Proceedings of the Edinburgh Mathematical Society (1976-77), **20**, 279-285 ©

SOME APPLICATIONS OF DOUBLE-NEGATION SHEAFIFICATION

by D. S. MACNAB

(Received 1st August 1975)

1.

In this note we point out that certain algebraic-topological constructions are particular cases of one construction, namely double-negation sheafification. The principal cases we have in mind are concerned with boolean powers, completions of boolean algebras, and maximal rings of quotients. We conjecture that several other constructions—particularly completion-type constructions—will turn out also to be examples of double-negation sheafification.

A general acquaintance with classical sheaf theory and its generalisation to topoi is assumed. This may be found in **(7)**, **(8)**, **(13)**, **(14)**, **(20)**, **(24)**, **(25)**, **(26)**, particularly **(7)** and **(26)** for the topos side. A discussion of the relationship between the espace étalé aspect of a classical sheaf and the categorical aspect is contained in **(3)**. We denote by $O(X)$ the open set lattice (category) of a topological space X and by $\mathrm{Top}(X)$ the category of classical sheaves on X. Elements of $\mathrm{Top}(X)$ will be regarded categorically—as contravariant set-valued functors on $O(X)$. If $A \in \mathrm{Top}(X)$, the value of A at $U \in O(X)$ will be denoted by $A(U)$, and the double-negation sheafification of A by $\neg\neg A$.

Double-negation sheaves in $\mathrm{Top}(X)$ may be characterised as follows.

Lemma 1. *If $B \in \mathrm{Top}(X)$, then B is a double-negation sheaf iff, for any $U, U_1 \in O(X)$ with U_1 dense in U (with respect to the induced topology), the canonical map $B(U) \to B(U_1)$ is a bijection.*

The proof follows standard lines.

Using this we obtain the following characterisation of double-negation sheafification.

Theorem 1. *If $A \in \mathrm{Top}(X)$, then, for $U \in O(X)$, we have*

$$\neg\neg A(U) = \varinjlim \{A(V)\colon V \text{ is dense in } U\}.$$

Proof. Let $F(U) = \varinjlim \{A(V)\colon V \text{ is dense in } U\}$. Then there are three more or less independent factors to demonstrate.

https://doi.org/10.1017/S001309150002650X Published online by Cambridge University Press

(A) The definition of $F(U)$ actually produces an element $F \in \text{Top}(X)$.
(B) F is a double-negation sheaf in Top(X).
(C) F is the reflection of A in the double-negation sheaf subcategory of Top(X).

The only difficult part of (A) is to show that F is collated. This follows from Lemma 2 below. Then (B) follows from Lemma 1, and (C) is straightforward.

Lemma 2. *If V_1, V_2 are any two open sets in a topological space, and if D is a dense open subset of $V_1 \cap V_2$ (in the induced topology), then there exist open sets D_1, D_2 dense in V_1, V_2 respectively, such that $D_1 \cap D_2 = D$.*

Part (A) of Theorem 1 extends and simplifies that of Ellerman (**4**), Appendix 1. We also note here that Theorem 1 can be extended to a wider class of modal operators on $O(X)$. This and other extensions of the Theorem will appear elsewhere.

The remainder of this note is concerned with applications of Theorem 1 to various algebraic and topological constructions.

2. Boolean powers and ultrapowers

The theory of boolean powers originated with Foster (**6**) and was developed by Mansfield (**17**), Ribenboim (**21**), and Daigneault (**2**), mainly for model-theoretic purposes.

Given an algebraic structure A and a complete boolean algebra B, the boolean power of A over B, denoted by $A^{(B)}$, is defined as the set of maps $d: A \to B$ such that $\bigcup_{x\in A} d(x) = 1$, and $x \neq y$ implies $d(x) \cap d(y) = 0$. Operations in A can be extended to $A^{(B)}$ as explained in (**17**) or (**2**).

If we define d_a, for $a \in A$, by $d_a(a) = 1$ and $d_a(b) = 0$ for $b \neq a$, then the map $a \to d_a$ is an embedding of A in $A^{(B)}$. If we factor out an ultrafilter D from B we obtain the boolean ultrapower $A^{(B)}/D$. From a model-theoretic point of view, $A^{(B)}/D$ is an elementary extension of A. Other model-theoretic properties of boolean ultrapowers are discussed in (**17**).

To see the connection between boolean powers and double-negation sheaves, we require a representation theorem due to Ribenboim but stated more clearly by Daigneault.

Let Z denote the dual space of the boolean algebra B, i.e. the space of ultrafilters of B with the Stone topology.

Theorem 2. *The boolean power $A^{(B)}$ is isomorphic to the set of functions $Z \to A$, (discrete topology on A), which are continuous on dense open subsets of Z and which are not continuously extendable to a larger open subset, with the operations induced pointwise by the operations of A.*

A discussion of aspects of this isomorphism is contained in (**16**).

Equivalently stated, $A^{(B)}$ is isomorphic to $\varinjlim \{A(V): V \text{ is dense in } Z\}$ with the operations induced by those of A.

Hence $A^{(B)}$ is isomorphic to the algebra of global sections of $\neg\neg A^*$ where A^* is the sheaf over Z associated with A; i.e. $A^*(U) = \{$the set of continuous functions $U \to A$, where A has the discrete topology$\}$.

We now look at the stalks of $\neg\neg A^*$.

Theorem 3. *The stalks of $\neg\neg A^*$ are the boolean ultrapowers of A.*

Proof. Denote $\neg\neg A^*$ by $\bar{A}$. Then $\bar{A}_x = \varinjlim \{\bar{A}(U): x \in U\}$. Since A as a topological space is discrete, $[f] = [f_1]$ in the limit iff $f(x) = f_1(x)$, where [ ] denotes an equivalence class. Hence, by **(2)** or **(16)**, $\bar{A}_x$ is the boolean ultrapower determined by the ultrafilter x.

Hence the boolean power $A^{(B)}$ is isomorphic to the algebra of global sections of the sheaf of its boolean ultrapowers over the dual space of B. (See also **(4)**.)

Remark. For the subcategory of double-negation sheaves in Top(Z), the internal logic object is $\Omega_{\neg\neg}$ whose underlying algebra $= \{U \in O(Z): U^{-0} = U\} = \{$regular open subsets of $Z\} = B$, (since B is complete).

3. The completion of a boolean algebra

If a boolean algebra B is represented as the algebra of closed-open subsets of its dual space Z, then the completion of B is (isomorphic to) the algebra of regular open subsets of Z.

Now B can also be represented as the set of continuous functions $Z \to \{0, 1\}$, where $\{0, 1\}$ has the discrete topology. Hence B becomes a member of Top(Z). We now show that the algebra of global sections of $\neg\neg B$ is the completion of B.

Theorem 4. *$\neg\neg B(Z)$ is isomorphic to the set of regular open subsets of Z.*

Proof. We identify $\neg\neg B(Z)$ with the set of functions $Z \to \{0, 1\}$ which are continuous on dense open subsets of Z and which are not continuously extendable to larger open sets.

(i) For V regular open in Z, let $D = V \cup V'^0$. Then D is dense in Z. Now define $f: D \to \{0, 1\}$ by

$$f(x) = \begin{cases} 1, & x \in V \\ 0, & x \in V'^0 \end{cases}.$$

Then f is continuous $D \to \{0, 1\}$ and hence (continuously extended if necessary) belongs to $\neg\neg B(Z)$.

(ii) Suppose $f \in \neg\neg B(Z)$; then there exists a dense open subset D of Z such that f is continuous $D \to \{0, 1\}$ and is not continuously extendable. Hence $f^{-1}(1)$ is closed-open with respect to the induced topology on D and hence is regular open in this topology. We have to show that $f^{-1}(1)$ is regular open in Z.

Let $V = f^{-1}(1)$. The closure of V in the induced topology on D is $V^- \cap D$. Hence V is regular open in D iff $(V^- \cap D)^0 = V$ iff $V^{-0} \cap D = V$. Hence it is sufficient to show that $V^{-0} \subseteq D$. If $V^{-0} \not\subseteq D$, then we can continuously extend f to $D \cup V^{-0}$, by setting $f(x) = 1$, for $x \in V^{-0} \setminus D$, which contradicts the definition of f.

Hence $V^{-0} \subseteq D$ and hence $f^{-1}(1)$ is regular in Z.

The above two constructions establish a 1-1 correspondence between elements of $\neg\neg B(Z)$ and regular open subsets of Z. That this is an isomorphism with respect to the boolean operations may easily be checked. Hence $\neg\neg B(Z)$ is the completion of B.

Remark. For L a distributive lattice with the discrete topology and X a topological space, the set of continuous functions $X \to L$ has the structure of a distributive lattice induced from L. If we denote the corresponding sheaf by A, then, for suitable choices of X, $\neg\neg A(X)$ is the completion of $A(X)$ as a distributive lattice.

4. Maximal rings of quotients

The details of the construction of maximal (or complete) rings of quotients of commutative rings and of maximal rings of right quotients of non-commutative rings may be found in (**12**) and (**23**). We will consider only the case of rings of continuous functions, see (**5**) and also (**1**). From (**5**) we take the following result.

Theorem 5. *The maximal ring of quotients of the ring of continuous real-valued functions over a completely regular hausdorff-space X may be identified with the ring of real-valued functions continuous on dense open subsets of X modulo identification on dense open sets.*

Banaschewski, (**1**), has extended this result to show that the maximal ring of quotients of any semi-prime commutative ring can be obtained in a similar manner by first representing the ring as a ring of continuous functions over its prime ideal space with the hull-kernel topology.

Hence in both these cases the maximal ring of quotients is obtainable as the ring of global sections of a double-negation sheaf. It is shown in (**12**) that the (MacNeille) completion of a boolean algebra can be identified with the maximal ring of quotients of its associated boolean ring.

Another extension of these results has been obtained by Mulvey (**19**). To explain this we need some definitions.

Let $A \in \mathrm{Top}(X)$ be such that for each $U \in O(X)$, $A(U)$ carries a ring structure and such that for $V \subseteq U$, the canonical map $A(U) \to A(V)$ is a ring homomorphism, these homomorphisms being compatible with the functorial nature of A. Then the pair (X, A) is called a ringed space. Suppose now that X is a hausdorff space. Then the global ring of sections $A(X)$ is called completely regular if (i) the canonical ring homomorphism $A(X) \to A_x$ is surjective, $\forall x \in X$, (ii) for each $x \in X$ and each open neighbourhood U of x, there exists $f \in A(X)$ such that $\{x: f_x \neq 0\} \subseteq U$ and f_x is the identity of the ring A_x, where f_x is the image of f under the homomorphism $A(X) \to A_x$. (In fact, (ii) implies (i).) If $A(X)$ is completely regular then (X, A) is also said to be completely regular. (X, A) is called a reduced ringed space if, for $U \in O(X)$ and $f \in A(U)$, f_x is a non-unit in A_x for each $x \in U$, then f is the zero element of $A(U)$.

We now can state Mulvey's theorem.

Theorem 6. *The maximal ring of right quotients of the ring of global sections of a completely regular reduced ringed space* (X, A) *is isomorphic to* $\varinjlim \{A(U)$: *U is dense in X*$\}$.

Hence such maximal rings of quotients are also constructible via double-negation sheaves.

Since many types of ring can be represented as rings of global sections of sheaves over suitable spaces, (see (**11**)), we conjecture that further maximal rings of quotients will turn out to be rings of global sections of double-negation sheaves, see also (**22**), §3.

5. Further applications

(1) The ultrasheaves of (**4**) are essentially double-negation sheaves. This observation makes some of the stated properties of such sheaves and their stalks, (called ultrastalks in (**4**)), more obvious.

(2) Given a sheaf space (S, π) over a topological space X, we can construct its section functor sheaf Γ and hence $\neg\neg\Gamma$. From $\neg\neg\Gamma$ we can construct a new sheaf space over X which we will denote by $(\neg\neg S_x, \pi')$. Then $\neg\neg S_x$ can be regarded as a new topological space constructed from S with the aid of X. For suitable choices of S and X we conjecture that the construction is an already known one—for example, a compactification or completion construction—thus bringing topological constructions also under the double-negation umbrella.

I should like to thank Dr H. Simmons for much helpful advice and in particular for encouraging me to prepare this paper for publication.

Note: I am indebted to the referee for bringing to my attention the following two related articles which are available as preprints.

(a) Boolean Powers as Algebras of Continuous Functions, B. Banaschewski and E. Nelson, Dept. of Mathematics, MacMaster University, Hamilton, Ontario, Canada.

(b) A Remark on the Prime Stalk Theorem, C. J. Mulvey, Dept. of Mathematics, Columbia University, New York.

REFERENCES

(1) B. BANASCHEWSKI, Maximal Rings of Quotients of Semi-simple Commutative Rings, *Archiv. der Math.*, **16** (1965), 414–420,

(2) A. DAIGNEAULT, Boolean Powers in Algebraic Logic, *Z. Math. Grundlagen Math.*, **17** (1971), 411–420.

(3) B. A. DAVEY, Sheaf Spaces and Sheaves of Universal Algebras, *Math. Z.* **134** (1973), 275–290.

(4) D. P. ELLERMAN, Sheaves of Structures and Generalised Ultraproducts, *Ann. Math. Logic*, **7** (1974), 163–195.

(5) N. J. FINE, L. GILLMAN and J. LAMBEK, *Rings of Quotients of Rings of Functions* (McGill University Press, Montreal, 1965).

(6) A. L. FOSTER, Generalised "Boolean" Theory of Universal Algebras, *Math. Z.* **58** (1953), 306–338 and **59** (1953), 191–199.

(7) P. J. FREYD, Aspects of Topoi, *Bull. Austral. Math. Soc.* **7** (1972), 1–76.

(8) R. GODEMENT, *Theorie des Faisceaux* (Hermann, Paris, 1964).

(9) G. GRATZER, *Universal Algebra* (Van Nostrand, Princeton, 1963).

(10) A. GROTHENDIECK, and J. L. VERDIER, *Theorie des Topos et Cohomologie Etale des Schemas* (Springer-Verlag, Lecture Notes 269, Berlin, 1972).

(11) K. H. HOFMANN, Representation of Algebras by Continuous Sections, *Bull. Amer. Math. Soc.*, **78** (1972), 291–373.

(12) J. LAMBEK, *Lectures on Rings and Modules* (Blaisdell, Mass., 1966).

(13) F. W. LAWVERE, Quantifiers and Sheaves, *Actes Congrès Intern. Math.* (1970), Tome 1, 329–334.

(14) F. W. LAWVERE, *Toposes, Algebraic Geometry and Logic* (Springer-Verlag, Lecture Notes 274, Berlin, 1972).

(15) I. G. MACDONALD, *Algebraic Geometry* (Benjamin, New York, 1968).

(16) D. S. MACNAB, *The Structure of Polyadic Algebras* (M.Sc. Thesis, Aberdeen, 1973).

(17) R. MANSFIELD, Theory of Boolean Ultrapowers, *Ann. Math. Logic*, **2** (1971), 297–323.

(18) B. MITCHELL, *Theory of Categories* (Academic Press, New York, 1965).

(19) C. J. MULVEY, *On Ringed Spaces* (Ph.D. Thesis, Sussex, 1970).

(20) C. J. MULVEY, Intuitionistic Algebra and the Representation of Rings, in Recent Advances in the Representation Theory of Rings and C^*-algebras, *Mem. Amer. Math. Soc.*, **148** (1974).

(21) P. RIBENBOIM, Boolean Powers, *Fund. Math.* **65** (1969), 243–268.

(22) J.-E. ROOS, Locally Distributive Spectral Categories and Strongly Regular Rings, *Reports of the Midwest Category Seminar I* (Springer-Verlag, Lecture Notes 47, Berlin, 1967).

(23) B. STENSTROM, *Rings and Modules of Quotients* (Springer-Verlag, Lecture Notes 237, Berlin, 1971).

(24) R. SWAN, *Theory of Sheaves* (University of Chicago Press, 1964).

(25) M. TIERNEY, *Sheaf Theory and the Continuum Hypothesis, Toposes, Algebraic Geometry and Logic* (Springer-Verlag, Lecture Notes 274, Berlin, 1972).

(26) G. C. WRAITH, *Lectures on Elementary Topoi, Model Theory and Topoi* (Springer-Verlag, Lecture Notes 445, Berlin, 1975).

COLLEGE OF EDUCATION
ABERDEEN

https://doi.org/10.1017/S001309150002650X Published online by Cambridge University Press