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ON SOME NON-HYPERFINITE FACTORS OF TYPE III

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Introduction. In 1967, Powers [7] proved that there exists a one-parameter family of pairwise non-isomorphic hyperfinite factors of type III. Powers' result on hyperfinite factors has been extended by Araki and Woods [1]. Connes [4], and Williams [11], with different proofs, showed that there exists a continuous family of mutually non-isomorphic non-hyperfinite factors of type III. Actually, this result was also established by Ching [3] and Sakai [8] independently in 1970, using a method derived from the classification of factors of type II_1. While the construction of groups in [3] and [8] are highly complicated, the computations in [11] are quite lengthy. On the other hand, Connes' elaborate and deep analysis [4] of factors of type III involves several sophisticated techniques in operator algebras developed recently. In particular, his new algebraic invariants S and T are based on the Tomita-Takesaki theory of modular operators of von Neumann algebras [10]. In this note, we shall give an elementary proof of the existence of a continuous family of mutually non-isomorphic non-hyperfinite factors of type III. In fact, we shall prove a special case of the main result in Williams [11] by restricting the finite factor in the tensor product to a group algebra. We follow the approach of Schwartz [9], but shall not use infinite tensor product of factors of type I.

In the following, 1 denotes the function constantly equal to 1, e the identity element in a group, N the set of natural numbers, δ_k the function on a set G with $\delta_k(k)=1$, and $\delta_k(g)=0$ for $g\neq k$, χ_S the characteristic function of a set S, B' the commutant of a set B of operators on a Hilbert space. All summations $\sum$ in this paper are over certain finite sets of indexes, all functions are complex-valued unless otherwise specified, and isomorphism means $*$-isomorphism.

Construction of R_λ^G, $0<\lambda<1$. Let $X_0=\{0,1\}$. Let μ_0 be the measure on X_0 with $\mu_0(0)=p$, $\mu_0(1)=q$, $p+q=1$, $0<p<q$, and $\lambda=p/q$. Let μ be the completion of the product measure $\mu'=\prod_{i\in N}\mu_i$ on the Cartesian product $X=\prod_{i\in N}X_i$, where $X_i=X_0$, $\mu_i=\mu_0$ for $i\in N$. Let Δ be the subset of X consisting of all functions on N which take the value 1 only finitely many times. For $\alpha\in\Delta$, $h(\alpha)=\max\{i\in N \mid \alpha(i)=1\}$ is called the *height* of α. For example, $\delta_n\in\Delta$, and $h(\delta_n)=n$, $n\in N$. Define for $i\in N$,

$$(x+y)(i) = x(i)+y(i)(\text{mod } 2), \quad \text{for} \quad x, y\in X.$$

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Δ is then an abelian group where each element is its own inverse. For each $\alpha \in \Delta$, we define the following: a measurable transformation $\alpha: x \mapsto x+\alpha$ on X, a measure $\mu_\alpha(E)=\mu(E+\alpha)$ for all μ-measurable subsets E of X. Let $d\mu_\alpha(x)/d\mu$ be the Radon-Nikodym derivative of μ_α with respect to μ, and $r_\alpha=(d\mu_\alpha(x)/d\mu)^{1/2}$. We have

$$(1) \qquad r_\beta(x+\alpha)r_\alpha(x) = r_{\alpha\beta}(x).$$

Let $H_\lambda=L^2(X, \mu)\otimes\ell^2(\Delta)$. Since H_λ is generated by all $F(x)\otimes\delta_\alpha$, $F(x) \in L^2(X, \mu)$, $\alpha \in \Delta$, for each $f(x) \in M(X)$, the algebra of all bounded μ-measurable functions on X, and $\beta \in \Delta$, the following uniquely define 4 bounded linear operators on H_λ:

$$L_f(F(x) \otimes \delta_\alpha) = f(x)F(x) \otimes \delta_\alpha, \qquad U_\beta(F(x) \otimes \delta_\alpha) = r_\beta(x)F(x+\beta) \otimes \delta_{\alpha+\beta},$$
$$M_f(F(x) \otimes \delta_\alpha) = f(x+\alpha)F(x) \otimes \delta_\alpha, \qquad V_\beta(F(x) \otimes \delta_\alpha) = F(x) \otimes \delta_{\alpha+\beta}.$$

A subset S in X of the form $\{x \in X \mid x(i_k)=a_{i_k}, (a_{i_k}=0 \text{ or } 1)\ k=1, \ldots, n\}$ is called a *cylinder set*, and $h(S)=\max\{i_k \mid k=1, \ldots, n\}$ is called the *height* of S. For example, $C_n^i=\{x \in X \mid x(n)=i\}$, $i=0, 1, n \in N$, are cylinder sets, and we write χ_n^i for $\chi_{C_n^i}$. Let $S(X)$ be the algebra of all linear combinations of characteristic functions of cylinder sets. For $f=\sum_{i=1}^n c_i\chi_{S_i}$ in $S(X)$, $h(f)=\max\{0, h(S_i) \mid c_i\neq 0, i=1, \ldots, n\}$ is called the *height* of f. We note that if S is a cylinder set and $n>h(S)$, then $\mu(S \cap C_n^1)=q\mu(S)$. Hence, for $f(x) \in S(X)$ and $n>h(f)$,

$$\int_{C_n^1} |f(x)|^2\, d\mu = q\, \|f\|_2^2, \quad \text{where } \|\cdot\|_2 \text{ is the } L^2\text{-norm}.$$

From the observation that

$$r_{\delta_n}(x) = \lambda^{i-1/2} \quad \text{for} \quad x \in C_n^i, \qquad i = 0, 1,$$

it is also not difficult to see that for each $\alpha \in \Delta$, $r_\alpha(x)$ assumes only finitely many values on a cylinder set, all of the form $\lambda^{\pm m/2}$, $m \in N$. Since the cylinder sets in X generate the σ-algebra of all μ-measurable subsets of X, $S(X)$ is dense in $M(X)$ in the strong operator topology. Operators of the form

$$\sum_{i=1}^n L_{f_1}U_{\beta_1}\left(\text{resp.} \sum_{i=1}^n M_{f_1}V_{\beta_1}\right),$$

where $f_i \in S(X)$, $\beta_i \in \Delta$, $i=1, \ldots, n$ form a self-adjoint algebra R_λ^0 (resp. R_λ^c) of operators on H_λ. Operators in R_λ^0 commute with operators in R_λ^c. Let R_λ be the strong closure of R_λ^0. By von Neumann [6], R_λ is a factor of type III, and the strong closure of R_λ^c is R_λ'. R_λ is isomorphic to the factor constructed in Powers [7] as pointed out in §4[7].

Let $\mathscr{A}(G)$ be a countable discrete group. Let $\mathscr{A}(G)$ (resp. $\mathscr{A}(G)'$) be the von Neumann algebra associated with the left (resp. right) regular representation of G on $\ell^2(G)$. We denote L_{g_0} (resp. R_{g_0}) the left (resp. right) translation by g_0^{-1} (resp. g_0) on $\ell^2(G)$. Let $R_\lambda^G=R_\lambda\otimes\mathscr{A}(G)$ be the tensor product of R_λ and $\mathscr{A}(G)$ on $H_\lambda^G=H_\lambda\otimes\ell^2(G)$. R_λ^G is purely infinite by Lemma 3 [2], where u is the identity

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representation, and $R_\lambda^{G'}=R'_\lambda\otimes\mathscr{A}(G)'$. A vector in H_λ^G of the form $\xi=\sum_{\alpha,g}f_{\alpha g}(x)\otimes\delta_\alpha\otimes\delta_g$, where all $f_{\alpha g}(x)\in S(X)$ is called *flat*, and $h(\xi)$, the *height* of ξ, is defined to be the largest of the heights of $f_{\alpha g}$'s and α's in the summation. Apparently, flat vectors are dense in H_λ^G.

For G and G_1 two countable discrete groups, and λ, $\lambda_1\in(0,1)$, we have the following:

THEOREM. *R_λ^G is not isomorphic to $R_{\lambda_1}^{G_1}$ if $\lambda\neq\lambda_1$.*

Proof. Let

$$J_n=L_{\chi_n^1}U_{\delta_n}\otimes L_e,\qquad K_n=M_{\chi_n^0}V_{\delta_n}\otimes R_e,\qquad n\in N.$$

Then $J_n\in R_\lambda^G$, $K_n\in R_\lambda^{G'}$, and

$$\|J_n\|=\|K_n\|=1,\quad\text{for}\quad n\in N. \tag{2}$$

We claim that for each $\xi\in H_\lambda^G$,

$$\|K_n\xi\|\to q^{1/2}\,\|\xi\|, \tag{3}$$

$$\|(J_n-\lambda^{1/2}K_n)\xi\|\to 0, \tag{4}$$

$$\|(\lambda^{1/2}J_n^*-K_n^*)\xi\|\to 0. \tag{5}$$

Because of (2), we only need to show (3)–(5) for an arbitrary flat vector $\xi=\sum_{\alpha,g}f_{\alpha g}(x)\otimes\delta_\alpha\otimes\delta_g$. We first observe that for $f\in S(X)$, $\alpha\in\Delta$, $n>\max\{h(f),h(\alpha)\}$, we have

$$f(x+\delta_n)=f(x),\qquad \chi_n^i(x+\alpha)=\chi_n^i(x),\qquad i=0,1,$$

and $\chi_n^0(x+\delta_n)=\chi_n^1(x)$, $\chi_n^1(x+\delta_n)=\chi_n^0(x)$ for all $x\in X$. Hence, for $n>h(\xi)$, we have

$$\begin{aligned}\|K_n\xi\|^2&=\sum_{\alpha,g}\|M_{\chi_n^0}V_{\delta_n}f_{\alpha g}(x)\otimes\delta_\alpha\|^2\\&=\sum_{\alpha,g}\|\chi_n^0(x+\alpha+\delta_n)f_{\alpha g}(x)\otimes\delta_{\alpha+\delta_n}\|^2\\&=\sum_{\alpha,g}|\chi_n^1(x)f_{\alpha g}(x)|^2=q\,\|\xi\|^2,\end{aligned}$$

$$\begin{aligned}\|(J_n-\lambda^{1/2}K_n)\xi\|^2&=\sum_{\alpha,g}\|(L_{\chi_n^1}U_{\delta_n}-\lambda^{1/2}M_{\chi_n^0}V_{\delta_n})f_{\alpha g}(x)\otimes\delta_\alpha\|^2\\&=\sum_{\alpha,g}\|\chi_n^1(x)r_{\delta_n}(x)f_{\alpha g}(x+\delta_n)-\lambda^{1/2}\chi_n^0(x+\alpha+\delta_n)f_{\alpha g}(x)\|_2^2=0,\end{aligned}$$

$$\begin{aligned}\|(\lambda^{1/2}J_n^*-K_n^*)\xi\|^2&=\sum_{\alpha,g}\|(\lambda^{1/2}U_{\delta_n}L_{\chi_n^1}-V_{\delta_n}M_{\chi_n^0})f_{\alpha g}(x)\otimes\delta_\alpha\|^2\\&=\sum_{\alpha,g}\|\lambda^{1/2}r_{\delta_n}(x)\chi_n^1(x+\delta_n)f_{\alpha g}(x+\delta_n)-\chi_n^0(x+\alpha)f_{\alpha g}(x)\|_2^2=0.\end{aligned}$$

This verifies (3)–(5).

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Now, suppose on the contrary that there is an isomorphism θ from R_λ^G onto $R_{\lambda_1}^{G_1}$. It is easy to see that $\xi_0=1\otimes\delta_e\otimes\delta_e$ is a cyclic and separating vector for both R_λ^G and $R_{\lambda_1}^{G_1}$ in H_λ^G and $H_{\lambda_1}^{G_1}$ respectively. By Theorem 3 on p. 222 of Dixmier [5], R_λ^G and $R_{\lambda_1}^{G_1}$ are then spacially isomorphic, i.e., there exists a unitary operator W from H_λ^G to $H_{\lambda_1}^{G_1}$ such that

$$\theta(T) = WTW^* \quad \text{for all} \quad T\in R_\lambda^G. \tag{6}$$

Clearly, (6) also defines an isomorphism from $R_\lambda^{G'}$ onto $R_{\lambda_1}^{G_1'}$. From (3)–(6), we have for an arbitrary $((\lambda/2)^{1/2}>)\varepsilon>0$, an $n_0\in N$ such that

$$\|\theta(K_{n_0})\xi_0\| > \tfrac{1}{2}q^{1/2}, \tag{7}$$

$$\|\theta(J_{n_0}-\lambda^{1/2}K_{n_0})\xi_0\| < \frac{\epsilon}{4}\, q^{1/2}. \tag{8}$$

$$\|\theta(\lambda^{1/2}J_{n_0}^*-K_{n_0}^*)\xi_0\| < \frac{\epsilon}{4}\, q^{1/2}. \tag{9}$$

Since $R_{\lambda_1}^0$ (resp. $R_{\lambda_1}^c$) is strongly dense in R_{λ_1} (resp. R_{λ_1}') and operators of the form $\sum_g c_gL_g$ (resp. $\sum_g c_gR_g$) are strongly dense in $\mathscr{A}(G)$ (resp. $\mathscr{A}(G)'$), by the Kaplansky density theorem (Theorem 3, p. 43 [5]), the hermitian parts of $R_{\lambda_1}^0$ and $R_{\lambda_1}^c$ are dense in the hermitian parts of R_{λ_1} and R_{λ_1}' respectively. Hence, after multiplying $2q^{-1/2}$ on (7)–(9), we can find two operators $P=\sum_{\alpha,g}L_{f_{\alpha g}}U_\alpha\otimes L_g$ and $Q=\sum_{\alpha,g}M_{\varphi_{\alpha g}}V_\alpha\otimes R_g$, where $f_{\alpha g}$, $\varphi_{\alpha g}\in S(X)$, such that

$$\|Q\xi_0\| > 1, \tag{10}$$

$$\|(P-\lambda^{1/2}Q)\xi_0\| < \epsilon, \tag{11}$$

$$\|(\lambda^{1/2}P^*-Q^*)\xi_0\| < \epsilon. \tag{12}$$

Without loss of generality, we can assume that the summations in P and Q are over the same finite subset $E\times F$ of $\Delta\times G$, and furthermore, $g\in F$ implies $g^{-1}\in F$. (11) and (12) are respectively the following:

$$\Big\|\sum_{\alpha,g}(f_{\alpha g}(x)r_\alpha(x)\otimes\delta_\alpha\otimes\delta_g-\lambda^{1/2}\varphi_{\alpha g}(x+\alpha)\otimes\delta_\alpha\otimes\delta_{g^{-1}})\Big\| \tag{13}$$
$$= \Big(\sum_{\alpha,g}\|f_{\alpha g}(x)r_\alpha(x)-\lambda^{1/2}\varphi_{\alpha g^{-1}}(x+\alpha)\|_2^2\Big) < \epsilon,$$

$$\Big\|\sum_{\alpha,g}(\lambda^{1/2}r_\alpha(x)f_{\alpha g}(x+\alpha)\otimes\delta_\alpha\otimes\delta_{g^{-1}}-\varphi_{\alpha g}(x)\otimes\delta_\alpha\otimes\delta_g)\Big\| \tag{14}$$
$$= \Big(\sum_{\alpha,g}\|\lambda^{1/2}r_\alpha(x)f_{\alpha g}(x+\alpha)-\varphi_{\alpha g^{-1}}(x)\|_2^2\Big)^{1/2}$$
$$= \Big(\sum_{\alpha,g}\|\lambda^{1/2}f_{\alpha g}(x)-r_\alpha(x)\varphi_{\alpha g^{-1}}(x+\alpha)\|_2^2\Big)^{1/2} < \epsilon.$$

The last equality follows from

$$\int_X f(x+\alpha)\, d\mu(x) = \int_X f(x) \frac{d\mu_\alpha}{d\mu}(x)\, d\mu(x),$$

and (1). By Minkowski inequality and (13), (14), we have

$$(15) \qquad \left(\sum_{\alpha,g} \int_X |r_\alpha(x)-\lambda^{1/2}|^2 \cdot |f_{\alpha g}(x)+\varphi_{\alpha g^{-1}}(x+\alpha)|^2\, d\mu(x)\right)^{1/2} < 2\epsilon,$$

and

$$(16) \qquad \left(\sum_{\alpha,g} \int_X |r_\alpha(x)+\lambda^{1/2}|^2 \cdot |\varphi_{\alpha g^{-1}}(x+\alpha)-f_{\alpha g}(x)|^2\, d\mu(x)\right)^{1/2} < 2\epsilon.$$

Since $r_\alpha(x)>0$, (16) implies that

$$A = \left(\sum_{\alpha,g} \|\varphi_{\alpha g^{-1}}(x+\alpha)-f_{\alpha g}(x)\|_2^2\right)^{1/2} < 2\epsilon\lambda^{-1/2} < 1.$$

Hence,

$$(17) \quad \left(\sum_{\alpha,g} \|f_{\alpha g}(x)+\varphi_{\alpha g^{-1}}(x+\alpha)\|_2^2\right)^{1/2}$$
$$\geq 2\left(\sum_{\alpha,g} \|\varphi_{\alpha g^{-1}}(x+\alpha)\|_2^2\right) - A = 2\,\|Q\xi_0\| - A > 1.$$

Let $\mathscr{F}$ be the finite collection of all cylinder sets whose characteristic functions are in the linear expansions of $f_{\alpha g}$'s and $\varphi_{\alpha g}$'s. (15) and (17) together yield that

$$\min_{\substack{x\in S\in\mathscr{F}\\ \alpha\in E}} |r_\alpha(x)-\lambda^{1/2}| < 2\epsilon.$$

But $r_\alpha(x)$ is always an integral power of $\lambda_1^{1/2}$ for $x \in S \in \mathscr{F}$, $\alpha \in E$. Since $0<\lambda$, $\lambda_1<1$, and ϵ can be arbitrarily small, we have $\lambda=\lambda_1^n$ for some $n \in N$. By symmetry, $\lambda_1=\lambda^m$ for some $m \in N$. Hence, $\lambda=\lambda_1$. This shows that R_λ^G is not isomorphic to $R_{\lambda_1}^{G_2}$ if $\lambda\neq\lambda_1$.

COROLLARY. *There exists a one-parameter family of non-isomorphic non-hyperfinite factors of type III.*

Proof. Let $G=G_1=\Phi_2$, the free group on two generators. R_λ^G and $R_{\lambda_1}^G$ are factors of type III (Lemma 2 [2] with u the identity representation). By the same argument as that in Lemma 9 [2], we see that R_λ^G and $R_{\lambda_1}^G$ are non-hyperfinite, for otherwise Φ_2 would admit a translation invariant measure of total mass 1. Hence, $\{R_\lambda^G \mid 0<\lambda<1\}$ is the required family of factors of type III.

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