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RESEARCH ARTICLE

A note on the splittings of finitely presented Bestvina–Brady groups

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Abstract
We show that when a finitely presented Bestvina–Brady group splits as an amalgamated product over a subgroup H, its defining graph contains an induced separating subgraph whose associated Bestvina–Brady group is contained in a conjugate of H.

1. Introduction

Let Γ be a finite simplicial graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. The associated *right-angled Artin group (RAAG)*, denoted by A_Γ, is generated by $V(\Gamma)$, and two generators v and w commute whenever they are connected by an edge. A common question in group theory is: when does a group split as an amalgamated product or an HNN extension over a subgroup? For RAAGs, the splittings over infinite cyclic subgroups and abelian subgroups were characterized by Clay [3] and by Groves and Hull [6], respectively. Recently, Hull [7] generalized the splittings of RAAGs over abelian subgroups to non-abelian subgroups.

Let $\phi\colon A_\Gamma \to \mathbb{Z}$ be a homomorphism that sends all the generators to 1. The kernel of ϕ is called the *Bestvina–Brady group* and is denoted by BB_Γ. We only focus on finitely presented Bestvina–Brady groups, which is equivalent to saying that the flag complexes on the defining graphs are simply connected [1]. The author in [2] characterized the splittings of finitely presented Bestvina–Brady groups over abelian subgroups. In this note, we prove a result for the splittings of finitely presented Bestvina–Brady groups over non-abelian subgroups.

Theorem 1.1. *Let Γ be a finite simplicial graph with no cut vertices and whose associated flag complex is simply connected. Suppose that BB_Γ splits as an amalgamated product over a subgroup H. Then Γ contains an induced subgraph Λ that separates Γ and BB_Λ is contained in a conjugate of H.*

In other words, Theorem 1.1 says that if BB_Γ acts on a tree which is not a line, then there is an induced subgraph Λ of Γ such that Λ separates Γ and BB_Λ fixes an edge of T.

If Γ contains an induced subgraph Λ such that $\Gamma \setminus \Lambda$ has more than one connected component, then A_Γ splits over A_Λ and BB_Γ splits over BB_Λ. In the language of Bass–Serre Theory, all the vertex groups and edge groups of this splitting for A_Γ are finitely presented, but this is not always the case for the corresponding splitting for BB_Γ; see Example 3.3. We remark that Γ contains a cut vertex if and only if BB_Γ splits as a free product.

The proof of Theorem 1.1 uses basic facts about groups acting on trees, and the idea of the proof is similar to those in [6], [2], and [7].

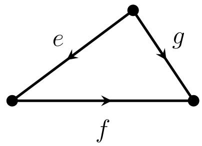

Figure 1. *A directed triangle.*

2. Preliminaries

2.1. Bestvina–Brady groups

Let Γ be a finite simplicial graph. The main result in [1] states that Γ is connected if and only if BB_Γ is finitely generated, and that the flag complex on Γ is simply connected if and only if BB_Γ is finitely presented. In the latter situation, Dicks and Leary found an explicit presentation:

Theorem 2.1. ([5, Corollary 3]) *Let Γ be a finite simplicial directed graph. If the flag complex on Γ is simply connected, then BB_Γ is generated by all the directed edges of Γ, and the relators are of the form ee^{-1}, where e^{-1} denotes the edge e with the opposite orientation, and $ef = g = fe$, where e, f, and g form a directed triangle; see Figure 1.*

2.2. Group acting on trees

Let G be a group acting on a tree T without inversions. We always assume that actions are minimal and nontrivial. An element $g \in G$ is called *elliptic* if it fixes a point in T; otherwise, it is called *hyperbolic*. When $g \in G$ is elliptic, the set of points fixed by g is a subtree of T and is denoted by $\mathrm{Fix}(g)$. When $g \in G$ is hyperbolic, it fixes a line in T on which it acts by translation. This line is called the *axis* of g and is denoted by $\mathrm{Axis}(g)$.

Lemma 2.2. ([4, Lemma 1.1, Corollary 1.5], [6, Lemma 1.1]) *Let G be a group acting on a tree, and let g and h be commuting elements in G.*

(1) If h is hyperbolic, then $\mathrm{Axis}(h) \subseteq \mathrm{Fix}(g)$.
(2) If both g and h are hyperbolic, then $\mathrm{Axis}(g) = \mathrm{Axis}(h)$.
(3) If both g and h are elliptic, then $\mathrm{Fix}(g) \cap \mathrm{Fix}(h) \neq \emptyset$.

3. Proof of Theorem 1.1

Throughout this section, we identify edges of a graph with elements in the associated Bestvina–Brady group.

Definition 3.1. *Let Γ be a finite simplicial graph. A* ***triangle path*** *P_Δ between two distinct edges e and f in Γ is a sequence of triangles $\Delta_1, \ldots, \Delta_n$ such that*

- *the edges e and f are contained in Δ_1 and Δ_n, respectively;*
- *the triangles Δ_i and Δ_{i+1} share a unique edge for each $i = 1, \ldots, n-1$;*
- *the triangles Δ_i and Δ_j do not share a common edge if $j \neq i-1$ or $j \neq i+1$.*

The edge shared by Δ_i and Δ_{i+1} is called an ***intermediate edge****.*

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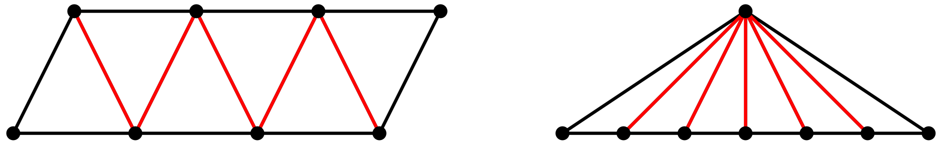

Figure 2. *Two triangle paths and their intermediate edges (red edges). The graph on the right illustrates that all the triangles in a triangle path share a common vertex.*

Two examples of triangle paths and their intermediate edges are given in Figure 2. Notice that triangle paths between two edges may not be unique, and all the triangles in a triangle path can share a common vertex.

Recall that a subgraph Λ of Γ separates two vertices (or two edges) if these two vertices (or edges) lie in different connected components of $\Gamma \setminus \Lambda$.

Lemma 3.2. *Let Γ be a finite simplicial graph without cut vertices and whose associated flag complex is simply connected. Let Λ be an induced subgraph of Γ. If every triangle path between e_1 and e_2 in $E(\Gamma)$ has an intermediate edge in $E(\Lambda)$, then Λ separates e_1 from e_2.*

Proof. Since Γ has no cut vertices and the associated flag complex is simply connected, every edge of Γ is contained in a triangle, and there is a triangle path between any two edges in Γ. Let $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$. Suppose that Λ does not separate e_1 from e_2. Then, without loss of generality, there is an edge path p between the vertices u_1 and u_2. This edge path p is contained in some triangle path P_Δ between e_1 and e_2, and therefore, every vertex of p, possibly except for the two end vertices, is a vertex of an intermediate edge of P_Δ. Since Λ contains an intermediate edge of P_Δ, removing Λ will disconnect the path p. Thus, the path p cannot exist. Hence, the subgraph Λ separates e_1 from e_2. □

We now prove the main theorem.

Proof of Theorem 1.1. Let BB_Γ act on a tree T, and let $e_h \in E(\Gamma)$ be hyperbolic. Since BB_Γ splits as an amalgamated product, the tree T is not a path. Let Λ be the induced subgraph of Γ such that $E(\Lambda)$ consists of all the elliptic edges of Γ that fix $\text{Axis}(e_h)$ pointwise. Let e be an edge of $\text{Axis}(e_h)$. Then BB_Λ fixes e and is contained in a conjugate of H.

We now show that Λ separates Γ. We claim that there is an edge $f \in E(\Gamma) \setminus E(\Lambda)$ such that every triangle path between e_h and f has an intermediate edge in $E(\Lambda)$. Suppose to the contrary that for every edge f' in $E(\Gamma) \setminus E(\Lambda)$, there is a triangle path $P_\Delta = \{\Delta_1, \ldots, \Delta_n\}$ between e_h and f' such that none of its intermediate edges is in $E(\Lambda)$. Denote by $\{f_1, \ldots, f_{n-1}\}$ the set of intermediate edges of P_Δ, where f_i is the edge shared by Δ_i and Δ_{i+1}. Since e_h and f_1 are contained in Δ_1, they are commuting elements. Since e_h is hyperbolic, the element f_1 is also hyperbolic. Otherwise, it follows from Lemma 2.2 (1) that $f_1 \in E(\Lambda)$. Therefore, Lemma 2.2 (2) implies $\text{Axis}(e_h) = \text{Axis}(f_1)$. Similarly, since f_1 is hyperbolic and commuting with f_2, the element f_2 is also hyperbolic and has the axis $\text{Axis}(f_2) = \text{Axis}(f_1)$. Continuing with the same argument, we have that the edges $e_h, f_1, \ldots, f_{n-1}, f'$ are all hyperbolic with the same axis $\text{Axis}(e_h)$. Now, every edge in $E(\Gamma) \setminus E(\Lambda)$ is hyperbolic and has the axis $\text{Axis}(e_h)$, which is fixed by $E(\Lambda)$ pointwise. Thus, the set $E(\Gamma)$ fixes $\text{Axis}(e_h)$, contradicting the fact that T is not a path. This proves the claim. Therefore, the subgraph Λ separates e_h from f by Lemma 3.2.

Next, suppose that every edge of Γ is elliptic. Since the action of BB_Γ on T has no global fixed points, it follows from [8, p.64, Corollary 2] that there are two edges e_α and e_β in $E(\Gamma)$ such that the intersection $\text{Fix}(e_\alpha) \cap \text{Fix}(e_\beta)$ is empty. Let L be the geodesic in T between $\text{Fix}(e_\alpha)$ and $\text{Fix}(e_\beta)$, and let e be an edge of L. Let Λ be an induced subgraph of Γ such that every edge of Λ fixes e. Then, BB_Λ fixes e and is contained in a conjugate of H. We now show that Λ separates Γ. Let $P_\Delta = \{\Delta_1, \ldots, \Delta_n\}$ be a triangle path between e_α and e_β. Denote by $\{f_1, \ldots, f_{n-1}\}$ the set of intermediate edges of P_Δ, where f_i is the edge shared by Δ_i and Δ_{i+1}. For convenience, we write $f_0 = e_\alpha$ and $f_n = e_\beta$. Since f_i and f_{i+1} are contained in the

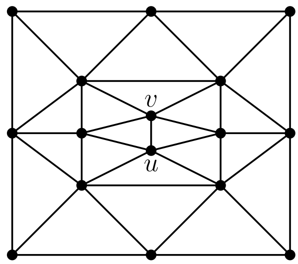

Figure 3. *A splitting of a finitely presented Bestvina–Brady group over a finitely generated but not finitely presented subgroup.*

triangle Δ_{i+1}, they are commuting elliptic elements. It follows from Lemma 2.2 (3) that the intersection $\text{Fix}(f_i) \cap \text{Fix}(f_{i+1})$ is nonempty for $i = 0, \ldots, n-1$. Thus, there is a path L' in T from $\text{Fix}(f_0) = \text{Fix}(e_\alpha)$ to $\text{Fix}(f_n) = \text{Fix}(e_\beta)$ lying entirely in $\bigcup_{i=0}^{n} \text{Fix}(f_i)$. Since T is a tree, we have $L' = L$. Then the edge e belongs to $\text{Fix}(f_i)$ for some i. That is, there is an intermediate edge f_i of P_Δ that belongs to $\text{E}(\Lambda)$. Since the choice of the triangle path P_Δ between e_α and e_β is arbitrary, the subgraph Λ separates e_α from e_β by Lemma 3.2. □

We end this section with one example.

Example 3.3. *Let* Γ *be the graph shown in Figure 3. Let* W *be the set of vertices that are adjacent to either u or v but different from u and v. Let* Λ, Γ_1, *and* Γ_2 *be the induced graphs on* W, $\text{V}(\Gamma) \setminus \{u, v\}$, *and* $W \cup \{u, v\}$, *respectively. Then* Λ *is an induced separating subgraph of* Γ *and* $\text{BB}_\Gamma \cong \text{BB}_{\Gamma_1} *_{\text{BB}_\Lambda} \text{BB}_{\Gamma_2}$. *However, the groups* BB_Γ *and* BB_{Γ_2} *are finitely presented, while* BB_{Γ_1} *and* BB_Λ *are finitely generated but not finitely presented.*

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