Canad. J. Math. Vol. **59** (6), 2007 pp. 1260–1283

Generic Extensions and Canonical Bases for Cyclic Quivers

Dedicated to Claus Michael Ringel on the occasion of his 60th birthday

Bangming Deng, Jie Du, and Jie Xiao

Abstract. We use the monomial basis theory developed by Deng and Du to present an elementary algebraic construction of the canonical bases for both the Ringel–Hall algebra of a cyclic quiver and the positive part $\mathbf{U}^+$ of the quantum affine $\mathfrak{sl}_n$. This construction relies on analysis of quiver representations and the introduction of a new integral PBW-like basis for the Lusztig $\mathbb{Z}[v, v^{-1}]$-form of $\mathbf{U}^+$.

1 Introduction

A landmark in Lie theory was G. Lusztig's introduction [18] of the canonical basis of the quantum enveloping algebra of a simple complex Lie algebra. He showed that this basis has some remarkable properties, such as the positivity property for structure constants (see [15] for Hecke algebras), the compatibility with various natural filtrations, and the fact that this basis is well adapted to all finite dimensional irreducible representations. In this case Lusztig actually gave two constructions of the canonical bases, namely, the elementary algebraic construction, involving analysis of quiver representations, and the geometric construction, based on perverse sheaves on representation varieties of a quiver. Nevertheless, the key steps in the proof of the existence of the canonical bases are the use of the Ringel–Hall algebra associated with the representation category of a quiver [25,27]. The geometric construction was soon extended [19, 20] to an arbitrary Kac–Moody algebra (see [23]). Though there are other elementary constructions including Kashiwara's crystal basis approach [14] for arbitrary Kac–Moody algebras and, in the affine case, the constructions given in [1, 2], the algebraic construction for the general case involving analysis of quiver representations remains unclear.

In this paper, we will present such a construction for cyclic quivers. The main ingredient in this construction is the strong monomial basis property established in [4]. This property is a systematic construction of many monomial bases for the subalgebra, the composition algebra, generated by simple modules of the generic (twisted) Ringel–Hall algebra of a cyclic quiver. It is proved [26] that the composition algebra is isomorphic to the positive part $\mathbf{U}^+$ of the quantum enveloping algebra $\mathbf{U}$ of the affine Lie algebra $\widehat{\mathfrak{sl}}_n$. This realization together with the strong monomial basis property allows us to introduce integral monomial/PBW-like bases for the Lusztig $\mathcal{Z}$-form

Received by the editors May 10, 2005.

Supported partially by ARC (Grant: DP0665124) and the NSF of China. The research was carried out while Deng and Xiao were visiting the University of New South Wales. The hospitality and support of UNSW are gratefully acknowledged.

AMS subject classification: Primary: 17B37; Secondary: 16G20.

$U_{\mathcal{Z}}^+$ ($\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$) of $\mathbf{U}^+$ and to see the triangular relations of the bar involution on these basis elements. In this way, a new basis is constructed through a standard linear algebra method. We then prove that this basis agrees with Lusztig's canonical basis; see [16, 30]. We further extend the approach to produce a similar construction for the canonical basis of the whole Ringel–Hall algebra. Note that the PBW bases constructed in this paper do not involve braid group actions. It would be interesting to find a relation between our PBW bases and those constructed in [2]. Note also that the construction for the cyclic quiver case is a key step towards the completion of a similar construction suitable for all affine Kac–Moody algebras with symmetric generalized Cartan matrices; see [17].

The paper is organized as follows. We start with nilpotent representations of a cyclic quiver Δ and their associated Ringel–Hall algebra $\mathcal{H}$ in §2. We investigate in §3 the generic extension monoid $\mathcal{M}$ of Δ through a minimal set I^e of generators consisting of simple and sincere semisimple representations. Thus we obtain a monoid epimorphism $\wp$ from the free monoid over I^e to $\mathcal{M}$. With $\wp$, we construct in §4 a distinguished word in every fibre of $\wp$, and discuss the strong monomial basis property for Ringel–Hall algebras in §5. From §6 onwards, we use the twisted Ringel–Hall algebra $H_{\mathcal{Z}}$ and its composition algebra $C_{\mathcal{Z}}$ as a realization of $U_{\mathcal{Z}}^+$ (§6) to introduce a new integral PBW basis from which we construct a so-called IC basis for $U_{\mathcal{Z}}^+$ (§7). In §8, we show that this elementarily constructed IC basis coincides with the (geometrically constructed) canonical basis for $U_{\mathcal{Z}}^+$, and in the last section, we further extend the construction to the whole Ringel–Hall algebra $H_{\mathcal{Z}}$.

1.1 Some Notation

For a finite dimensional quiver representation (or a finite dimensional module over an algebra) M, let $\operatorname{soc}^1 M = \operatorname{soc} M$ (resp. $\operatorname{rad}^1 M = \operatorname{rad} M$) denote the socle (resp. radical) of M. Let $\operatorname{soc}^0 M = 0$, $\operatorname{rad}^0 M = M$ and, for $i > 1$, let $\operatorname{soc}^i M$ be the inverse image of $\operatorname{soc}(M/\operatorname{soc}^{i-1} M)$ in M under the natural projection $M \to M/\operatorname{soc}^{i-1} M$ and $\operatorname{rad}^i M = \operatorname{rad}(\operatorname{rad}^{i-1} M)$. We also set $\operatorname{top} M = M/\operatorname{rad} M$.

Let $\mathrm{Ll}(M)$ denote the *Loewy length* of M, that is,

$$\mathrm{Ll}(M) = \min\{s \mid \operatorname{rad}^s M = 0\} = \min\{t \mid \operatorname{soc}^t M = M\}.$$

Then M admits two natural filtrations: *the radical filtration*

$$M \supseteq \operatorname{rad} M \supseteq \cdots \supseteq \operatorname{rad}^{l-1} M \supseteq \operatorname{rad}^l M = 0$$

and *the socle filtration*

$$M = \operatorname{soc}^l M \supseteq \operatorname{soc}^{l-1} M \supseteq \cdots \supseteq \operatorname{soc}^1 M \supseteq 0,$$

where $l = \mathrm{Ll}(M)$. We have obviously the following lemma.

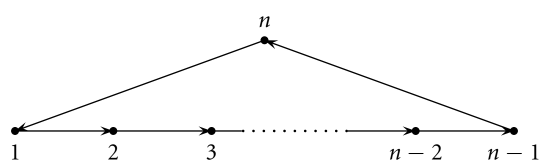
Lemma 1.1 *For each $0 \leq s \leq l$, $\operatorname{soc}^s M$ is the unique maximal submodule of M of Loewy length s, while $M/\operatorname{rad}^s M$ is the unique maximal quotient module of M of Loewy*

https://doi.org/10.4153/CJM-2007-054-7 Published online by Cambridge University Press

length $l-s$. In other words, any filtration $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{l-1} \supseteq M_l = 0$ satisfying the property that M_s (resp. M/M_s) is a maximal submodule (resp. quotient module) of Loewy length s (resp. $l-s$) coincides with the socle (resp. radical) filtration of M.

2 Nilpotent Representations and Ringel–Hall Algebras

Let $\Delta = \Delta(n)$ be the cyclic quiver



with vertex set $I := \mathbb{Z}/n\mathbb{Z} = \{1, 2, \ldots, n\}$ and arrow set $\{i \to i+1 \mid 1 \le i \le n\}$, and let $k\Delta$ be the path algebra of Δ over a field k. For a representation $M = (V_i, f_i)_i$ of Δ, let $\dim M = \sum_{i=1}^n \dim V_i$ and $\mathbf{dim}\, M = (\dim V_1, \ldots, \dim V_n) \in \mathbb{N}^n$ denote the dimension and dimension vector of M, respectively, and let $[M]$ denote the isoclass (isomorphism class) of M. Further, for each $a \ge 1$, we write

$$aM := \underbrace{M \oplus \cdots \oplus M}_{a}.$$

If $a = 0$, we let $aM = 0$ by convention.

A representation $M = (V_i, f_i)_i$ of Δ over k (or a $k\Delta$-module) is called *nilpotent* if the composition $f_n \cdots f_2 f_1 \colon V_1 \to V_1$ is nilpotent, or equivalently, one of the $f_{i-1} \cdots f_n f_1 \cdots f_i \colon V_i \to V_i$ $(2 \le i \le n)$ is nilpotent. Let $\mathbb{T}_k = \mathbb{T}_k(n)$ denote the category of finite-dimensional nilpotent representations of Δ over k, and let $S_i = (S_i)_k$, $i \in I$ (resp. $S_i[l]_k$, $i \in I$ and $l \ge 1$) be the irreducible (resp. indecomposable) objects in $\mathbb{T}_k$. Here $S_i[l]_k$ is the (unique) indecomposable object with top $(S_i)_k$ and length, *i.e.*, dimension, l.

Following [16], a (cyclic) multisegment is a formal finite sum

$$\pi = \sum_{i \in I,\, l \ge 1} \pi_{i,l}[i; l),$$

where $\pi_{i,l} \in \mathbb{N}$. Let Π denote the set of all multisegments. Then each multisegment $\pi = \sum_{i,l} \pi_{i,l}[i; l) \in \Pi$ defines a representation in $\mathbb{T}_k$

$$M_k(\pi) = \bigoplus_{i \in I,\, l \ge 1} \pi_{i,l} S_i[l]_k.$$

In this way we obtain a bijection between Π and the set of isoclasses of representations in $\mathbb{T}_k$. Note that this bijection is independent of the field k. Thus, throughout, the subscripts k are often dropped for notational simplicity. We shall also write $\mathrm{End}(M)$, $\mathrm{Hom}(M, N)$, etc. for $\mathrm{End}_{k\Delta}(M)$, $\mathrm{Hom}_{k\Delta}(M, N)$, etc.

Remark 2.1 In [4, 28], nilpotent representations of Δ are parametrized by n-tuples of partitions. In fact, if we identify a multisegment $\pi = \sum_{i,l} \pi_{i,l}[i;l) \in \Pi$ as the n-tuple $(\pi^{(1)}, \pi^{(2)}, \dots, \pi^{(n)})$ of partitions, where, for each $1 \leq i \leq n$, $\pi^{(i)}$ is the partition dual to the partition $(m^{\pi_{i,m}}, \dots, 2^{\pi_{i,2}}, 1^{\pi_{i,1}})$, where m is the maximal l for which $\pi_{i,l} \neq 0$ (*i.e.*, $m = \mathrm{Ll}(M(\pi))$), then the two parametrizations coincide.

A multisegment $\pi = \sum_{i,l} \pi_{i,l}[i;l)$ in Π is called *aperiodic*[1] (see [19, p. 417]) if, for each $l \geq 1$, there is some $i \in I$ such that $\pi_{i,l} = 0$. Otherwise, π is called *periodic*. By Π^a we denote the set of aperiodic multisegments. A representation M in $\mathbb{T}$ is called *aperiodic* (resp. *periodic*) if $M \cong M(\pi)$ for some $\pi \in \Pi^a$ (resp. $\pi \in \Pi \backslash \Pi^a$).

For $\mathbf{d} \in \mathbb{N}^n$, let

$$\Pi_{\mathbf{d}} = \{\lambda \in \Pi \mid \mathbf{dim}\, M(\lambda) = \mathbf{d}\} \quad \text{and} \quad \Pi^a_{\mathbf{d}} = \Pi^a \cap \Pi_{\mathbf{d}}.$$

Associated to a cyclic quiver, or more precisely, to $\mathbb{T}$, Ringel introduced an associative algebra, the *Ringel–Hall algebra*, which can be defined at two levels: the integral level and the generic level.

Let k be a *finite* field of q_k elements and, for L, M, N in $\mathbb{T}_k$, let F^L_{MN} be the number of submodules V of L such that $V \cong N$ and $L/V \cong M$. More generally, given modules $M, N_1, \dots, N_m$ in $\mathbb{T}_k$, we let $F^M_{N_1 \cdots N_m}$ be the number of the filtrations

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{m-1} \supseteq M_m = 0,$$

such that $M_{t-1}/M_t \cong N_t$ for all $1 \leq t \leq m$. By [13, 26], $F^M_{N_1 \cdots N_m}$ is a polynomial in q_k. In other words, for $\pi, \mu_1, \dots, \mu_m$ in Π, there is a polynomial $\varphi^\pi_{\mu_1 \cdots \mu_m}(q) \in \mathcal{A} := \mathbb{Z}[q]$ such that for any finite field k of q_k elements

$$\varphi^\pi_{\mu_1 \cdots \mu_m}(q_k) = F^{M_k(\pi)}_{M_k(\mu_1) \cdots M_k(\mu_m)}.$$

The (generic) *Ringel–Hall algebra* $\mathcal{H} = \mathcal{H}_{\mathcal{A}}(n)$ of $\Delta(n)$ is by definition the free $\mathcal{A}$-module with basis $\{u_\pi \mid \pi \in \Pi\}$ and multiplication given by

$$u_\mu \circ u_\nu = \sum_{\pi \in \Pi} \varphi^\pi_{\mu\nu}(q) u_\pi.$$

By specializing q to the prime power q_k, we obtain the integral Ringel–Hall algebra associated with $\mathbb{T}_k$.

In practice, we sometimes write $u_\pi = u_{[M(\pi)]}$ in order to make certain calculations in terms of modules. Denote by $\mathcal{C} = \mathcal{C}_{\mathcal{A}}(n)$ the subalgebra of $\mathcal{H}$ generated by $u_i := u_{[S_i]}$, $i \in I$. This is called the (generic) *composition algebra* of $\Delta(n)$. It is easy to see that $\mathcal{C}$ is a proper subalgebra of $\mathcal{H}$. Moreover, both $\mathcal{H}$ and $\mathcal{C}$ admit a natural $\mathbb{N}^n$-grading by dimension vectors:

$$\mathcal{H} = \bigoplus_{\mathbf{d} \in \mathbb{N}^n} \mathcal{H}_{\mathbf{d}} \quad \text{and} \quad \mathcal{C} = \bigoplus_{\mathbf{d} \in \mathbb{N}^n} \mathcal{C}_{\mathbf{d}}, \tag{2.1}$$

where $\mathcal{H}_{\mathbf{d}}$ is spanned by all u_λ with $\lambda \in \Pi_{\mathbf{d}}$ and $\mathcal{C}_{\mathbf{d}} = \mathcal{C} \cap \mathcal{H}_{\mathbf{d}}$.

[1] In [26, 4.1], the corresponding n-tuple of partitions is called *separated*.

3 The Generic Extension Monoid of a Cyclic Quiver

Let $\mathcal{M}$ (resp. $\mathcal{M}_c$) be the set of all isoclasses of representations (resp. aperiodic representations) in $\mathbb{T}$. Given two objects M, N in $\mathbb{T}$, there exists a unique (up to isomorphism) extension G of M by N with minimal $\dim \operatorname{End}(G)$ [3, 4, 24]. The extension G is called the *generic extension*[2] of M by N and is denoted by $G =: M * N$. Thus, if we define $[M] * [N] = [M * N]$, then it is known from [4] that $*$ is associative and $(\mathcal{M}, *)$ is a monoid with identity $[0]$.

Every semisimple module in $\mathbb{T}$ has the form $S_{\mathbf{a}} = \bigoplus_{i=1}^n a_i S_i$ for some $\mathbf{a} = (a_i) \in \mathbb{N}^n$. We shall see below that every module in $\mathbb{T}$ is a sequence of generic extensions by semisimple modules.

For each multisegment $\pi = \sum_{i,l} \pi_{i,l}[i;l)$ and each $i \in I$, we define

$$i * \pi = \pi - [i+1; l_0) + [i; l_0 + 1),$$

where l_0 is maximal such that $\pi_{i+1,l_0} \neq 0$. Then by [4, Proposition 3.7], we have

$$S_i * M(\pi) \cong M(i * \pi).$$

Further, for each $i \in I$, we set $\pi^{(i)} = \sum_{l \geq 1} \pi_{i,l}[i;l)$. Then $\pi = \pi^{(1)} + \pi^{(2)} + \cdots + \pi^{(n)}$. Finally, for every $\mathbf{a} = (a_i) \in \mathbb{N}^n$, we define

$$\mathbf{a} * \pi = \sum_{i \in I} \underbrace{i * i * \cdots * i}_{a_i} * \pi^{(i+1)}.$$

Lemma 3.1 *Let $\mathbf{a} \in \mathbb{N}^n$ and $\pi \in \Pi$. Then we have $M(\mathbf{a} * \pi) \cong S_{\mathbf{a}} * M(\pi)$. Dually, a similar result holds for the generic extension of a module by a semisimple one.*

Proof For each $1 \leq i \leq n$, we set

$$M_i(\pi) = M(\pi^{(i)}) = \bigoplus_{l \geq 1} \pi_{i,l} S_i[l].$$

Then $M(\pi) = \bigoplus_{i=1}^n M_i(\pi)$. Since $\operatorname{Ext}^1(S_i, M_j(\pi)) = 0$ for all $j \neq i+1$, we have

$$S_{\mathbf{a}} * M(\pi) = \bigoplus_{i \in I} (a_i S_i) * M_{i+1}(\pi).$$

Applying [4, Proposition 3.7] repeatedly gives

$$(a_i S_i) * M_{i+1}(\pi) \cong \underbrace{S_i * \cdots * S_i}_{a_i} * M(\pi^{(i+1)}) \cong M(\underbrace{i * \cdots * i}_{a_i} * \pi^{(i+1)}).$$

Hence, $M(\mathbf{a} * \pi) \cong S_{\mathbf{a}} * M(\pi)$. ■

[2] Geometrically, when k is algebraically closed, each isoclass $[M]$ of dimension vector $\mathbf{d} = \mathbf{d}_M = (d_i) \in \mathbb{N}^n$ corresponds to a unique $\mathrm{GL}(\mathbf{d})$-orbit $\mathcal{O}_M$ in the representation variety $R(\mathbf{d}) = \prod_{i=1}^n \operatorname{Hom}_k(k^{d_i}, k^{d_{i+1}})$ on which $\mathrm{GL}(\mathbf{d}) = \prod_{i=1}^n \mathrm{GL}_{d_i}(k)$ acts by conjugation. Thus, $M * N$ of dimension vector $\mathbf{d} = \mathbf{d}_M + \mathbf{d}_N$ corresponds to the unique dense orbit $\mathcal{O}$ (of maximal dimension) in the extension variety $\mathcal{E}(M, N) = \{x \in R(\mathbf{d}) \mid x \text{ defines an extension of } M \text{ by } N\}$.

By Lemma 3.1, we obtain the following, *cf.* [22].

Corollary 3.2 *Let $M \in \mathbb{T}$ with $l = \mathrm{Ll}(M)$. Then we have*

$$M \cong (M/\operatorname{rad} M) * (\operatorname{rad} M/\operatorname{rad}^2 M) * \cdots * (\operatorname{rad}^{l-2} M/\operatorname{rad}^{l-1} M) * (\operatorname{rad}^{l-1} M)$$

and

$$M \cong (M/\operatorname{soc}^{l-1} M) * (\operatorname{soc}^{l-1} M/\operatorname{soc}^{l-2} M) * \cdots * (\operatorname{soc}^2 M/\operatorname{soc} M) * (\operatorname{soc} M).$$

In particular, we have for each $0 < s \leq l$,

$$(M/\operatorname{rad}^s M) * (\operatorname{rad}^s M) \cong M \cong (M/\operatorname{soc}^s M) * (\operatorname{soc}^s M).$$

A semi-simple module $S_{\mathbf{a}} = \bigoplus_{i=1}^n a_i S_i$ is called *sincere* if all $a_i \geq 1$. Clearly, sincere semi-simple modules are in one-to-one correspondence with *sincere vectors* $\mathbf{a} = (a_i) \in \mathbb{N}^n$. Let $I^e = I \cup \{\text{all sincere vectors in } \mathbb{N}^n\}$.

We have already proved the following result, *cf.* [22].

Proposition 3.3 *The generic extension monoid $\mathcal{M}$ is generated by $[S_{\mathbf{a}}], \mathbf{a} \in I^e$, and this generating set is minimal.*

In [22], the structure of the monoids $\mathcal{M}$ and $\mathcal{M}_c$ in terms of generators and relations is investigated.

Let Σ (resp. Ω) denote the set of all words on the alphabet I^e (resp. I). For each $w = \mathbf{a}_1\mathbf{a}_2 \cdots \mathbf{a}_m \in \Sigma$, we set $M(w) = S_{\mathbf{a}_1} * S_{\mathbf{a}_2} * \cdots * S_{\mathbf{a}_m}$. Then there is a unique $\pi \in \Pi$ such that $M(w) \cong M(\pi)$, and we set $\wp(w) = \pi$. In this way we obtain a surjective map $\wp : \Sigma \to \Pi,\ w \mapsto \pi = \wp(w)$. Note that the map $\wp$ is independent of the field k and that $\wp$ induces a surjection $\wp : \Omega \twoheadrightarrow \Pi^a$ (see [4, Theorem 4.1]).

Besides the monoid structure, $\mathcal{M}$ has also a poset structure. For two representations $M, N \in \mathbb{T}$, we say that M degenerates to N (or N is a *degeneration* of M), following [3], and write $M \leq_{\deg} N$, if $\dim \operatorname{Hom}(X, M) \leq \dim \operatorname{Hom}(X, N)$ for all X in $\mathbb{T}$ (see also [31]).

Since the order relation is independent of the field k, we may turn Π into a poset with the opposite partial order $\leq := (\leq_{\deg})^{\mathrm{op}}$ defined by setting[3]

$$\mu \leq \lambda \Longleftrightarrow M(\lambda) \leq_{\deg} M(\mu).$$

4 Distinguished Words and Distinguished Decompositions

We recall from [26, 2.3] and [4, Section 5] the definitions of a reduced filtration and distinguished words in Ω. We now generalize them to the words in Σ.

For $\mathbf{a} \in I^e$, we set $u_{\mathbf{a}} = u_{[S_{\mathbf{a}}]}$. Let $w = \mathbf{a}_1\mathbf{a}_2 \cdots \mathbf{a}_m$ be a word in Σ and let $\varphi_w^\lambda(q)$ be the Hall polynomial $\varphi^\lambda_{\mu_1 \cdots \mu_m}(q)$ with $M(\mu_r) \cong S_{\mathbf{a}_r}$. Then w can be uniquely expressed

[3]Geometrically, this ordering coincides with the Bruhat type ordering: $\mu \leq \lambda$ if and only if $\mathcal{O}_{M(\mu)} \subseteq \overline{\mathcal{O}}_{M(\lambda)}$, the closure of $\mathcal{O}_{M(\lambda)}$; see [4, §3].

https://doi.org/10.4153/CJM-2007-054-7 Published online by Cambridge University Press

in the *tight form* $w = \mathbf{b}_1^{e_1}\mathbf{b}_2^{e_2}\cdots\mathbf{b}_t^{e_t}$, where $e_r = 1$ if $\mathbf{b}_r \in I^e\backslash I$, and e_r is the number of consecutive occurrences of $\mathbf{b}_r$ if $\mathbf{b}_r \in I$. A filtration

$$M = M_0 \supset M_1 \supset \cdots \supset M_{t-1} \supset M_t = 0$$

of a nilpotent representation M is called a *reduced* filtration of type w if $M_{r-1}/M_r \cong e_r S_{\mathbf{b}_r}$ for all $1 \le r \le t$. By $\gamma_w^\lambda(q)$ we denote the Hall polynomial $\varphi^\lambda_{\mu_1\cdots\mu_t}(q)$, where $M(\mu_r) = e_r S_{\mathbf{b}_r}$. Thus, for any finite field k of q_k elements, $\gamma_w^\lambda(q_k)$ is the number of the reduced filtrations of $M_k(\lambda)$ of type w. A word w is called *distinguished* if the Hall polynomial $\gamma_w^{\wp(w)}(q) = 1$. Note that w is distinguished if and only if, for an algebraically closed field k, $M_k(\wp(w))$ has a unique reduced filtration of type w.

For each multisegment $\pi = \sum_{i,l} \pi_{i,l}[i;l)$, we define

$$p(\pi) = \max\{l \mid \pi_{i,l} \neq 0 \text{ for all } 1 \le i \le n\}.$$

If no such an l exists, we set $p(\pi) = 0$. This is exactly the case where π is aperiodic. In particular, a multisegment π is called *strongly periodic* if $\pi_{i,l} = 0$ for all $i \in I$ and $l > p(\pi)$. Clearly, we have

$$p(\mathbf{a} * \pi) = p(\pi) + 1 \text{ whenever } \mathbf{a} \in \mathbb{N}^n \text{ is sincere.} \tag{4.1}$$

Let $\pi \in \Pi$ with $p = p(\pi)$ and consider the submodule $M' = \operatorname{soc}^p M(\pi)$ of $M(\pi)$. Then

$$M'' := M(\pi)/M' = \bigoplus_{i\in I}\bigoplus_{l>p} \pi_{i,l} S_i[l-p].$$

Let $\pi', \pi'' \in \Pi$ be such that $M(\pi') \cong M'$ and $M(\pi'') \cong M''$. Then, obviously, π' is strongly periodic, π'' is aperiodic, and both π' and π'' are uniquely determined by π. We call (π', π'') the *associated pair* of π. We have the following.

Lemma 4.1 *Maintain the notation introduced above. We have $M(\pi) \cong M(\pi'') * M(\pi')$. Moreover, M' is the unique submodule of $M(\pi)$ isomorphic to $M(\pi')$.*

Proof The isomorphism follows from Corollary 3.2, while the uniqueness follows from Lemma 1.1, since $M' = \operatorname{soc}^p M(\pi)$. ■

We have the following characterization of a strongly periodic multisegment.

Lemma 4.2 *Let $\pi \in \Pi$ and $M = M(\pi)$. Then π is strongly periodic with $p = p(\pi)$ if and only if $p = \operatorname{Ll}(M)$ and every subquotient $S_{\mathbf{a}_s} \cong \operatorname{soc}^{p-s+1} M/\operatorname{soc}^{p-s} M$, $1 \le s \le p$, in the socle filtration of M is sincere. Moreover, putting $y_\pi = \mathbf{a}_1\cdots\mathbf{a}_p$, we have $\wp(y_\pi) = \pi$, and any filtration*

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{p-1} \supseteq M_p = 0$$

satisfying $M_{s-1}/M_s \cong S_{\mathbf{a}_s}$ for all $1 \le s \le p$ is the socle filtration of M.

Proof The sufficient part follows from (4.1). To see the necessary part, we apply induction on p; the case for $p = 1$ is trivial. Assume now $p > 1$ and let $\pi = \sum_{i,l} \pi_{i,l}[i; l)$. Then $\mathbf{a}_1 = (\pi_{1,p}, \dots, \pi_{n,p})$ is sincere. Putting

$$\pi_1 = \pi - \sum_i \pi_{i,p}[i; p) + \sum_i \pi_{i,p}[i+1; p-1),$$

we have $\pi = \mathbf{a}_1 * \pi_1$, and π_1 is strongly periodic with $p(\pi_1) = p - 1$. Hence $S_{\mathbf{a}_1} * M(\pi_1) \cong M(\pi)$ by Lemma 3.1. Clearly, $M(\pi_1)$ is isomorphic to a maximal submodule M_1 of $M(\pi)$ with Loewy length $p - 1$. Hence, $M_1 = \operatorname{soc}^{p-1} M(\pi)$ by Lemma 1.1, and the assertion follows from induction. ■

For an aperiodic $\pi \in \Pi^a$, we have the following which was not explicitly stated in [4].

Proposition 4.3 *For any $\pi \in \Pi^a$, there exists a distinguished word*

$$w_\pi = j_1^{e_1} \cdots j_t^{e_t} \in \Omega \cap \wp^{-1}(\pi),$$

where $j_{r-1} \neq j_r$, $e_r \geq 1$ for all r, that is, $M(\pi)$ has a unique filtration

$$M(\pi) = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{t-1} \supseteq M_t = 0$$

satisfying $M_{r-1}/M_r \cong e_r S_{j_r}$ for all $1 \leq r \leq t$.

Proof Let $\pi = \sum_{i,l} \pi_{i,l}[i; l)$ be an aperiodic multisegment. For each $i \in I$, we set $M_i = \bigoplus_{l \geq 1} S_i[l]$. Then there is a $j \in I$ (not necessarily unique) such that $\mathrm{Ll}(M_j) > \mathrm{Ll}(M_{j+1})$. We write

$$M_j = S_j[l_1] \oplus S_j[l_2] \oplus \cdots \oplus S_j[l_r]$$

with $l_1 \geq l_2 \geq \cdots \geq l_r \geq 1$. Choose $e \geq 1$ such that $l_e > l_{e+1}$ and $l_e > \mathrm{Ll}(M_{j+1})$, and define

$$\mu = \pi - \big([j; l_1) + \cdots + [j; l_e)\big) + \big([j+1; l_1 - 1) + \cdots + [j+1; l_e - 1)\big).$$

By [4, Lemma 5.4], there is a unique submodule X of $M(\pi)$ such that $X \cong M(\mu)$ and $M(\pi)/X \cong eS_j$. We may assume that μ is aperiodic. For example, taking the maximal index e with the property $l_e > l_{e+1}$ and $l_e > \mathrm{Ll}(M_{j+1})$ ensures that μ is aperiodic. By induction, there is a distinguished word $w_1 \in \Omega \cap \wp^{-1}(\mu)$. Then $w := j^e w_1$ is a distinguished word in $\Omega \cap \wp^{-1}(\pi)$, as desired. ■

Note that by [4, Theorem 5.5], every distinguished word in $\wp^{-1}(\pi)$ can be obtained in the above way.

Example 4.4 Let $n = 3$ and $\pi = [1;4) + [1;3) + [2;2) + [2;1) + 2\,[3;1)$. Then π is aperiodic. From the proof of Proposition 4.3, we can take $j_1 = 1$ or 2. Moreover, if $j_1 = 1$, then $e_1 = 1$ or 2, and if $j_1 = 2$, then $e_1 = 1$. If we fix $j_1 = 1$ and e_1=2, then $j_2 = 2$ and $e_2 = 1$ or 3. Continuing this process, we finally get all the seven distinguished words in $\wp^{-1}(\pi)$:

$$1213^3 2^3 13^2, \quad 12^2 13^4 2^2 13, \quad 1^2 23^3 2^3 13^2, \quad 1^2 2^3 3^5 21,$$

$$21213^4 2^2 13, \quad 21^2 23^4 2^2 13, \quad 21^2 2^2 3^5 21.$$

In general, for any $\pi \in \Pi$ with $p = p(\pi)$, let (π', π'') be the associated pair, where π' is strongly periodic with $p(\pi') = p$ and π'' is aperiodic. By Lemma 4.2 and Proposition 4.3, there are distinguished words

$$w_{\pi''} = j_1^{e_1} \cdots j_t^{e_t} \in \Omega \cap \wp^{-1}(\pi'') \quad \text{and} \quad y_{\pi'} = \mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_p \in \Sigma \cap \wp^{-1}(\pi') \tag{4.2}$$

associated to π' and π''. Thus, we obtain a word

$$w_\pi = w_{\pi''} y_{\pi'} = j_1^{e_1} \cdots j_t^{e_t} \mathbf{a}_1 \cdots \mathbf{a}_p \in \wp^{-1}(\pi), \tag{4.3}$$

and a decomposition $M(\pi) = e_1 S_{j_1} * \cdots * e_t S_{j_t} * S_{\mathbf{a}_1} * \cdots * S_{\mathbf{a}_p}$. We shall call such a decomposition a *distinguished* decomposition because of the following.

Proposition 4.5 *For any $\pi \in \Pi$, the word w_π defined in* (4.3) *is distinguished.*

Proof The existence of a reduced filtration of type w_π obtained by refining $M(\pi) \supseteq M' = \operatorname{soc}^p M(\pi) \supseteq 0$, follows from Lemmas 4.1 and 4.2, and Proposition 4.3. Suppose now that $M(\pi)$ has another filtration

$$M(\pi) = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_{t-1} \supseteq N_t \supseteq \cdots \supseteq N_{t+p-1} \supseteq N_{t+p} = 0,$$

satisfying $N_{s-1}/N_s \cong e_s S_{j_s}$ for $1 \le s \le t$ and $N_{t+i-1}/N_{t+i} \cong S_{\mathbf{a}_i}$ for $1 \le i \le p$. Then we have $\mathrm{Ll}(N_t) \le p$. Since M' is the maximal submodule of $M(\pi)$ of Loewy length p, we infer $N_t \subseteq M'$, and consequently, $N_t = M'$ as $\mathbf{dim}\, N_t = \mathbf{dim}\, M'$. Now the uniqueness of the filtrations given in Lemma 4.2 and Proposition 4.3 forces that the filtration above must be unique. Hence, w_π is distinguished. ■

5 The Strong Monomial Basis Property

For $m \ge 1$, let $[\![m]\!]^! = [\![1]\!][\![2]\!] \cdots [\![m]\!]$, where $[\![e]\!] = \frac{q^e - 1}{q-1}$.

For any $w = \mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_m \in \Sigma$, let $u_w = u_{\mathbf{a}_1} \circ u_{\mathbf{a}_2} \circ \cdots \circ u_{\mathbf{a}_m}$. The proof of the following is entirely similar to that of [4, Proposition 9.1].

Lemma 5.1 *For each $w \in \Sigma$ with the tight form $\mathbf{b}_1{}^{e_1} \mathbf{b}_2{}^{e_2} \cdots \mathbf{b}_t{}^{e_t}$, we have*

$$u_w = \prod_{r=1}^{t} [\![e_r]\!]^! \sum_{\lambda \le \wp(w)} \gamma_w^\lambda(q) u_\lambda. \tag{5.1}$$

In other words, we have the relation $\varphi_w^\lambda(q) = \prod_{r=1}^{t} [\![e_r]\!]^! \gamma_w^\lambda(q)$. Moreover, the coefficients appearing in the sum are all non-zero.

For any $\pi \in \Pi$, choose an arbitrary $w_\pi \in \wp^{-1}(\pi)$. We shall call the set $\{w_\pi \mid \pi \in \Pi\}$ a *section* of Σ over Π. Similarly, we may define a section of Ω over Π^a. A section is called *distinguished* if all its members are distinguished words. By the invertibility of the matrix arising from (5.1) over the components $\mathcal{H}_{\mathbf{d}}$, Lemma 5.1 implies immediately the following strong monomial basis property for the Ringel–Hall algebra associated with a cyclic quiver; see [4, 8.1] and [5, 1.1] for the quantum group case.

Theorem 5.2 *Let $\mathcal{H}_{\mathbb{Q}(q)} = \mathcal{H} \otimes_{\mathcal{A}} \mathbb{Q}(q)$ and, for $w \in \Sigma$, let*

$$u^{(w)} = \frac{1}{\prod_{r=1}^{t} [\![e_r]\!]^!} u_w.$$

(i) *If $\{w_\pi \mid \pi \in \Pi\}$ is a section of Σ over Π. Then the set $\{u_{w_\pi} \mid \pi \in \Pi\}$ forms a basis for $\mathcal{H}_{\mathbb{Q}(q)}$. In particular, the Ringel–Hall algebra $\mathcal{H}_{\mathbb{Q}(q)}$ is generated by $u_{\mathbf{a}}$, $\mathbf{a} \in I^e$.*

(ii) *If the section $\{w_\pi \mid \pi \in \Pi\}$ is distinguished, then $\{u^{(w_\pi)} \mid \pi \in \Pi\}$ forms an integral basis for $\mathcal{H}$.*

6 Twisted Ringel–Hall Algebras and Quantum Affine $\mathfrak{sl}_n$

Let $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$ be the Laurent polynomial ring over $\mathbb{Z}$ in indeterminate v. For each $m \geq 1$, let

$$[m] = \frac{v^m - v^{-m}}{v - v^{-1}} \quad \text{and} \quad [m]^! = [1][2]\cdots[m].$$

The *twisted Ringel–Hall algebra* $H_{\mathcal{Z}} = H_{\mathcal{Z}}(n)$ of $\Delta(n)$ is by definition the free $\mathcal{Z}$-module with basis $\{u_\pi = u_{[M(\pi)]} \mid \pi \in \Pi\}$ and multiplication defined by

$$u_\mu u_\nu = v^{\varepsilon(\mu,\nu)}(u_\mu \circ u_\nu) = v^{\varepsilon(\mu,\nu)} \sum_{\pi \in \Pi} \varphi^\pi_{\mu\nu}(v^2) u_\pi.$$

Here $\varepsilon(\mu,\nu) = \varepsilon(\mathbf{dim}\, M(\mu), \mathbf{dim}\, M(\nu))$ is the Euler form $\varepsilon(-,-)\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ associated with the cyclic quiver Δ and defined by

$$\varepsilon(\mathbf{a},\mathbf{b}) = \sum_{i=1}^n a_i b_i - \sum_{i=1}^n a_i b_{i+1},$$

for $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n)$ (noting $n+1 = 1$ in I). It is well known that for two representations $M, N \in \mathbb{T}$, there holds

$$\varepsilon(\mathbf{dim}\, M, \mathbf{dim}\, N) = \dim_k \operatorname{Hom}(M,N) - \dim_k \operatorname{Ext}^1(M,N).$$

The $\mathcal{Z}$-subalgebra $C_{\mathcal{Z}}$ of $H_{\mathcal{Z}}$ generated by $u_i^{(m)} := \frac{u_i^m}{[m]^!}$, $i \in I$ and $m \geq 1$, is called the *twisted composition algebra*. Then $C_{\mathcal{Z}}$ is also generated by $u_{[mS_i]}$, $i \in I$, $m \geq 1$, since $u_i^{(m)} = v^{m(m-1)}u_{[mS_i]}$. Clearly, both $H_{\mathcal{Z}}$ and $C_{\mathcal{Z}}$ inherit the grading given in (2.1).

Let $\mathbf{H} = H_{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathbb{Q}(v)$ and $\mathbf{C} = C_{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathbb{Q}(v)$. Let $\mathbf{U} = \mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ be the quantum $\widehat{\mathfrak{sl}}_n$ ($n \geq 2$) over $\mathbb{Q}(v)$, and let $E_i, F_i, K_i^{\pm 1}$ ($i \in I$) be its generators; see [21]. Then $\mathbf{U}$ admits a triangular decomposition $\mathbf{U} = \mathbf{U}^-\mathbf{U}^0\mathbf{U}^+$ where $\mathbf{U}^+$ (resp. $\mathbf{U}^-$, $\mathbf{U}^0$) is the subalgebra generated by the E_i's (resp. F_i's, $K_i^{\pm 1}$'s). We denote by $U_{\mathcal{Z}}^+$ the Lusztig form of $\mathbf{U}^+$, that is, $U_{\mathcal{Z}}^+$ is generated by all the divided powers $E_i^{(m)} := E_i^m/[m]^!$. We have the following important results.

Theorem 6.1

(i) *There is a $\mathcal{Z}$-algebra isomorphism*

$$C_{\mathcal{Z}} \xrightarrow{\sim} U_{\mathcal{Z}}^+, \; u_i^{(m)} \longmapsto E_i^{(m)}, \; i \in I, m \geq 1,$$

and hence a $\mathbb{Q}(v)$-algebra isomorphism $\mathbf{U}^+ \cong \mathbf{C}$ [26].

(ii) *The algebra $\mathbf{H}$ is isomorphic to $\mathbf{U}^+ \otimes_{\mathbb{Q}(v)} \mathbb{Q}(v)[x_1, x_2, \ldots]$, where $\mathbb{Q}(v)[x_1, x_2, \ldots]$ is an infinite polynomial algebra over $\mathbb{Q}(v)$ with x_r central of degree $(r, \ldots, r)$ [29].*

In the sequel, we will identify the two algebras $U_{\mathcal{Z}}^+$ and $C_{\mathcal{Z}}$. In particular, we shall identify $u_i^{(m)}$ with $E_i^{(m)}$, etc. Note that the Ringel–Hall algebra notation u_λ will be used to facilitate calculations involving modules.

The elementary construction of the canonical bases for $U_{\mathcal{Z}}$ and $H_{\mathcal{Z}}$ uses (integral) monomials which we now define. For each $w = i_1 \cdots i_m = j_1^{e_1} \cdots j_t^{e_t} \in \Omega$ with $j_{r-1} \neq j_r$ for all r, let[4]

$$\mathfrak{m}_w = u_{i_1} \cdots u_{i_m} = E_{i_1} \cdots E_{i_m} \quad \text{and} \quad \mathfrak{m}^{(w)} = u_{j_1}^{(e_1)} \cdots u_{j_t}^{(e_t)} = E_{j_1}^{(e_1)} \cdots E_{j_t}^{(e_t)}. \tag{6.1}$$

Then we have by Lemma 5.1

$$\mathfrak{m}_w = v^{\delta_1(w)} u_w = v^{\delta_1(w)} \sum_{\lambda \leq \wp(w)} \varphi_w^\lambda(v^2) u_\lambda, \tag{6.2}$$

where $\delta_1(w) = \sum_{1 \leq r < s \leq m} \varepsilon(\mathbf{dim}\, S_{i_r}, \mathbf{dim}\, S_{i_s})$. If we put

$$\delta_2(w) = \sum_{r=1}^{t} \frac{e_r(e_r - 1)}{2} \quad \text{and} \quad \delta(w) = \delta_1(w) + \delta_2(w), \tag{6.3}$$

then $\prod_{r=1}^t [e_r]^! = v^{-\delta_2(w)} \prod_{r=1}^t [\![e_r]\!]^!$, and

$$\begin{aligned} \mathfrak{m}^{(w)} &= \Big(\prod_{r=1}^{t} [e_r]^!\Big)^{-1} \mathfrak{m}_w = \Big(\prod_{r=1}^{t} [\![e_r]\!]^!\Big)^{-1} v^{\delta_1(w) + \delta_2(w)} u_w \\ &= v^{\delta(w)} \sum_{\lambda \leq \wp(w)} \gamma_w^\lambda(v^2) u_\lambda. \end{aligned} \tag{6.4}$$

[4] The element $\mathfrak{m}_w$ is denoted as E_w in [5].

https://doi.org/10.4153/CJM-2007-054-7 Published online by Cambridge University Press

We now define (integral) monomials in $H_{\mathcal{Z}}$. For each $\mathbf{a} = (a_i) \in \mathbb{N}^n$, we set $\|\mathbf{a}\| = \sum_i a_i^2$ and $|\mathbf{a}| = \sum_i a_i$, and define

$$\tilde{u}_{\mathbf{a}} = v^{\dim \operatorname{End}(S_{\mathbf{a}}) - \dim S_{\mathbf{a}}} u_{\mathbf{a}} = v^{\|\mathbf{a}\| - |\mathbf{a}|} u_{\mathbf{a}} \in H_{\mathcal{Z}}.$$

In particular, for $i \in I$ and $e \geq 1$, we have

$$\tilde{u}_{ei} = v^{e^2 - e} u_{ei} = u_i^{(e)},$$

where $u_{ei} = u_{[eS_i]}$. Note that if $\mathbf{a} = (a_i) \in \mathbb{N}^n$ is insincere, say $a_i = 0$, then

$$\tilde{u}_{\mathbf{a}} = \tilde{u}_{a_{i-1}(i-1)} \cdots \tilde{u}_{a_1 1}\, \tilde{u}_{a_n n} \cdots \tilde{u}_{a_{i+1}(i+1)}$$

is a monomial in $U_{\mathcal{Z}}^+$ defined above.

In general, for a given word $w = \mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_m \in \Sigma$ with tight form $\mathbf{b}_1{}^{e_1} \mathbf{b}_2{}^{e_2} \cdots \mathbf{b}_t{}^{e_t}$, we define monomials in $H_{\mathcal{Z}}$ (*cf.* (6.1))

$$\mathfrak{m}_w = u_{\mathbf{a}_1} \cdots u_{\mathbf{a}_m} \quad \text{and} \quad \mathfrak{m}^{(w)} = \tilde{u}_{e_1 \mathbf{b}_1} \cdots \tilde{u}_{e_t \mathbf{b}_t}.$$

Again by Lemma 5.1, we obtain

$$\mathfrak{m}_w = v^{\sum_{1 \leq r < s \leq m} \varepsilon(\mathbf{dim}\, S_{\mathbf{a}_r}, \mathbf{dim}\, S_{\mathbf{a}_s})} \sum_{\lambda \leq \wp(w)} \varphi_w^\lambda(v^2) u_\lambda$$

and

$$\mathfrak{m}^{(w)} = v^{\delta'(w)} \sum_{\lambda \leq \wp(w)} \gamma_w^\lambda(v^2) u_\lambda, \tag{6.5}$$

where

$$\delta'(w) = \sum_{r=1}^{m} \big(e_r^2 \|\mathbf{b}_r\| - e_r |\mathbf{b}_r| - \frac{e_r(e_r - 1)}{2} \big) + \sum_{1 \leq r < s \leq m} \varepsilon(\mathbf{dim}\, S_{\mathbf{a}_r}, \mathbf{dim}\, S_{\mathbf{a}_s}).$$

Note that if $w \in \Omega$, then all $\|\mathbf{b}_r\| = |\mathbf{b}_r| = 1$, and so $\delta'(w) = \delta(w)$. Since δ' extends δ, we will use the same letter δ for δ' in the sequel.

Here are the twisted version and the (non-integral) quantum group version of the strong monomial basis property (Theorem 5.2). The integral quantum group version of this property was not given in [4] and will be discussed in the next section as a key step to the elementary construction.

Theorem 6.2

(i) *For each $\pi \in \Pi$, choose a word $w_\pi \in \wp^{-1}(\pi)$. Then the set $\{\mathfrak{m}_{w_\pi} | \pi \in \Pi\}$ is a $\mathbb{Q}(v)$-basis of $\mathbf{H}$. Moreover, if all w_π are chosen to be distinguished, then the set $\{\mathfrak{m}^{(w_\pi)} | \pi \in \Pi\}$ is a $\mathcal{Z}$-basis of $H_{\mathcal{Z}}$.*

https://doi.org/10.4153/CJM-2007-054-7 Published online by Cambridge University Press

(ii) *Let* $\{w_\pi \mid \pi \in \Pi^a\}$ *be a section of* Ω *over* Π^a. *Then the set* $\{\mathfrak{m}_{w_\pi} \mid \pi \in \Pi^a\}$ *is a* $\mathbb{Q}(v)$*-basis of* $\mathbf{U}^+$ [4, 8.1].

By [30, Proposition 7.5]), there is a $\mathbb{Z}$-linear ring involution $\iota\colon H_{\mathcal{Z}} \to H_{\mathcal{Z}}$ satisfying $\iota(v) = v^{-1}$ and $\iota(\mathfrak{m}^{(w_\pi)}) = \mathfrak{m}^{(w_\pi)}$.

Remark 6.3 The construction of the ring involution ι is not algebraic and elementary, though its restriction to $U_{\mathcal{Z}}^+$ can be seen easily through the Drinfeld–Jimbo presentation. However, if we note that the ring homomorphism condition is not required in the (linear algebra) construction of IC bases, then we may use the basis $\{\mathfrak{m}^{(w)} \mid w \in \mathcal{D}\}$ for $H_{\mathcal{Z}}$ described in Theorem 6.2(i) to define a semi-linear involution $\iota(\mathcal{D})$ on $H_{\mathcal{Z}}$, and then to construct an IC basis with respect to $\iota(\mathcal{D})$ (see [9]). By Theorem 9.2, we shall see that the resulting IC bases constructed from the semi-linear maps $\iota(\mathcal{D})$ are the same. This in turn shows that the definition of $\iota = \iota(\mathcal{D})$ is independent of the selection of distinguished sections (*cf.* Corollary 8.3). Hence, this definition for ι is also somehow natural.

7 Integral PBW and Canonical Bases for Quantum Affine $\mathfrak{sl}_n$

In this section, we give two applications of Theorem 6.2(ii). First, it can be used to prove that the $\mathcal{Z}$-form $U_{\mathcal{Z}}^+$ is $\mathcal{Z}$-free with many monomial bases determined by distinguished words. Second, from every such a monomial basis, we may construct an integral PBW basis for $U_{\mathcal{Z}}^+$ from which the canonical basis can be constructed by a standard linear algebra argument.

Lemma 7.1 *Let* $\mathbf{P}$ *be the subspace of* $\mathbf{H}$ *spanned by all* u_λ *with* $\lambda \in \Pi\backslash\Pi^a$. *Then as a vector space* $\mathbf{H} = \mathbf{U}^+ \oplus \mathbf{P}$.

Proof If suffices to prove that for each $\mathbf{d} \in \mathbb{N}^n$, $\mathbf{H_d} = \mathbf{U_d^+} \oplus \mathbf{P_d}$, where $\mathbf{P_d}$ is the $\mathbb{Q}(v)$-subspace of $\mathbf{H_d}$ spanned by all u_λ with $\lambda \in \Pi_{\mathbf{d}}\backslash\Pi_{\mathbf{d}}^a$.

First, we show $\mathbf{U_d^+} \cap \mathbf{P_d} = 0$. Take an $x \in \mathbf{U_d^+} \cap \mathbf{P_d}$ and suppose $x \neq 0$. Since $x \in \mathbf{U_d^+}$, we use the basis $\{\mathfrak{m}_{w_\pi} \mid \pi \in \Pi_{\mathbf{d}}^a\}$ for $\mathbf{U_d^+}$ constructed in Theorem 6.2 to write

$$x = \sum_{\pi \in \Pi_{\mathbf{d}}^a} a_\pi \mathfrak{m}_{w_\pi}$$

for some $a_\pi \in \mathbb{Q}(v)$. Now let $\mu \in \Pi_{\mathbf{d}}^a$ be maximal such that $a_\mu \neq 0$. Using (6.2), we can rewrite $x = \sum_{\lambda \in \Pi_{\mathbf{d}}} b_\lambda u_\lambda$. By the maximality of μ, we have $b_\mu = a_\mu v^{\delta_1(w_\mu)} \varphi_{w_\mu}^\mu(v^2) \neq 0$. This contradicts the fact that $x \in \mathbf{P_d}$. Hence, $\mathbf{U_d^+} \cap \mathbf{P_d} = 0$. Now a dimension comparison forces $\mathbf{H_d} = \mathbf{U_d^+} \oplus \mathbf{P_d}$. ∎

For each $\pi \in \Pi^a$, we now fix a distinguished word $w_\pi \in \Omega \cap \wp^{-1}(\pi)$ (see Proposition 4.3). Since $\gamma_{w_\pi}^\pi(v^2) = 1$, we may rewrite (6.4) as

$$\mathfrak{m}^{(w_\pi)} = v^{\delta(w_\pi)} u_\pi + v^{\delta(w_\pi)} \sum_{\lambda < \pi} \gamma_{w_\pi}^\lambda(v^2) u_\lambda. \tag{7.1}$$

https://doi.org/10.4153/CJM-2007-054-7 Published online by Cambridge University Press

Definition 7.2 For each given distinguished section $\mathcal{D} = \{w_\pi \mid \pi \in \Pi^a\}$, we define inductively the elements $E_\pi = E_\pi(\mathcal{D})$, $\pi \in \Pi^a$, as follows. For any $\mathbf{d} \in \mathbb{N}^n$ and $\pi \in \Pi^a_{\mathbf{d}}$, if π is minimal, put $E_\pi = \mathfrak{m}^{(w_\pi)} \in U^+_{\mathbf{d}} := \mathbf{U_d} \cap U^+_{\mathcal{Z}}$. Assume in general that $E_\lambda \in U^+_{\mathbf{d}}$ have been defined for all $\lambda \in \Pi^a_{\mathbf{d}}$ with $\lambda < \pi$. Then we define

$$E_\pi = \mathfrak{m}^{(w_\pi)} - \sum_{\lambda \in \Pi^a_{\mathbf{d}},\, \lambda < \pi} v^{\delta(w_\pi)-\delta(w_\lambda)} \gamma^\lambda_{w_\pi}(v^2) E_\lambda \in U^+_{\mathbf{d}}. \tag{7.2}$$

In other words, we have

$$\mathfrak{m}^{(w_\pi)} = E_\pi + \sum_{\lambda \in \Pi^a_{\mathbf{d}},\, \lambda < \pi} v^{\delta(w_\pi)-\delta(w_\lambda)} \gamma^\lambda_{w_\pi}(v^2) E_\lambda. \tag{7.3}$$

Lemma 7.3 *Let $\{w_\pi \mid \pi \in \Pi^a\}$ be a given distinguished section. For each $\mathbf{d} \in \mathbb{N}^n$ and each $\pi \in \Pi^a_{\mathbf{d}}$, we have*

$$E_\pi = v^{\delta(w_\pi)} u_\pi + \sum_{\lambda \in \Pi_{\mathbf{d}} \setminus \Pi^a_{\mathbf{d}},\, \lambda < \pi} \xi^\pi_\lambda u_\lambda$$

for some $\xi^\pi_\lambda \in \mathcal{Z}$.

Proof By (7.1), we have

$$\mathfrak{m}^{(w_\pi)} - \sum_{\lambda \in \Pi^a_{\mathbf{d}}, \lambda \le \pi} v^{\delta(w_\pi)} \gamma^\lambda_{w_\pi}(v^2) u_\lambda = \sum_{\lambda \in \Pi_{\mathbf{d}} \setminus \Pi^a_{\mathbf{d}},\, \lambda < \pi} v^{\delta(w_\pi)} \gamma^\lambda_{w_\pi}(v^2) u_\lambda \in \mathbf{P}.$$

On the other hand, replacing $\mathfrak{m}^{(w_\pi)}$ in the left-hand side by (7.2) yields

$$\mathfrak{m}^{(w_\pi)} - \sum_{\lambda \in \Pi^a_{\mathbf{d}},\, \lambda \le \pi} v^{\delta(w_\pi)} \gamma^\lambda_{w_\pi}(v^2) u_\lambda = \sum_{\lambda \in \Pi^a_{\mathbf{d}},\, \lambda \le \pi} v^{\delta(w_\pi)-\delta(w_\lambda)} \gamma^\lambda_{w_\pi}(v^2)(E_\lambda - v^{\delta(w_\lambda)} u_\lambda) \in \mathbf{P}.$$

Now an inductive argument concludes $E_\pi - v^{\delta(w_\pi)} u_\pi \in \mathbf{P}$. Hence,

$$E_\pi = v^{\delta(w_\pi)} u_\pi + \sum_{\lambda \in \Pi_{\mathbf{d}} \setminus \Pi^a_{\mathbf{d}}, \lambda < \pi} \xi^\pi_\lambda u_\lambda \quad \text{for some } \xi^\pi_\lambda \in \mathcal{Z},$$

as required. ∎

Example 7.4 Let $n = 3$ and $\mathbf{d} = (1, 2, 3)$. Then $\Pi_{\mathbf{d}}$ consists of 18 elements, *i.e.*, there are 18 isoclasses of nilpotent representations of $\Delta(3)$ of dimension vector $\mathbf{d}$. Let $\pi = [1; 3] + [2; 1] + 2[3; 1] \in \Pi_{\mathbf{d}}$, *i.e.*, $M(\pi) = S_1[3] \oplus S_2 \oplus 2S_3$. Then

$$\Pi^{\le \pi}_{\mathbf{d}} := \{\pi' \in \Pi_{\mathbf{d}} \mid \pi' \le \pi\} = \{\pi, \lambda, \mu, \nu, \tau\}$$

such that

$$M(\lambda) = S_1[2] \oplus S_2[2] \oplus 2S_3, \qquad M(\mu) = S_1[2] \oplus S_2 \oplus 3S_3,$$
$$M(\nu) = S_1 \oplus S_2[2] \oplus S_2 \oplus 2S_3, \qquad M(\tau) = S_1 \oplus 2S_2 \oplus 3S_3.$$

Clearly, π, λ, μ are aperiodic, ν, τ are periodic, and the Hasse diagram of $(\Pi_{\mathbf{d}}^{\leq \pi}, \leq)$ has the form

$$\begin{tikzcd}[column sep=small]
 & \pi \arrow[d, dash] & \\
 & \lambda \arrow[dl, dash] \arrow[dr, dash] & \\
\mu \arrow[dr, dash] & & \nu \arrow[dl, dash] \\
 & \tau &
\end{tikzcd}$$

Take distinguished words $w_1 = 1 2 3^3 2 \in \wp^{-1}(\pi)$, $w_2 = 2 1 3^3 2 \in \wp^{-1}(\lambda)$, and $w_3 = 1 3^3 2^2 \in \wp^{-1}(\mu)$. Then we get

$$\mathfrak{m}^{(w_1)} = v^2 u_\pi + v^2 u_\lambda + (v^4 + v^2) u_\mu + v^2 u_\nu + (v^4 + v^2) u_\tau,$$
$$\mathfrak{m}^{(w_2)} = v^3 u_\lambda + v^3 u_\mu + v^3 u_\nu + (v^5 + v^3) u_\tau,$$
$$\mathfrak{m}^{(w_3)} = v^6 u_\mu + v^6 u_\tau.$$

Thus, with respect to the chosen distinguished words w_1, w_2, w_3, we obtain

$$E_\pi = v^2 u_\pi - v^4 u_\tau, \qquad \mathfrak{m}^{(w_1)} = E_\pi + v^{-1} E_\lambda + (v^{-2} + v^{-4}) E_\mu,$$
$$E_\lambda = v^3 u_\lambda + v^3 u_\nu + v^5 u_\tau, \qquad \mathfrak{m}^{(w_2)} = E_\lambda + v^{-3} E_\mu,$$
$$E_\mu = v^6 u_\mu + v^6 u_\tau, \qquad \mathfrak{m}^{(w_3)} = E_\mu.$$

Theorem 7.5 *For each given distinguished section $\mathcal{D} = \{w_\pi \mid \pi \in \Pi^a\}$ of Ω over Π^a, each of the following sets forms a $\mathcal{Z}$-basis for $U_{\mathcal{Z}}^+$:*

(i) $\{\mathfrak{m}^{(w_\pi)} \mid \pi \in \Pi^a\}$;

(ii) $\{E_\lambda \mid \lambda \in \Pi^a\}$, *where* $E_\lambda = E_\lambda(\mathcal{D})$.

In particular, $U_{\mathcal{Z}}^+$ is a free $\mathcal{Z}$-module.

Proof By Theorem 6.2(ii), $\mathfrak{m}^{(w_\pi)}$, $\pi \in \Pi^a$, are $\mathcal{Z}$-linearly independent. It suffices to prove that for any dimension vector $\mathbf{d} \in \mathbb{N}^n$, the $\mathcal{Z}$-module $U_{\mathbf{d}}^+ = U_{\mathcal{Z}}^+ \cap \mathbf{U}_{\mathbf{d}}^+$ is spanned by $\{\mathfrak{m}^{(w_\pi)} \mid \pi \in \Pi_{\mathbf{d}}^a\}$, or equivalently, spanned by $\{E_\pi \mid \pi \in \Pi_{\mathbf{d}}^a\}$ by (7.3).

Let $x \in U_{\mathbf{d}}^+$ and write

$$x \equiv \sum_{\pi \in \Pi_{\mathbf{d}}^a} a_\pi u_\pi \pmod{\mathbf{P}},$$

where $a_\pi \in \mathcal{Z}$. Then we get by Lemma 7.3 that

$$x - \sum_{\pi \in \Pi_{\mathbf{d}}^a} v^{-\delta(w_\pi)} a_\pi E_\pi = \sum_{\pi \in \Pi_{\mathbf{d}}^a} v^{-\delta(w_\pi)} a_\pi (v^{\delta(w_\pi)} u_\pi - E_\pi) \in U_{\mathbf{d}}^+ \cap \mathbf{P}.$$

Since $U_{\mathbf{d}}^+ \cap \mathbf{P} = 0$ by Lemma 7.1, we have $x - \sum_{\pi \in \Pi_{\mathbf{d}}^a} v^{-\delta(w_\pi)} a_\pi E_\pi = 0$, as required. ■

With the basis $\{E_\pi\}_{\pi\in\Pi^a}$, we may follow the standard linear algebra method to define (uniquely) an IC basis $\{C_\pi\}_\pi$ as follows (see [10]).

The involution $\iota\colon H_{\mathcal{Z}} \to H_{\mathcal{Z}}$ defined at the end of the last section restricts to an involution $\iota\colon U_{\mathcal{Z}}^+ \to U_{\mathcal{Z}}^+$ taking $E_i^{(m)} \mapsto E_i^{(m)}$ and $v \mapsto v^{-1}$. For each polynomial $f \in \mathcal{Z}$, we will denote $\iota(f)$ by $\bar{f}$.

By restricting to $\Pi_{\mathbf{d}}^a$, (7.3) gives a transition matrix $(f_{\lambda,\pi})_{\lambda,\pi\in\Pi_{\mathbf{d}}^a}$ for each fixed dimension vector $\mathbf{d} \in \mathbb{N}^n$. This matrix has an inverse $(g_{\lambda,\pi})_{\lambda,\pi\in\Pi_{\mathbf{d}}^a}$ satisfying $g_{\lambda,\lambda} = 1$ and $g_{\lambda,\pi} = 0$ unless $\lambda \le \pi$. Thus we have

$$E_\pi = \mathfrak{m}^{(w_\pi)} + \sum_{\lambda<\pi} g_{\lambda,\pi}\mathfrak{m}^{(w_\lambda)}.$$

Applying ι, we get

$$\iota(E_\pi) = \mathfrak{m}^{(w_\pi)} + \sum_{\lambda<\pi} \bar{g}_{\lambda,\pi}\mathfrak{m}^{(w_\lambda)} = E_\pi + \sum_{\lambda<\pi} r_{\lambda,\pi}E_\lambda.$$

By [18, 7.10] (see [10] for more details), the system

$$p_{\lambda,\pi} = \sum_{\lambda\le\mu\le\pi} r_{\lambda,\mu}\bar{p}_{\mu,\pi} \qquad \text{for } \lambda \le \pi, \lambda, \pi \in \Pi_{\mathbf{d}}^a$$

has a unique solution satisfying $p_{\lambda,\lambda} = 1$, $p_{\lambda,\pi} \in v^{-1}\mathbb{Z}[v^{-1}]$ for $\lambda < \pi$. Moreover, the elements

$$C_\pi = \sum_{\lambda\le\pi,\ \lambda\in\Pi_{\mathbf{d}}^a} p_{\lambda,\pi}E_\lambda, \quad \pi \in \Pi_{\mathbf{d}}^a,$$

form a $\mathcal{Z}$-basis of $U_{\mathbf{d}}^+$. We shall prove in the next section that $\{C_\pi \mid \pi \in \Pi^a\}$ is in fact the canonical basis of $\mathbf{U}^+$ constructed in [20].

8 A Comparison of Canonical Bases for Quantum Affine $\mathfrak{sl}_n$

We first recall the geometric construction of Lusztig's canonical basis for the (generic twisted) Ringel–Hall algebra $H_{\mathcal{Z}}$.

For each $\pi \in \Pi$, we denote by $\mathcal{O}_\pi$ the orbit corresponding to the module $M(\pi)$ (see footnote 2). Let χ_π be the characteristic function of $\mathcal{O}_\pi$ and put

$$\langle \mathcal{O}_\pi \rangle = v^{\dim \mathcal{O}_\pi}\chi_\pi.$$

Thus, the Ringel–Hall algebra $H_{\mathcal{Z}}^L$ defined geometrically by Lusztig (see [16, 3.2]) has the (twisted) multiplication

$$\langle \mathcal{O}_\lambda \rangle\langle \mathcal{O}_\mu \rangle = \sum_\pi v^{\alpha(\lambda,\mu,\pi)}\varphi_{\lambda\mu}^\pi(v^{-2})\langle \mathcal{O}_\pi \rangle,$$

where
$$\alpha(\lambda,\mu,\pi)=\dim\mathcal{O}_\lambda+\dim\mathcal{O}_\mu-\dim\mathcal{O}_\pi+m(\lambda,\mu)$$
with $m(\lambda,\mu)=\sum_{i=1}^n\lambda_i\mu_i+\sum_{i=1}^n\lambda_i\mu_{i+1}$.

If we define for each $\pi\in\Pi$

$$\tilde{u}_\pi=v^{\dim\operatorname{End}(M(\pi))-\dim M(\pi)}u_\pi, \tag{8.1}$$

then we have the following.

Lemma 8.1 *For $\lambda,\mu,\pi\in\Pi$, let $\psi^\pi_{\lambda\mu}(v)\in\mathcal{Z}$ satisfy $\tilde{u}_\lambda\tilde{u}_\mu=\sum_\pi\psi^\pi_{\lambda\mu}(v)\tilde{u}_\pi$. Then*
$$\psi^\pi_{\lambda\mu}(v)=v^{-\alpha(\lambda,\mu,\pi)}\varphi^\pi_{\lambda\mu}(v^2).$$
Thus, we have a ring isomorphism $L\colon H_\mathcal{Z}\to H^L_\mathcal{Z}$ sending v to v^{-1} and $\tilde{u}_\lambda$ to $\langle\mathcal{O}_\lambda\rangle$.

Proof By [16, 3.3(7)], we have
$$\alpha(\lambda,\mu,\pi)=-\big(\dim\operatorname{End}M(\lambda)+\dim\operatorname{End}M(\mu)-\dim\operatorname{End}M(\pi)\big)+\varepsilon(\lambda,\mu).$$
Now the equality follows from the definition. ■

We further recall the geometric construction of the canonical basis for $H^L_\mathcal{Z}$ at $v=0$. Let $H^i_{\mathcal{O}_\lambda}(IC_{\mathcal{O}_\pi})$ be the stalk at a point of $\mathcal{O}_\pi$ of the i-th intersection cohomology sheaf of the closure $\overline{\mathcal{O}_\lambda}$ of $\mathcal{O}_\lambda$, and let
$$\mathfrak{b}^L_\pi=\sum_{i,\,\lambda\le\pi}v^{-i+\dim\mathcal{O}_\pi-\dim\mathcal{O}_\lambda}\dim H^i_{\mathcal{O}_\lambda}(IC_{\mathcal{O}_\pi})\langle\mathcal{O}_\lambda\rangle.$$
Then the set $\{\mathfrak{b}^L_\pi\mid\pi\in\Pi\}$ is the canonical basis (at $v=0$) of $H^L_\mathcal{Z}$ introduced in [16, 30]. Denote by $\mathfrak{b}_\pi$ the corresponding basis for $H_\mathcal{Z}$, that is, $\mathfrak{b}_\pi$ is sent to $\mathfrak{b}^L_\pi$ under the map L. Then the set $\{\mathfrak{b}_\pi\mid\pi\in\Pi\}$ is the canonical basis (at $v=\infty$) for $H_\mathcal{Z}$ and the elements $\mathfrak{b}_\pi$ with $\pi\in\Pi$ are characterized as the *unique* elements of $\mathfrak{L}$ such that

$$\iota(\mathfrak{b}_\pi)=\mathfrak{b}_\pi,\quad \mathfrak{b}_\pi\in\sum_{\lambda\le\pi}\mathbb{Z}[v^{-1}]\tilde{u}_\lambda \text{ and } \mathfrak{b}_\pi\equiv\tilde{u}_\pi(\operatorname{mod}v^{-1}\mathfrak{L}), \tag{8.2}$$

where ι is an involution on $H_\mathcal{Z}$ satisfying $\iota(v)=v^{-1}$ and $\iota(\tilde{u}_\mathbf{a})=\tilde{u}_\mathbf{a}$ for all $\mathbf{a}\in\mathbb{N}^n$, and $\mathfrak{L}$ is the $\mathbb{Z}[v^{-1}]$-submodule of $H_\mathcal{Z}$ spanned by $\tilde{u}_\pi$, $\pi\in\Pi$. In other words, for any $\lambda\le\pi$ in Π, the Laurent polynomials
$$P_{\lambda,\pi}:=\sum_i v^{i-\dim\mathcal{O}_\pi+\dim\mathcal{O}_\lambda}\dim H^i_{\mathcal{O}_\lambda}(IC_{\mathcal{O}_\pi})$$
satisfy $P_{\pi,\pi}=1$, $P_{\lambda,\pi}\in v^{-1}\mathbb{Z}[v^{-1}]$ for $\lambda<\pi$, and $\mathfrak{b}_\pi=\sum_{\lambda\le\pi}P_{\lambda,\pi}\tilde{u}_\lambda$.

Note that it is shown in [20] that the subset $\{\mathfrak{b}_\pi\mid\pi\in\Pi^a\}$ over Π^a is a basis for $U_\mathcal{Z}$ and is called the canonical basis of $U^+_\mathcal{Z}$. We now use the uniqueness to prove that the basis $\{C_\pi\}_{\pi\in\Pi^a}$ coincides with the basis $\{\mathfrak{b}_\pi\mid\pi\in\Pi^a\}$. We need a lemma.

Lemma 8.2 *Let $\pi \in \Pi$ be aperiodic. Then for each distinguished word $w \in \Omega \cap \wp^{-1}(\pi)$, we have*

$$\delta(w) = \delta_1(w) + \delta_2(w) = \dim \mathrm{End}(M(\pi)) - \dim M(\pi).$$

Proof Let $w = i_1 i_2 \cdots i_m \in \wp^{-1}(\pi)$ be distinguished with the tight form $j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t}$. We write $j = j_1$ and $e = e_1$, and let $w' = j^{e-1} j_2^{e_2} \cdots j_t^{e_t}$. From the definition of a distinguished word, we have that w' is again distinguished. We further set $\mu = \wp(w')$. Thus, $S_j * M(\mu) = M(\pi)$.

We use induction on the length m of w. If $m = 1$, it is clear. Now let $m > 1$. By induction hypothesis, we have for w'

$$\delta(w') = \delta_1(w') + \delta_2(w') = \dim \mathrm{End}(M(\mu)) - \dim M(\mu).$$

On the other hand, we have clearly (see (6.2))

$$\delta_1(w) = \sum_{s=2}^{m} \varepsilon(\mathbf{dim}\, S_1, \mathbf{dim}\, S_{i_s}) + \delta_1(w') = \delta_1(w') + \varepsilon(\mathbf{dim}\, S_1, \mathbf{dim}\, M(\mu))$$

and (see (6.3))

$$\delta_2(w) = \frac{e(e-1)}{2} + \delta_2(w') - \frac{(e-1)(e-2)}{2} = \delta_2(w') + e - 1.$$

Thus, we obtain

$$\delta(w) = \delta(w') + \varepsilon(\mathbf{dim}\, S_1, \mathbf{dim}\, M(\mu)) + e - 1. \tag{8.3}$$

Let $\mu = \sum_{i,l} \mu_{i,l}[i;l)$ and take l_0 maximal such that $\mu_{j+1,l_0} \neq 0$. Then $j * \mu = \pi$ implies $\nu := \pi - [j; l_0+1) = \mu - [j+1; l_0)$. In other words, $M(\pi) \cong M(\nu) \oplus S_j[l_0+1]$ and $M(\mu) \cong M(\nu) \oplus S_{j+1}[l_0]$. Thus, we have

$$\begin{aligned}\dim \mathrm{Hom}(M(\nu), S_j[l_0+1]) - \dim \mathrm{Hom}(M(\nu), S_{j+1}[l_0]) &= \sum_{l > l_0} \nu_{j,l} \\ &= (\pi_{j,l_0+1} - 1) + \sum_{l > l_0+1} \pi_{j,l}.\end{aligned}$$

Since w is distinguished, $M(\pi)$ has a unique submodule isomorphic to

$$M(\wp(j_2^{e_2} \cdots j_t^{e_t})).$$

Equivalently, $M_j(\pi) = M(\pi^{(j)})$ has a unique submodule N with $M_j(\pi)/N \cong eS_j$. By [4, Lemma 5.4], the uniqueness implies $e = \sum_{l > l_0} \pi_{j,l}$. Hence,

$$\dim \mathrm{Hom}(M(\nu), S_j[l_0+1]) - \dim \mathrm{Hom}(M(\nu), S_{j+1}[l_0]) = e - 1.$$

https://doi.org/10.4153/CJM-2007-054-7 Published online by Cambridge University Press

From the maximality of l_0, we compute

$$\dim \operatorname{Hom}(S_j[l_0+1], M(\nu)) - \dim \operatorname{Hom}(S_{j+1}[l_0], M(\nu)) = s - t,$$

where s denotes the multiplicity of S_j in $\operatorname{soc} M(\nu)$ and $t = \sum_{l\geq 1} \nu_{j+1,l}$. This is because each map from $S_j[l_0+1]$ into $\operatorname{soc} M(\nu)$ is zero when restricted to $S_{j+1}[l_0]$; while each surjective map from $S_{j+1}[l_0]$ onto each summand $S_{j+1}[l]$ of $M(\nu)$ cannot be lifted to $S_j[l_0+1]$. Furthermore, we have

$$\dim \operatorname{End}(S_j[l_0+1]) = \begin{cases} \dim \operatorname{End}(S_{j+1}[l_0]) + 1 & \text{if } n | l_0, \\ \dim \operatorname{End}(S_{j+1}[l_0]) & \text{otherwise,} \end{cases}$$

since $\operatorname{soc} S_j[l_0+1] = S_j$ if and only if $n|l_0$. Altogether, we obtain

$$\dim \operatorname{End}(M(\pi)) = \begin{cases} \dim \operatorname{End}(M(\mu)) + s + e - t & \text{if } n|l_0, \\ \dim \operatorname{End}(M(\mu)) + s + e - t - 1 & \text{otherwise.} \end{cases}$$

We also have

$$\begin{aligned} \varepsilon(\mathbf{dim}\, S_j, \mathbf{dim}\, M(\mu)) &= \dim \operatorname{Hom}(S_j, M(\mu)) - \dim \operatorname{Ext}^1(S_j, M(\mu)) \\ &= \begin{cases} s - t & \text{if } n|l_0, \\ s - t - 1 & \text{otherwise.} \end{cases} \end{aligned}$$

Finally, putting everything into (8.3), we obtain that

$$\delta(w) = \delta_1(w) + \delta_2(w) = \dim \operatorname{End}(M(\pi)) - \dim M(\pi),$$

as required. ■

For each $\pi \in \Pi^a$, we pick a distinguished word $w_\pi \in \Omega \cap \wp^{-1}(\pi)$ to form a distinguished section $\mathcal{D} = \{w_\pi \mid \pi \in \Pi^a\}$, and let $\{E_\pi \mid \pi \in \Pi^a\}$ be the basis of $U_{\mathcal{Z}}^+$ defined with respect to $\mathcal{D}$ in Definition 7.2. Then by Lemmas 7.3 and 8.2, we have for each $\pi \in \Pi_{\mathbf{d}}^a$

$$\begin{aligned} E_\pi &= \tilde{u}_\pi + \sum_{\lambda \in \Pi_{\mathbf{d}} \setminus \Pi_{\mathbf{d}}^a,\, \lambda < \pi} \eta_\lambda^\pi \tilde{u}_\lambda \quad (\eta_\lambda^\pi \in \mathcal{Z}), \\ \mathfrak{m}^{(w_\pi)} &= E_\pi + \sum_{\lambda \in \Pi_{\mathbf{d}}^a,\, \lambda < \pi} f_{\lambda,\pi} E_\lambda \quad (f_{\lambda,\pi} = v^{\delta(w_\pi) - \delta(w_\lambda)} \gamma_{w_\pi}^\lambda (v^2) \in \mathcal{Z}). \end{aligned} \tag{8.4}$$

Corollary 8.3 *The basis $\{E_\pi \mid \pi \in \Pi^a\}$ is independent of the selection of distinguished sections.*

Proof Suppose $\mathcal{D}$ and $\mathcal{D}'$ are two distinguished sections. Then $E_\pi(\mathcal{D}) - E_\pi(\mathcal{D}')$ is a linear combination of u_λ, $\lambda \in \Pi \setminus \Pi^a$, *i.e.*, $E_\pi(\mathcal{D}) - E_\pi(\mathcal{D}') \in \mathbf{U}^+ \cap \mathbf{P}$. Hence it is zero by Lemma 7.1. ■

Lemma 8.4 *For $\pi \in \Pi^a_{\mathbf{d}}$ and $\lambda \in \Pi \backslash \Pi^a$ with $\lambda < \pi$, we have $\eta^\pi_\lambda \in v^{-1}\mathbb{Z}[v^{-1}]$, that is, $E_\pi \in \mathfrak{L}$ and $E_\pi \equiv \tilde{u}_\pi (\mathrm{mod}\, v^{-1}\mathfrak{L})$.*

Proof First, let $\pi \in \Pi^a_{\mathbf{d}}$ be minimal. Then

$$E_\pi - \mathfrak{b}_\pi = \sum_{\lambda \in \Pi_{\mathbf{d}} \backslash \Pi^a_{\mathbf{d}},\, \lambda < \pi} (\eta^\pi_\lambda - P_{\lambda,\pi})\tilde{u}_\lambda \in \mathbf{U}^+_{\mathbf{d}} \cap \mathbf{P}_{\mathbf{d}}.$$

By Lemma 7.1, $E_\pi - \mathfrak{b}_\pi$ must be zero, that is, $E_\pi = \mathfrak{b}_\pi$ and $\eta^\pi_\lambda = P_{\lambda,\pi} \in v^{-1}\mathbb{Z}[v^{-1}]$ for all $\lambda < \pi$ with $\lambda \notin \Pi^a_{\mathbf{d}}$. Now let $\pi \in \Pi^a_{\mathbf{d}}$ and assume that the result is true for all $\mu \in \Pi^a_{\mathbf{d}}$ with $\mu < \pi$, that is, for such a μ, we have $\eta^\mu_\nu \in v^{-1}\mathbb{Z}[v^{-1}]$ for all $\nu \in \Pi_{\mathbf{d}} \backslash \Pi^a_{\mathbf{d}}$ with $\nu < \mu$. Consider the element

$$\begin{aligned}
\mathfrak{b}_\pi - \sum_{\lambda \in \Pi^a_{\mathbf{d}},\, \lambda \le \pi} P_{\lambda,\pi} E_\lambda &= (\tilde{u}_\pi - E_\pi) + \sum_{\mu \in \Pi^a_{\mathbf{d}},\, \mu < \pi} P_{\mu,\pi}(\tilde{u}_\mu - E_\mu) + \sum_{\sigma \in \Pi_{\mathbf{d}} \backslash \Pi^a_{\mathbf{d}},\, \sigma < \pi} P_{\sigma,\pi}\tilde{u}_\sigma \\
&= - \sum_{\lambda \in \Pi_{\mathbf{d}} \backslash \Pi^a_{\mathbf{d}},\, \lambda < \pi} \eta^\pi_\lambda \tilde{u}_\lambda - \sum_{\substack{\mu \in \Pi^a_{\mathbf{d}},\, \mu < \pi \\ \nu \in \Pi_{\mathbf{a}} \backslash \Pi^a_{\mathbf{d}},\, \nu < \mu}} P_{\mu,\pi}\eta^\mu_\nu \tilde{u}_\nu + \sum_{\sigma \in \Pi_{\mathbf{d}} \backslash \Pi^a_{\mathbf{d}},\, \sigma < \pi} P_{\sigma,\pi}\tilde{u}_\sigma
\end{aligned}$$

which is clearly in $U^+_{\mathbf{d}} \cap \mathbf{P}_{\mathbf{d}}$. Again, by Lemma 7.1, we must have

$$\mathfrak{b}_\pi - \sum_{\lambda \in \Pi^a_{\mathbf{d}},\, \lambda < \pi} P_{\lambda,\pi} E_\lambda = 0,$$

that is,

$$\sum_{\lambda \in \Pi_{\mathbf{d}} \backslash \Pi^a_{\mathbf{d}},\, \lambda < \pi} \eta^\pi_\lambda \tilde{u}_\lambda = - \sum_{\substack{\mu \in \Pi^a_{\mathbf{d}}, \mu < \pi \\ \nu \in \Pi_{\mathbf{d}} \backslash \Pi^a_{\mathbf{d}}, \nu < \mu}} P_{\mu,\pi}\eta^\mu_\nu \tilde{u}_\nu + \sum_{\sigma \in \Pi_{\mathbf{d}} \backslash \Pi^a_{\mathbf{d}},\, \sigma < \pi} P_{\sigma,\pi}\tilde{u}_\sigma.$$

This implies by induction $\eta^\pi_\lambda \in v^{-1}\mathbb{Z}[v^{-1}]$ for all $\lambda < \pi, \lambda \notin \Pi^a_{\mathbf{d}}$. ■

With what we have done above, the following comparison now follows easily.

Theorem 8.5 *For each $\pi \in \Pi^a$, we have $C_\pi = \mathfrak{b}_\pi$.*

Proof From the construction, we have $\iota(C_\pi) = C_\pi$ and

$$C_\pi = E_\pi + \sum_{\lambda < \pi, \lambda \in \Pi^a} p_{\lambda,\pi} E_\lambda,$$

where $p_{\lambda,\pi} \in v^{-1}\mathbb{Z}[v^{-1}]$. By Lemma 8.4, we see that

$$C_\pi \in \sum_{\lambda \le \pi} \mathbb{Z}[v^{-1}]\tilde{u}_\lambda \quad \text{and} \quad C_\pi \equiv E_\pi \equiv \tilde{u}_\pi (\mathrm{mod}\ v^{-1}\mathfrak{L}).$$

Thus, $\{C_\pi \mid \pi \in \Pi^a\}$ also satisfies the three properties in (8.2). Hence, $C_\pi = \mathfrak{b}_\pi$ for each $\pi \in \Pi^a$. ■

https://doi.org/10.4153/CJM-2007-054-7 Published online by Cambridge University Press

Remark 8.6 (i) The basis $\{E_\pi \mid \pi \in \Pi^a\}$ plays a role as a PBW basis. It would be interesting to know if the PBW type basis (for affine type A) constructed in [2, 3.9, 3.39], involving braid group actions, is the same as the basis E_π presented here. It would be also interesting to know the meaning of the coefficients η_λ^π given in (8.4).

(ii) This elementary construction is an important component in a more general elementary construction [17] of canonical bases for quantum groups associated to all symmetric affine Kac–Moody Lie algebras. It is expected that one can extend this elementary construction to the symmetrizable affine case using the theory developed in [7,8], or the new approach developed in [6].

9 An Algebraic Construction of the Canonical Basis for $H_{\mathcal{Z}}$

In this section, we shall use distinguished words of the form in (4.3) to present an algebraic construction of the canonical basis for the whole Ringel–Hall algebra $H_{\mathcal{Z}}$.

Let $\pi \in \Pi$ with (π', π'') its associated pair. Choose a distinguished pair as in (4.2):

$$w_{\pi''} = j_1^{e_1} \cdots j_t^{e_t} \in \Omega \cap \wp^{-1}(\pi'') \quad (j_{r-1} \neq j_r, \forall r),$$
$$y_{\pi'} = \mathbf{a}_1 \cdots \mathbf{a}_p \in \Sigma \cap \wp^{-1}(\pi'),$$

and form $w_\pi = w_{\pi''} y_{\pi'}$. By (6.4) and (6.5), we have

$$\mathfrak{m}^{(w_{\pi''})} = \tilde{u}_{e_1 j_1} \cdots \tilde{u}_{e_t j_t} = v^{\delta(w_{\pi''})} \sum_{\mu \leq \pi''} \gamma^\mu_{w_{\pi''}}(v^2) u_\mu,$$
$$\mathfrak{m}^{(y_{\pi'})} = \tilde{u}_{\mathbf{a}_1} \cdots \tilde{u}_{\mathbf{a}_p} = v^{\delta(y_{\pi'})} \sum_{\nu \leq \pi'} \gamma^\nu_{y_{\pi'}}(v^2) u_\nu,$$

where

$$\delta(y_{\pi'}) := \sum_{s=1}^{p} (\|\mathbf{a}_s\| - |\mathbf{a}_s|) + \sum_{1 \leq s < t \leq p} \varepsilon(\mathbf{dim}\, S_{\mathbf{a}_s}, \mathbf{dim}\, S_{\mathbf{a}_t}).$$

Finally, we get

$$\begin{aligned} \mathfrak{m}^{(w_\pi)} &= \mathfrak{m}^{(w_{\pi''})} \mathfrak{m}^{(y_{\pi'})} \\ &= v^{\delta(w_{\pi''}) + \delta(y_{\pi'}) + \varepsilon(\mathbf{dim}\, M(\pi''), \mathbf{dim}\, M(\pi'))} \sum_{\lambda \leq \pi} \gamma^\lambda_{w_\pi}(v^2) u_\lambda. \end{aligned} \tag{9.1}$$

A key step in such a construction is to prove that the coefficient of $\tilde{u}_\pi$ in (9.1) is 1 (see (8.1)).

Proposition 9.1 *Let $\pi = \sum_{i \in I, l \geq 1} \pi_{i,l}[i; l] \in \Pi$ with $p = p(\pi)$ and (π', π'') be the associated pair. For each distinguished word $w_{\pi''} \in \Omega \cap \wp^{-1}(\pi'')$, we have*

$$\dim \mathrm{End}(M(\pi)) - \dim M(\pi) = \delta(w_{\pi''}) + \delta(y_{\pi'}) + \varepsilon(\mathbf{dim}\, M(\pi''), \mathbf{dim}\, M(\pi')).$$

https://doi.org/10.4153/CJM-2007-054-7 Published online by Cambridge University Press

Proof We prove the proposition by induction on p. If $p = 0$, *i.e.,* π is aperiodic, this is the case treated in Lemma 8.2. Suppose now $p \geq 1$ and write $M = M(\pi)$. Let $y_{\pi'} = \mathbf{a}_1\mathbf{a}_2 \cdots \mathbf{a}_p$. Then $\operatorname{soc} M \cong S_{\mathbf{a}_p}$. Let $\mu \in \Pi$ be such that $M(\mu) \cong M/\operatorname{soc} M$. Then $\mu'' = \pi''$ and $y_{\mu'} = \mathbf{a}_1\mathbf{a}_2 \cdots \mathbf{a}_{p-1}$. In particular, $w_{\pi''}y_{\mu'}$ is a distinguished word in $\wp^{-1}(\mu)$. Since $p(\mu) = p - 1$, we have by induction that

$$\dim \operatorname{End}(M(\mu)) - \dim M(\mu) = \delta(w_{\pi''}) + \delta(y_{\mu'}) + \varepsilon(\mathbf{dim}\, M(\pi''), \mathbf{dim}\, M(\mu')).$$

It is clear that

$$\dim M = \dim M(\mu) + \dim \operatorname{soc} M = \dim M(\mu) + |\mathbf{a}_p|,$$
$$\delta(y_{\pi'}) = \delta(y_{\mu'}) + \|\mathbf{a}_p\| - |\mathbf{a}_p| + \varepsilon(\mathbf{a}_1 + \cdots + \mathbf{a}_{p-1}, \mathbf{a}_p),$$

and

$$\begin{aligned}\varepsilon(\mathbf{dim}\, M(\pi''), \mathbf{dim}\, M(\pi')) = \varepsilon(&\mathbf{dim}\, M(\pi''), \mathbf{dim}\, M(\mu')) \\ &+ \varepsilon(\mathbf{dim}\, M(\pi''), \mathbf{dim} \operatorname{soc} M).\end{aligned}$$

On the other hand, since each indecomposable summand of M is uniserial, we have

$$\begin{aligned}\dim \operatorname{End}(M) &= \dim \operatorname{End}(M/\operatorname{soc} M) + \dim \operatorname{Hom}(M, \operatorname{soc} M) \\ &= \dim \operatorname{End}(M(\mu)) + \dim \operatorname{Hom}(\operatorname{top} M, \operatorname{soc} M).\end{aligned}$$

Note that $\|\mathbf{a}_p\| = \dim \operatorname{End}(\operatorname{soc} M)$, $\mathbf{a}_p = \mathbf{dim} \operatorname{soc} M$ and

$$\varepsilon(\mathbf{a}_1 + \cdots + \mathbf{a}_{p-1}, \mathbf{a}_p) + \varepsilon(\mathbf{dim}\, M(\pi''), \mathbf{dim} \operatorname{soc} M) = \varepsilon(\mathbf{dim}\, M(\mu), \mathbf{dim} \operatorname{soc} M).$$

Hence, it remains to show that

$$\dim \operatorname{Hom}(\operatorname{top} M, \operatorname{soc} M) = \varepsilon(\mathbf{dim}\, M(\mu), \mathbf{dim} \operatorname{soc} M) + \dim \operatorname{End}(\operatorname{soc} M).$$

Now let $l = \operatorname{Ll}(M)$ and for each $1 \leq r \leq l$, set $\operatorname{soc}^{l-r+1} M/\operatorname{soc}^{l-r} M = S_{\mathbf{d}_r}$ for some $\mathbf{d}_r \in \mathbb{N}^n$. In particular, $\operatorname{soc} M = S_{\mathbf{d}_l}$, *i.e.,* $\mathbf{d}_l = \mathbf{a}_p$. Now, for $\mathbf{a} = (a_i), \mathbf{b} = (b_i) \in \mathbb{N}^n$, we define $\tau\mathbf{a} = (a_n, a_1, \ldots, a_{n-1})$ and $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^n a_i b_i$. Then by definition we have that $\varepsilon(\mathbf{a}, \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} - \tau\mathbf{a} \cdot \mathbf{b} = (\mathbf{a} - \tau\mathbf{a}) \cdot \mathbf{b}$. Furthermore, we have $\operatorname{top} M = S_{\mathbf{c}}$ with

$$\mathbf{c} = \mathbf{dim} \operatorname{top} M = (\sum_{l'} \pi_{1,l'}, \ldots, \sum_{l'} \pi_{n,l'}) = \sum_{l'=1}^{l} (\pi_{1,l'}, \ldots, \pi_{n,l'}).$$

Hence, $\mathbf{c} = \mathbf{d}_1 + (\mathbf{d}_2 - \tau\mathbf{d}_1) + \cdots + (\mathbf{d}_l - \tau\mathbf{d}_{l-1})$. Finally, we obtain

$$\begin{aligned}\dim \operatorname{Hom}(\operatorname{top} M, \operatorname{soc} M) = \mathbf{c} \cdot \mathbf{d}_l &= \sum_{r=1}^{l-1} (\mathbf{d}_r - \tau\mathbf{d}_r) \cdot \mathbf{d}_l + \mathbf{d}_l \cdot \mathbf{d}_l \\ &= \varepsilon(\mathbf{d}_1 + \cdots + \mathbf{d}_{l-1}, \mathbf{d}_l) + \|\mathbf{d}_l\| \\ &= \varepsilon(\mathbf{dim}\, M/\operatorname{soc} M, \mathbf{dim} \operatorname{soc} M) + \dim \operatorname{End}(\operatorname{soc} M),\end{aligned}$$

as desired. ∎

https://doi.org/10.4153/CJM-2007-054-7 Published online by Cambridge University Press

We now have all the ingredients for the elementary construction of a canonical basis. First, the Ringel–Hall algebra $H_{\mathcal{Z}}$ admits the involution ι; see (8.2). Second, we use the basis $\{\tilde{u}_\pi \mid \pi \in \Pi\}$ as a PBW basis. To see the triangular relation when applying ι to $\tilde{u}_\pi$, we use a monomial basis of the form $\{\mathfrak{m}^{(w_\pi)} \mid \pi \in \Pi\}$ constructed in (9.1) whose members are fixed by ι. Thus, for each $\pi \in \Pi$ with the associated pair (π', π''), we fix a distinguished word $w_{\pi''} \in \Omega \cap \wp^{-1}(\pi'')$. By Proposition 4.5, the word $w_\pi = w_{\pi''} y_{\pi'}$ is also distinguished. By Proposition 9.1, (9.1) becomes

$$\mathfrak{m}^{(w_\pi)} = \tilde{u}_\pi + \sum_{\lambda<\pi} \theta_{\lambda,\pi} \tilde{u}_\lambda, \tag{9.2}$$

where $\theta_{\lambda,\pi} = v^{\dim \operatorname{End}(M(\pi)) - \dim \operatorname{End}(M(\lambda))} \gamma^{\lambda}_{w_\pi}(v^2)$. Solving (9.2) gives

$$\tilde{u}_\pi = \mathfrak{m}^{(w_\pi)} + \sum_{\lambda<\pi} \zeta_{\lambda,\pi} \mathfrak{m}^{(w_\lambda)}.$$

Now, applying the standard construction at the end of §7 yields a new basis $\{\mathfrak{c}_\pi \mid \pi \in \Pi\}$ of $H_{\mathcal{Z}}$ satisfying

$$\mathfrak{c}_\pi = \sum_{\lambda \leq \pi} \sigma_{\lambda,\pi} \tilde{u}_\lambda,$$

where $\sigma_{\pi,\pi} = 1$ and $\sigma_{\lambda,\pi} \in v^{-1}\mathbb{Z}[v^{-1}]$ for $\lambda < \pi$. Since the basis $\{\mathfrak{b}_\pi \mid \pi \in \Pi\}$ satisfies the same property (see (8.2)), the uniqueness of the canonical basis implies the following theorem (*cf.* Theorem 8.5).

Theorem 9.2 *For each $\pi \in \Pi$, we have $\mathfrak{c}_\pi = \mathfrak{b}_\pi$. In particular, we have, for each $\pi \in \Pi^a$, $\mathfrak{c}_\pi = C_\pi$.*

Acknowledgement The authors would like to thank the referee for the nice proof of Proposition 9.1 and the suggestion to use the multisegment parametrization of nilpotent representations, which simplifies the ideas in the proofs involving the calculation of generic extensions throughout the paper.

References

[1] J. Beck, V. Chari and A. Pressley, *An algebraic characterization of the affine canonical basis.* Duke Math. J. **99**(1999), no. 3, 455–487.
[2] J. Beck and H. Nakajima, *Crystal bases and two sided cells of quantum affine algebras.* Duke Math. J. **123**(2004), no. 2, 335–402.
[3] K. Bongartz, *On degenerations and extensions of finite-dimensional modules.* Adv. Math. **121**(1996), no. 2, 245–287.
[4] B. Deng and J. Du, *Monomial bases for quantum affine $\mathfrak{sl}_n$.* Adv. Math. **191**(2005), no. 2, 276–304.
[5] ———, *bases of quantized enveloping algebras.* Pacific J. Math. **220**(2005), no. 1, 33–48.
[6] ———, *Frobenius morphisms and representations of algebras.* Trans. Amer. Math. Soc. **358**(2006), no. 8, 3591–3622.
[7] V. Dlab and C. M. Ringel, *On algebras of finite representation type.* J. Algebra **33**(1975), 306–394.
[8] ———, *Indecomposable representations of graphs and algebras.* Memoirs Amer. Math. Soc. **6**(1976), no. 173.
[9] J. Du, *A matrix approach to IC bases.* In: Representations of Algebras, CMS Conf. Proc. 14, American Mathematical Society, Providence, RI, 1993, pp. 165–174.

[10] ———, *IC bases and quantum linear groups.* In: Algebraic Groups and Their Generalizations: Quantum and Infinite-Dimensional Methods, Proc. Sympos. Pure Math. 56, American Mathematical Society, Providence, RI, 1994, pp. 135–148.

[11] J. Du and B. Parshall, *Monomial bases for q-Schur algebras.* Trans. Amer. Math. Soc. **355**(2003), no. 4, 1593–1620.

[12] I. Grojnowski and G. Lusztig, *A comparison of bases of quantized enveloping algebras.* In: Linear Algebraic Groups and Their Representations, Contemp. Math. 153, American Mathematical Society, Providence, RI, 1993, 11–19.

[13] J. Y. Guo, *The Hall polynomials of a cyclic serial algebra.* Comm. Algebra **23**(1995), no. 2, 743–751.

[14] M. Kashiwara, *On cystal bases of the Q-analogue of universal enveloping algebras.* Duke Math. J. **63**(1991), no. 2, 465–516.

[15] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras.* Invent. Math. **53**(1979), no. 2, 165–184.

[16] B. Leclerc, J.-Y. Thibon, and E. Vasserot, *Zelevinsky's involution at roots of unity.* J. Reine Angew. Math. **513**(1999), 33–51.

[17] Z. Lin, J. Xiao and G. Zhang, *Representations of tame quivers and affine canonical bases.* arXiv:0706.1444v3.

[18] G. Lusztig, *Canonical bases arising from quantized enveloping algebras,* J. Amer. Math. Soc. **3** (1990), no. 2, 447–498.

[19] ———, *Quivers, perverse sheaves, and quantized enveloping algebras,* J. Amer. Math. Soc. **4**(1991), 365–421.

[20] ———, *Affine quivers and canonical bases.* Inst. Hautes Études Sci. Publ. Math. **76**(1992), 111–163.

[21] ———, *Introduction to Quantum Groups,* Progress in Mathematics 110, Birkhäuser Boston, Boston MA, 1993.

[22] A. Mah, *Generic extension monoids for cyclic quivers.* Ph.D. thesis, University of New South Wales, Sydney, 2006.

[23] R. V. Moody and A. Pianzola, *Lie algebras with triangular decompositions.* John Wiley and Sons, New York 1995.

[24] M. Reineke, *Generic extensions and multiplicative bases of quantum groups at $q = 0$.* Represent. Theory **5**(2001), 147–163 (electronic).

[25] C. M. Ringel, *Hall algebras and quantum groups.* Invent. Math. **101**(1990), no. 3, 583–591.

[26] ———, *The composition algebra of a cyclic quiver.* Proc. London Math. Soc. **66**(1993), no. 3, 507–537.

[27] ———, *Hall algebras revisited.* In: Quantum Deformations of Algebras and Their Representations. Israel Math. Conf. Proc. 7, Bar-Ilan Univ., Ramat Gan, 1993, pp. 171–176.

[28] ———, *The Hall algebra approach to quantum groups.* In: XI Latin American School of Mathematics (Spanish). Aportaciones Mat. Comun. 15, Soc. Mat. Mexicana, Mxico, 1995. pp. 85–114.

[29] O. Schiffmann, *The Hall algebra of a cyclic quiver and canonical bases of Fock spaces.* Internat. Math. Res. Notices (2000), no. 8, 413–440.

[30] M. Varagnolo and E. Vasserot, *On the decomposition matrices of the quantized Schur algebra.* Duke Math. J. **100** (1999), no. 2, 267–297.

[31] G. Zwara, *Degenerations for modules over representation-finite biserial algebras.* J. Algebra **198**(1997), no. 2, 563–581.

School of Mathematical Sciences
Beijing Normal University
Beijing 100875
P.R. China
e-mail: dengbm@bnu.edu.cn

School of Mathematics and Statistics
University of New South Wales
Sydney, NSW 2052
Australia
e-mail: j.du@unsw.edu.au

Department of Mathematical Sciences
Tsinghua University
Beijing 100084
P.R. China
e-mail: jxiao@math.tsinghua.edu.cn