ON THE LOCATION OF ZEROS OF POLYNOMIALS

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1. Introduction. The different results proved in this paper do not have very much in common. Since they all deal with the location of the zeros of a polynomial, we have decided to put them in one place. Improving upon a classical result of Cauchy we obtain in § 2 a circle containing all the zeros of a polynomial. In § 3 we obtain an extension of the well known theorem of Eneström and Kakeya concerning the zeros of a polynomial whose coefficients are non-negative and monotonic. We devote § 4 to the study of the trinomial equation $1 - z + cz^n = 0$. Finally, in § 5 we present an elementary proof of a theorem of M. Zedek [5] on the zeros of linear combinations of polynomials.

2. A classical result of Cauchy on the location of the zeros of the polynomial

(1) $$p(z) = z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \dots + a_o$$

states that all the zeros are in the circle

(2) $$|z| \leq 1 + A$$

where $A = \max\limits_{0 \leq j < n} |a_j|$.

Here we give a smaller circle containing all the zeros of the polynomial.

THEOREM 1. If $B = \max\limits_{0 \leq j < n-1} |a_j|$ then all the zeros of the polynomial (1) are contained in the circle

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(3) $$|z| \le \frac{1}{2} \{1 + |a_{n-1}| + \sqrt{(1-|a_{n-1}|)^2 + 4B}\} .$$

The expression in (3) takes a very simple form if $a_{n-1} = 0$. If $|a_{n-1}| = 1$ it reduces to $1 + \sqrt{B}$.

Proof of Theorem 1. If $|z| > \frac{1}{2} \{1 + |a_{n-1}| + \sqrt{(1-|a_{n-1}|)^2 + 4B}\}$ then $|z| > 1$ and

$$(|z| - 1)(|z| - |a_{n-1}|) - B > 0 .$$

Multiplying by $|z|^{n-1}$ and dividing by $(|z| - 1)$ we get

$$|z|^n - |a_{n-1}| |z|^{n-1} - B|z|^{n-1} /(|z| - 1) > 0 .$$

But

$$B|z|^{n-1} / (|z| - 1) > B(1 + |z| + |z|^2 + \ldots + |z|^{n-2})$$

$$\ge |a_{n-2} z^{n-2} + a_{n-3} z^{n-3} + \ldots + a_o|$$

and

$$|z|^n - |a_{n-1}||z|^{n-1} \le |z^n + a_{n-1} z^{n-1}| .$$

Hence we have

$$|p(z)| \ge |z^n + a_{n-1} z^{n-1}| - |a_{n-2} z^{n-2} + \ldots + a_o| > 0$$

and the proposition is proved.

By applying Theorem 1 to the polynomial $z^n P(1/z)$ we can deduce the following

COROLLARY 1. If $\beta = \max_{2 \le j \le n} |a_j|$ then the polynomial

$$P(z) = 1 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

has no zeros in the circle

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$$|z| \leq 2 / \{1 + |a_1| + \sqrt{(1-|a_1|)^2 + 4\beta}\} .$$

The next corollary is obtained by applying the above theorem to the polynomial $(z - a_{n-1})\, p(z)$.

COROLLARY 2. The polynomial (1) has all its zeros in the circle

$$|z| \leq \frac{1}{2}(1 + \sqrt{1 + 4B'})$$

where

$$B' = \max_{0 \leq k \leq n-1} |a_{n-1} a_k - a_{k-1}| , \qquad (a_{-1} = 0) .$$

The following corollary is also an immediate consequence of our Theorem.

COROLLARY 3. The polynomial (1) has all its zeros in the circle

$$|z| \leq 1 + \sqrt{B''}$$

where

$$B'' = \max_{0 \leq k \leq n-1} |(1 - a_{n-1})a_k + a_{k-1}| , \qquad (a_{-1} = 0) .$$

In order to prove Corollary 3 we may apply Theorem 1 to the polynomial $(z + 1 - a_{n-1})\, p(z)$.

We also prove the following

THEOREM 2. Let

$$\mathcal{B} = \left(\sum_{j=0}^{n-2} |a_j|^p \right)^{1/p}, \qquad p > 1 .$$

Then all the roots of the polynomial (1) are contained in the circle $|z| \leq k$ where $k \geq \max(1, |a_{n-1}|)$ is a root of the equation

(4) $$(|z| - |a_{n-1}|)^q (|z|^q - 1) - \mathcal{B}^q = 0 , \qquad \frac{1}{p} + \frac{1}{q} = 1 .$$

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Proof of Theorem 2. It is clear that

$$|p(z)| \geq |z^n| - |a_{n-1}|\,|z^{n-1}| - |a_{n-2}z^{n-2} + \ldots + a_1 z + a_o|,$$

and

$$|a_{n-2}z^{n-2} + \ldots + a_1 z + a_o| \leq \left(\sum_{j=0}^{n-2} |a_j|^p\right)^{1/p} \left(\sum_{j=0}^{n-2} |z|^{jq}\right)^{1/q}$$

$$= \mathcal{B}\left(\frac{|z|^{q(n-1)} - 1}{|z|^q - 1}\right)^{1/q}$$

$$< \mathcal{B}\frac{|z|^{n-1}}{(|z|^q - 1)^{1/q}} .$$

Hence

$$|p(z)| \geq |z|^n - |a_{n-1}|\,|z|^{n-1} - \mathcal{B}\frac{|z|^{n-1}}{(|z|^q - 1)^{1/q}} > 0 .$$

if

$$(|z| - |a_{n-1}|)^q (|z|^q - 1) > \mathcal{B}^q .$$

From this the desired result follows.

3. The theorem of Eneström and Kakeya [2] mentioned in the introduction states that if

(5) $$a_n \geq a_{n-1} \geq a_{n-2} \geq \ldots \geq a_2 \geq a_1 \geq a_o \geq 0$$

then the polynomial

(6) $$f(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \ldots + a_2 z^2 + a_1 z + a_o$$

has all its zeros in the unit circle. If we do not assume the coefficients to be non-negative the conclusion does not hold. However, we prove

THEOREM 3. *If*

(7) $$a_n \geq a_{n-1} \geq a_{n-2} \geq \ldots \geq a_2 \geq a_1 \geq a_o$$

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then the polynomial (6) has all its zeros in the circle

(8) $$|z| \leq (a_n - a_o + |a_o|) / |a_n| .$$

If $a_o \geq 0$, this result reduces to the theorem of Eneström and Kakeya.

Proof of Theorem 3. Consider the polynomial $(1 - z)f(z)$ which can be written as

$$- a_n z^{n+1} + \phi(z)$$

where

$$\phi(z) \equiv (a_n - a_{n-1}) z^n + (a_{n-1} - a_{n-2}) z^{n-1} + \ldots + (a_1 - a_o) z + a_o .$$

If $|z| = 1$, then

$$|\phi(z)| \leq |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \ldots + |a_1 - a_o| + |a_o|$$

$$= (a_n - a_{n-1} + a_{n-1} - a_{n-2} + \ldots + a_1 - a_o) + |a_o|$$

$$= a_n - a_o + |a_o| .$$

It is clear that also

(9) $$|z^n \phi(1/z)| \leq a_n - a_o + |a_o|$$

on the unit circle. Since the function $z^n \phi(1/z)$ is analytic in $|z| \leq 1$ the inequality (9) holds also inside the unit circle, i.e.

$$|\phi(1/z)| \leq (a_n - a_o + |a_o|) / |z|^n$$

for $|z| \leq 1$. Replacing z by $1/z$ we get

$$|\phi(z)| \leq (a_n - a_o + |a_o|)|z|^n$$

for $|z| \geq 1$. Hence if $|z| > (a_n - a_o + |a_o|) / |a_n|$, then

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$$|(1 - z) f(z)| = |- a_n z^{n+1} + \phi(z)| \geq |a_n| \, |z|^{n+1}$$

$$- (a_n - a_o + |a_o|) \, |z|^n > 0 ,$$

i.e. the function $(1 - z) f(z)$ has all its zeros in (8). The same is therefore true for $f(z)$ and the theorem is proved.

4. Let us consider the trinomial equation

$$(10) \qquad 1 - z + c z^n = 0 \qquad (c \neq 0) .$$

For every $n \geq 2$ this equation is known [1, 4] to have a root in both the regions $|z - 1| \geq 1$, $|z - 1| \leq 1$. To start with we present a very simple proof of this fact.

It is easy to verify that the result is true for $n = 2$. In fact, if we put $\xi = z - 1$ the equation reduces to

$$c \xi^2 + (2c - 1)\xi + c = 0 .$$

The product of the moduli of the roots of this equation is 1 . Hence both the roots cannot lie in $|\xi| < 1$. They cannot both lie in $|\xi| > 1$ either. From this the result follows.

So let $n \geq 3$ and suppose if possible that all the roots of the trinomial equation (10) lie in $|z - 1| \leq 1$. By the Gauss-Lucas theorem all the roots of the derived equation

$$(11) \qquad cn z^{n-1} - 1 = 0$$

also lie in $|z - 1| \leq 1$. It is however obvious that if $n \geq 3$ this equation cannot have all its roots in $|z - 1| \leq 1$. Thus we get a contradiction and the original equation must have a root in $|z - 1| > 1$.

In order to prove that it also has a root in $|z - 1| \leq 1$ we may prove that the equation

$$- \xi + c(\xi + 1)^n = 0$$

has a root in $|\xi| \leq 1$ or that the equation

$$- \xi^{n-1} + c(\xi + 1)^n = 0$$

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has a root in $|\xi| \geq 1$. This is equivalent to the fact that the equation

$$(12) \qquad - (z - 1)^{n-1} + c z^n = 0$$

has a root in $|z - 1| \geq 1$. We prove this again by contradiction. If the equation (12) has all its roots in $|z - 1| < 1$ then the roots of the successively derived equations will also lie in $|z - 1| < 1$. In particular, the roots of the equation

$$1 - z + \frac{cn}{2} z^2 = 0$$

lie in $|z - 1| < 1$. But we have proved above that this equation has a root in $|z - 1| \geq 1$ whatever be the value of $\frac{cn}{2}$. Hence we get a contradiction which proves that the trinomial equation (10) has a root in $|z - 1| \leq 1$.

We also prove

THEOREM 4. _If $n \geq 3$ the trinomial equation (10) has a root outside every circle which passes through the origin._

Proof of Theorem 4. Suppose if possible that there exists a circle

$$|z - \alpha| = |\alpha|$$

passing through the origin which includes all the roots of the trinomial equation (10). Putting $z = \alpha\xi$ we conclude that the circular region

$$|\xi - 1| \leq 1$$

contains all the roots of the equation

$$1 - \alpha\xi + c\,\alpha^n \xi^n = 0$$

and so also of the derived equation

$$-\alpha + cn\,\alpha^n \xi^{n-1} = 0$$

by the Gauss-Lucas theorem. But all the roots of this last equation cannot lie in $|\xi - 1| \leq 1$ if $n \geq 3$. This is a contradiction which proves the theorem.

Theorem 4 is a special case of the following more general principle which is an immediate consequence of the Gauss-Lucas theorem.

If $P(z)$ *is a polynomial of degree* n *such that* $P^{(k)}(0) = 0$ *for some* k, *where* $0 < k < n$, *then every convex region excluding the origin also excludes a zero of* $P(z)$.

Thus a lacunary polynomial cannot have all its zeros in a convex region which excludes the origin.

5. *Linear combinations of two polynomials.* The following theorem is due to M. Zedek ([5], Theorem 4).

THEOREM 5. *Let* $f_m(z) \equiv z^m + a_{m-1} z^{m-1} + \ldots + a_o$ *and* $g_n(z) \equiv z^n + b_{n-1} z^{n-1} + \ldots + b_o$ *be two polynomials whose zeros lie, respectively, in the discs* $|z - c_1| \leq R_1$ *and* $|z - c_2| \leq R_2$ *and suppose* $m > n \geq 1$. *For a fixed* λ *let* $F(z, \lambda) \equiv f_m(z) + \lambda g_n(z)$. *Then:*

I. *If* ρ_1 *is the unique positive root of the equation*

$$C(x) \equiv x^m - |\lambda|(x + |c_2 - c_1| + R_1 + R_2)^n = 0, \tag{13}$$

then the m *zeros of* $F(z, \lambda)$ *lie in* $|z - c_1| \leq R_1 + \rho_1$.

II. *Setting* $L = m^n(|c_2 - c_1| + R_1 + R_2)^m / n^n(m-n)^{m-n}$, *the equation*

$$D(x) \equiv |\lambda| x^n - (x + |c_2 - c_1| + R_1 + R_2)^m = 0 \tag{14}$$

has two positive roots $\rho_2, \rho_3 (\rho_2 \leq \rho_3)$, *provided* $|\lambda| \geq L$. *At least* n *zeros of* $F(z, \lambda)$ *lie in* $|z - c_2| \leq R_2 + \rho_2$.

Part I of this theorem has been proved independently and by a different method by Z. Rubinstein ([3], Theorem 2). We give a very simple proof of Theorem 5. It is perhaps the simplest one can think of.

Proof of Theorem 5. Let us denote by $\xi_1, \xi_2, \ldots, \xi_m$ the zeros of $f_m(z)$ and by $z_1, z_2, \ldots, z_n$ the zeros of $g_n(z)$.

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If $|z - c_1| = R_1 + \rho$ then for $1 \leq j \leq n$

$$|z - z_j| = |z - c_1 + c_1 - c_2 - (z_j - c_2)|$$
$$\leq |z - c_1| + |c_2 - c_1| + |z_j - c_2|$$
$$\leq R_1 + \rho + |c_2 - c_1| + R_2 .$$

Consequently

(15) $$|\lambda g_n(z)| = |\lambda| \prod_{j=1}^{n} |z - z_j| \leq |\lambda| (\rho + |c_2 - c_1| + R_1 + R_2)^n .$$

Again for $|z - c_1| = R_1 + \rho$, we have

$$|f_m(z)| = \prod_{j=1}^{m} |z - \xi_j|$$

(16) $$\geq \prod_{j=1}^{m} (|z - c_1| - |\xi_j - c_1|)$$

$$\geq \prod_{j=1}^{m} (R_1 + \rho - R_1)$$

$$= \rho^m .$$

It is clear that equation (13) has only one positive root ρ_1 Thus if $\rho > \rho_1$,

$$|\lambda| (\rho + |c_2 - c_1| + R_1 + R_2)^n < \rho^m ,$$

i.e.

$$|\lambda g_n(z)| < |f_m(z)|$$

if $|z - c_1| = R_1 + \rho$ where $\rho > \rho_1$. The polynomial $f_m(z) + \lambda g_n(z)$ cannot therefore vanish for any z such that $|z - c_1| = R_1 + \rho$ and $\rho > \rho_1$, i.e. it has all its zeros in $|z - c_1| \leq R_1 + \rho_1$.

In order to prove the second part of Theorem 5 let

$|\lambda| \geq L$ so that the equation $D(x) = 0$ has two positive roots ρ_2, ρ_3 $(\rho_2 \leq \rho_3)$. There are two different possibilities.

Case (i). $\rho_2 \neq \rho_3$. For $\rho_2 < \rho < \rho_3$

(17) $$|\lambda|\rho^n > (\rho + |c_2 - c_1| + R_1 + R_2)^m .$$

Hence if $|z - c_2| = R_2 + \rho$ where $\rho_2 < \rho < \rho_3$, then

(18) $$|\lambda\, g_n(z)| \geq |\lambda|\, \rho^n > (\rho + |c_2 - c_1| + R_1 + R_2)^m \geq f_m(z) .$$

By Rouché's theorem the functions $f_m(z) + \lambda\, g_n(z)$ and $g_n(z)$ have the same number of zeros in $|z - c_2| \leq R_2 + \rho$, i.e. n. On account of (18), $f_m(z) + \lambda g_n(z) \neq 0$ in $\rho_2 < \rho < \rho_3$, hence it has precisely n zeros in $|z - c_2| \leq R_2 + \rho_2$.

Case (ii). $\rho_2 = \rho_3 = \rho'$ (say). In this case $\lambda = L$. Now suppose if possible that only $n'(<n)$ zeros of $f_m(z) + \lambda\, g_n(z)$ lie in $|z - c_2| \leq R_2 + \rho'$. There exists $\rho'' > \rho'$ such that $f_m(z) + \lambda\, g_n(z)$ has no zero in the annulus $R_2 + \rho' < |z - c_2| < R_2 + \rho''$. Since the zeros of a polynomial vary continuously with the coefficients, we may increase λ by such a small amount ϵ that the smaller of the two distinct positive roots which

(14') $$|\lambda + \epsilon|x^n - (x + |c_2 - c_1| + R_1 + R_2)^m = 0$$

has is less than $(\rho' + \rho'')/2$ and that $f_m(z) + (\lambda + \epsilon)\, g_n(z)$ has exactly n' zeros in $|z - c_2| \leq R_2 + (\rho' + \rho'')/2$. But from Case (i), $f_m(z) + (\lambda + \epsilon)\, g_n(z)$ must have at least n zeros in $|z - c_2| \leq R_2 + (\rho' + \rho'')/2$. Thus we get a contradiction.

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