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PURE SUBFIELDS OF PURELY INSEPARABLE FIELD EXTENSIONS

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1. Introduction. The notion of pure subgroups is due to Prufer [**7**]. It has proven extremely useful in establishing structural properties of abelian groups. In a recent paper [**9**], Waterhouse introduced the concept of a pure subfield of a purely inseparable extension. Let L be a purely inseparable modular extension of k, and let K be an intermediate field. K is called *pure* if K and $k(L^{p^n})$ are linearly disjoint over $k(K^{p^n})$ for all n. Waterhouse used this concept to establish the existence of basic subfields [**9**]. The purpose of this paper is to examine the properties of pure subfields, and in particular to determine when a pure subfield is a tensor factor of L/k, i.e. when there exists an intermediate field K' such that $L = K \otimes_k K'$. Theorem 4 states that if K is pure and L/K is of bounded exponent, then K is a tensor factor of L/K. This result yields an elementary proof that a finite dimensional modular extension is a tensor product of simple extensions. A further application gives a simple proof of a theorem in [**3**].

Theorem 8 states that if K is pure and of bounded exponent over k, then K is a tensor factor of L/k. Theorem 8 is used to establish a conjecture in [**2**] regarding the distinguished intermediate fields of the purely inseparable Galois theory developed in [**1**]. Assume L is a finite dimensional modular extension of k. If there exists a subbase $T = T_1 \cup \ldots \cup T_n$ for L over k, the elements of T_i being of exponent i over k, such that $K = K \cap k(T_1) \otimes \ldots \otimes K \cap k(T_n)$, then K is also modular over k; and moreover, there exists a subbase $\{x_1, \ldots, x_n\}$ for L over k such that $K = k(x_1^{p^{r_1}}) \otimes \ldots \otimes k(x_t^{p^{r_t}})$.

2. Pure subfields. L is a modular extension of k if L^{p^n} and k are linearly disjoint. We assume throughout that L is a purely inseparable modular extension of k. We will use the following definition originally due to Waterhouse [**9**].

Definition 1. Let K be a subfield of L containing k. Then K is *pure* if and only if K and $k(L^{p^n})$ are linearly disjoint over $k(K^{p^n})$ for all n.

We will need the following two results.

THEOREM 2 [**8**, p. 206]. *Let L/k be purely inseparable, and let K be an intermediate field. The following are equivalent.*

a) *K and $k(L^{p^n})$ are linearly disjoint over $k(K \cap L^{p^n})$ for all (positive) n, and L is modular over k.*

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b) *$K \cap L^{p^n}$ and k are linearly disjoint for all n and L is modular over K.*

THEOREM 3 [**5**, Proposition 3.3, p. 94]. *Let $L \supseteq K \supseteq k$ be fields and assume L is of exponent e over K. The following conditions are equivalent.*

a) *There exists an intermediate field J of L/k such that $L = K \otimes_k J$ and J is modular over k.*

b) *There exists a canonical generating system $B = B_1 \cup \ldots \cup B_e$ of L over K such that $B_i^{p^i} \subseteq (L^{p^i} \cap k)((K(B_{i+1}, \ldots, B_e))^{p^i})$.*

The following result was first observed by Waterhouse; however, a proof is given here for completeness.

LEMMA 3. *If L is modular over k and K is pure, then L is modular over K.*

Proof. Consider the following chain of fields: $k(K^{p^n}) \subseteq k(K \cap L^{p^n}) \subseteq K \cap k(L^{p^n})$. If K is pure, these fields must all be equal. Theorem 2 shows L is modular over K.

Note that if K is pure in L over k, then K is also pure in L' over k where $L \supseteq L' \supseteq K$. Thus any such L' which is modular over k must also be modular over K. The following result gives the first condition for a pure subfield to be a tensor factor. It corresponds to the result in abelian group theory which states: If B is a pure subgroup of the p-group A and A/B is of bounded order, then B is a direct summard of A.

THEOREM 4. *Let L be a purely inseparable modular extension of k and let K be an intermediate field such that L/K is of bounded exponent. Then K is pure if and only if K is a tensor factor of L over k* (i.e. $L = K \otimes_k K'$ *for some* K').

Proof. Assume $L = K \otimes_k K'$. Then K and K' are linearly disjoint over k. By [**4**, Lemma, p. 162], K and $k(K^{p^n})(K')$ are linearly disjoint over $k(K^{p^n})$, and this implies K and $k(K^{p^n})(K'^{p^n}) = k(L^{p^n})$ are linearly disjoint over $k(K^{p^n})$. Thus K is pure.

Conversely, assume K is pure. By Lemma 3, L/K is modular. Since K is pure, $k(K \cap L^{p^n}) = k(K^{p^n})$ and by Theorem 2, we can conclude $K \cap L^{p^n}$ and k are linearly disjoint over $k \cap L^{p^n}$ for all n. Let $B_1 \cup \ldots \cup B_n$ be a cannonical generating system for L/K (that one exists follows since L/K is of bounded exponent [**5**, Corollary 1.35, p. 30]), where the elements t_i of B_i are of exponent i over K. In view of Theorem 3, it suffices to show $t_i^{p^i} \in (L^{p^i} \cap k)(K^{p^i})$. Since K is pure, $t_i^{p^i} \in k(K^{p^i})$ i.e. $t_i^{p^i} = \sum a_r y_r^{p^i}$. We may assume $\{y_r^{p^i}\}$ is linearly independent over k.

Then $\{t_i^{p^i}\} \cup \{y_r^{p^i}\}$ is a subset of $K \cap L^{p^i}$ which is dependent over k, and hence must be dependent over $k \cap L^{p^i}$. Since $\{y_r^{p^i}\}$ is independent over k and hence over $k \cap L^{p^i}$, this relation shows $t_i^{p^i} \in (k \cap L^{p^i})(K^{p^i})$, and hence there exists J such that $L = K \otimes_k J$. (Note that J is also modular over k.)

COROLLARY 5. *Assume L is modular over k of bounded exponent. Let $x \in L$*

be of exponent e over k. Then $k(x)$ *is a tensor factor of L over k if and only if* $x^{p^{e-1}} \notin k(L^{p^e})$.

Proof. This follows by direct application of Theorem 4.

Theorem 4 and its corollary provides an elementary proof that a finite dimensional modular extension L of k is a tensor product of simple extensions. Pick an element x in L of maximal exponent over k. The condition of Corollary 5 must then automatically be satisfied, and we can write $L = k(x) \otimes_k J$ and, as noted after Theorem 4, we may assume J is modular over k. By induction, the result now follows since $[J : k] < [L : k]$.

Theorem 4 also provides a simple proof of the following known result.

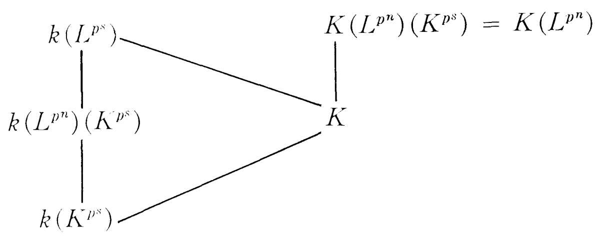
COROLLARY 6 [**3**, Theorem 11, p. 339]. *Any modular field extension L over k where, for some finite n,* $k(L^{p^n}) = k(L^{p^{n+1}})$, *is isomorphic to* $\bigcap k(L^{p^i}) \otimes_k M$ *where M is a modular subfield of L of finite exponent.*

Proof. Since $k(L^{p^n}) = k(L^{p^{n+1}})$, $k(L^{p^n}) = \bigcap k(L^{p^i})$. It is straightforward that $k(L^{p^n})$ is thus a pure subfield. Since $L/k(L^{p^n})$ is of exponent n, Theorem 4 applies.

We now wish to establish the analogue to the result that a bounded pure subgroup is a direct summand.

LEMMA 7. *Let K be a pure subfield of L of exponent n over k. Then* $K(L^{p^n})$ *is pure in L over* $k(L^{p^n})$.

Proof. We need to show $K(L^{p^n})$ and $k(L^{p^n})(L^{p^s})$ are linearly disjoint over $k(L^{p^n})(K^{p^s})$. If $s \geqq n$, this is obvious. If $s < n$, $k(L^{p^s}) \supseteq k(L^{p^n})(K^{p^s}) \supseteq k(K^{p^s})$. Since K is pure, K and $k(L^{p^s})$ are linearly disjoint over $k(K^{p^s})$. By applying the familiar theorem on linear disjointness [**4**, Lemma, p. 162] to the following diagram, we obtain the desired result.



THEOREM 8. *Let L be modular over k and let K be a subfield of bounded exponent over k. Then K is pure if and only if it is a tensor factor of L over k.*

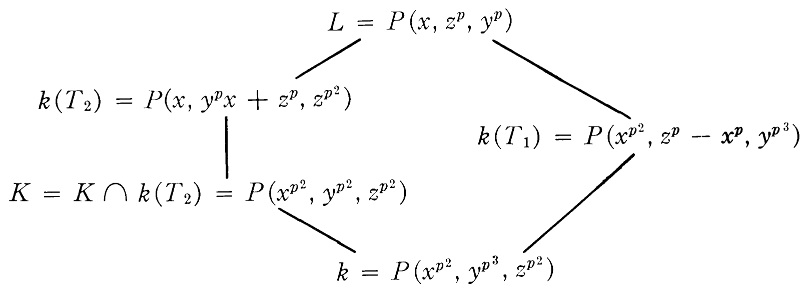
Proof. The if part follows as in Theorem 4. Now let n be the exponent of K over k. Since K is pure in L/k, K and $k(L^{p^n})$ are linearly disjoint over $k(K^{p^n}) = k$. Thus $K(L^{p^n}) = K \otimes_k k(L^{p^n})$. By the previous lemma, $K(L^{p^n})$ is pure in L over $k(L^{p^n})$. Since this extension is of bounded exponent by Theorem 4, $L = K(L^{p^n}) \otimes_{k(L^{p^n})} K'$. Thus $L = K \otimes_k k(L^{p^n}) \otimes_{k(L^{p^n})} K' \approx K \otimes_k K'$.

This result was proven by Waterhouse [**9**, Proposition 2.6] under the added assumption that K is modular over k.

As an application of the theorems on pure subfields, we consider the following problem. In [**2**], L is called an *equiexponential modular extension of* k if there exists a subbase for L over k each of whose elements has the same fixed exponent over k. It was shown in [**2**, Theorem 4.4] that if L is a finite dimensional equiexponential modular extension of k, and K is an intermediate field such that L is modular over K, then K must also be modular over k, and moreover there must exists ome subbase $\{x_1, \ldots, x_t\}$ for L over k such that $K = k(x_1^{p^{r_1}}) \otimes \ldots \otimes k(x_t^{p^{r_t}})$. Thus the intermediate fields over which L is modular are completely determined in the equiexponential case.

The obvious way to generalize this result for non-equiexponential modular extensions is to consider intermediate fields K (L/K modular) for which there exists a subbase $T = T_1 \cup \ldots \cup T_n$ of L over k, the elements of T_i being of exponent i over k, such that $K = K \cap k(T_1) \otimes \ldots \otimes K \cap k(T_n)$. (Such fields are called *homogeneous*.) Then by considering the "pieces" $k(T_i) \supseteq K \cap k(T_i) \supseteq k$, these fields could be characterized. However, as seen in the following example, $k(T_i)$ need not be modular over $K \cap k(T_i)$.

Example 9. Let P be a perfect field (char. $p \neq 0$) and let $\{x, y, z\}$ be algebraically independent over P. Consider the following diagram



Elementary calculations show $L = k(T_1) \otimes k(T_2)$ and L is modular over K. However $K(T_2)$ is not modular over $K \cap k(T_2) = K$. However, if we replace $T_2 = \{x, y^p x + z^p\}$ by $T_2' = \{x, y^p\}$, then $K = K \cap k(T_2')$ and $k(T_2')/K$ is now modular. We shall now show that if K is homogeneous, then we can always find some T_i' such that $K = K \cap k(T_1') \otimes \ldots \otimes K \cap k(T_n')$ where $k(T_i')$ is modular over $K \cap k(T_i')$. Thus, using [**2**, Theorem 4.4], the homogeneous intermediate fields will be completely determined.

THEOREM 10. *Let L/k be a finite dimensional purely inseparable modular extension and let K be a homogeneous intermediate field. Then K is also modular over k and there exists a subbase $\{x_1, \ldots, x_t\}$ for L over k such that $K = k(x_1^{p^{r_1}}) \otimes \ldots \otimes k(x_t^{p^{r_t}})$.*

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Proof. Since K is homogeneous, $K = K \cap k(T_1) \otimes \ldots \otimes K \cap k(T_n)$. Let $K_i = K \cap k(T_i)$. Since L is modular over both K and $k(T_i)$, L is modular over K_i [**9**, Proposition 1.2]. Let $K_i' = k(T_1) \otimes \ldots \otimes k(T_{i-1}) \otimes k(T_{i+1}) \otimes \ldots \otimes k(T_n)$. Since $k(T_i)$ and K_i' are linearly disjoint over k, $k(T_i)$ and $K_i(K_i')$ are linearly disjoint over K_i. Thus $L = k(T_i) \otimes_{K_i} K_i(K_i')$. By Theorem 8, $K_i(K_i')$ is pure in L over K_i, and hence $L = J \otimes_{K_i} K_i(K_i')$ where J is modular over K_i. Now since K_i and K_i' are linearly disjoint over k, $K_i(K_i') = K_i \otimes_k K_i'$. Thus $L = J \otimes_{K_i} (K_i \otimes_k K_i') \approx J \otimes_k K_i'$. Since L is equiexponential modular over K_i', a degree argument shows that J is equiexponential modular over k of exponent i. Thus by [**2**, Theorem 4.4], K_i is modular over k and there exists $t_{i_1}, \ldots, t_{i_s}$ such that $K_i = k(t_{i_1}^{p^{r_1}}) \otimes \ldots \otimes k(t_{i_s}^{p^{r_s}})$. The result now follows.

This theorem answers a conjecture given in [**2**] and also gives a new description of the distinguished intermediate fields for the Galois theory in [**1**] and [**2**].

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