operatorname{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n) = \operatorname{Alg}\mathcal{L}_1 \otimes_w \cdots \otimes_w \operatorname{Alg}\mathcal{L}_n.$$

∎

The following corollary is one of the main results in [**3**].

COROLLARY 8. ([**3**, Theorem 17].) *Let $\mathcal{L}_i \subseteq \mathcal{B}(\mathcal{H}_i)(i = 1, \dots, n)$ be completely distributive commutative subspace lattices. Then*

$$\operatorname{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n) = \operatorname{Alg}\mathcal{L}_1 \otimes_w \cdots \otimes_w \operatorname{Alg}\mathcal{L}_n.$$

PROOF: It follows from [**3**, Theorem 10] that $\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n$ is a completely distributive commutative subspace lattice. Thus, by virtue of [**5**, Theorem 3], $\operatorname{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$ is weakly generated by the rank-one operators in itself. So the corollary follows from Theorem 6. ∎

COROLLARY 9. ([**1**, Theorem 2.6].) *Let $\mathcal{N}_i(i = 1, \dots, n)$ be nests. Then*

$$\operatorname{Alg}(\mathcal{N}_1 \otimes \cdots \otimes \mathcal{N}_n) = \operatorname{Alg}\mathcal{N}_1 \otimes_w \cdots \otimes_w \operatorname{Alg}\mathcal{N}_n.$$

If $\mathcal{L}$ is a subspace lattice, let $\mathcal{R}(\mathcal{L})$ denote the linear span of rank-one operators in $\operatorname{Alg}\mathcal{L}$ and $\overline{\mathcal{R}(\mathcal{L})}$ the norm closure of $\mathcal{R}(\mathcal{L})$. If $\mathcal{S}_i \subseteq \mathcal{B}(\mathcal{H}_i)(i = 1, \dots, n)$ are subspaces, $\mathcal{S}_1 \overline{\otimes} \cdots \overline{\otimes} \mathcal{S}_n$ denotes the norm closure of $\mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n$.

PROPOSITION 10. *Let $\mathcal{L}_i \subseteq \mathcal{B}(\mathcal{H}_i)(i = 1, \dots, n)$ be subspace lattices. Then*

$$\overline{\mathcal{R}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)} = \mathcal{R}(\mathcal{L}_1) \overline{\otimes} \cdots \overline{\otimes} \mathcal{R}(\mathcal{L}_n).$$

PROOF: The result is essentially implied in the proof of Theorem 6. ∎

Note that in Proposition 10 we do not need the hypothesis that $\operatorname{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$ is weakly generated by rank-one operators in itself.

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