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On Hodge Theory of Singular Plane Curves

Nancy Abdallah

Abstract. The dimensions of the graded quotients of the cohomology of a plane curve complement $U = \mathbb{P}^2 \smallsetminus C$ with respect to the Hodge filtration are described in terms of simple geometrical invariants. The case of curves with ordinary singularities is discussed in detail. We also give a precise numerical estimate for the difference between the Hodge filtration and the pole order filtration on $H^2(U, \mathbb{C})$.

1 Introduction

The Hodge theory of the complement of projective hypersurfaces has received much attention; see, for instance, Griffiths [10] in the smooth case, and Dimca–Saito [5] and Sernesi [13] in the singular case. In this paper we consider the case of plane curves and continue the study initiated by Dimca–Sticlaru [7] in the nodal case and by the author [1] in the case of plane curves with ordinary singularities of multiplicity up to 3.

In the second section we compute the Hodge–Deligne polynomial of a plane curve C, the irreducible case in Proposition 2.1 and the reducible case in Proposition 2.2. Using this we determine the Hodge–Deligne polynomial of $U = \mathbb{P}^2 \smallsetminus C$ and then deduce in Theorem 2.7 the dimensions of the graded quotients of $H^2(U)$ with respect to the Hodge filtration.

In Section 3 we consider the case of arrangements of curves having ordinary singularities and intersecting transversely at smooth points. We obtain a formula in Theorem 3.1 generalizing the formulas obtained in [7] and in [1] (for these curves). In fact, the results in [1] show that this formula holds in the more general case of plane curves with ordinary singularities of multiplicity up to 3 (without assuming transverse intersection).

In the fourth section we show that the case of plane curves with ordinary singularities of multiplicity up to 4 (without assuming transverse intersection) is definitely more complicated, and the formula in Theorem 3.1 has to be replaced by the formula in Theorem 4.1 containing a correction term coming from triple points on one component through which another component of C passes.

In the final section we state and prove our main result, Theorem 5.1, which expresses the difference between the Hodge filtration and the pole order filtration on $H^2(U, \mathbb{C})$ in terms of numerical invariants easy to compute in given situations. An example involving a free divisor concludes this note.

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2 Hodge Theory of Plane Curve Complements

For the general theory of mixed Hodge structures we refer to [2,15]. Recall the definition of the Hodge–Deligne polynomial of a quasi-projective complex variety X:

$$P(X)(u,v) = \sum_{p,q} E^{p,q}(X)u^p v^q,$$

where $E^{p,q}(X) = \sum_s (-1)^s h^{p,q}(H_c^s(X))$, with

$$h^{p,q}\big(H_c^s(X)\big) = \dim Gr_F^p Gr_{p+q}^W H_c^s(X,\mathbb{C}),$$

the mixed Hodge numbers of $H_c^s(X)$.

This polynomial is additive with respect to constructible partitions, *i.e.*, $P(X) = P(X \smallsetminus Y) + P(Y)$ for a closed subvariety Y of X. In this section we determine $P(C)$ for a (reduced) plane curve C.

Suppose first that the curve C is irreducible, of degree N. Denote by a_k, $k = 1,\dots,p$ the singular points of C, and let $r(C,a_k)$ be the number of irreducible branches of the germ (C,a_k). Let $\nu\colon \widetilde{C} \to C$ be the normalization mapping. Using the normalization map ν and the additivity of the Hodge–Deligne polynomial, it follows that

$$\begin{aligned} P(C)(u,v) &= P(C\backslash(C)_{\text{sing}}) + P\big((C)_{\text{sing}}\big) = P\big(\widetilde{C}\backslash(\cup_k \nu^{-1}(a_k)\big) + p \\ &= P(\widetilde{C}) - \sum_k P\big(\nu^{-1}(a_k)\big) + p = uv - gu - gv + 1 - \sum_k \big(r(C,a_k) - 1\big). \end{aligned}$$

Indeed, it is known that for the smooth curve $\widetilde{C}$, the genus $g = g(\widetilde{C})$ is exactly the Hodge number $h^{1,0}(\widetilde{C}) = h^{0,1}(\widetilde{C})$. Moreover, it is known that one has the formula

$$g = \frac{(N-1)(N-2)}{2} - \sum_k \delta(C,a_k), \tag{2.1}$$

relating the genus, the degree and the local singularities of C, and the δ-invariants can be computed using the formula

$$2\delta(C,a_k) = \mu(C,a_k) + r(C,a_k) - 1, \tag{2.2}$$

where $\mu(C,a_k)$ is the Milnor number of the singularity (C,a_k). For both formulas above, see [11, p. 85]. This proves the following result.

Proposition 2.1 *With the above notation and assumptions, we have the following for an irreducible plane curve $C \subset \mathbb{P}^2$.*

(i) *The Hodge–Deligne polynomial of C is given by*

$$P(C)(u,v) = uv - gu - gv + 1 - \sum_k \big(r(C,a_k) - 1\big),$$

with g given by the formula (2.1).

(ii) $H^0(C) = \mathbb{C}$ *is pure of type* $(0,0)$.

(iii) $H^2(C) = \mathbb{C}$ *is pure of type* $(1,1)$.

(iv) *The mixed Hodge numbers of the MHS on $H^1(C)$ are given by*

$$h^{0,0}\big(H^1(C)\big) = \sum_k \big(r(C,a_k) - 1\big), \quad h^{1,0}\big(H^1(C)\big) = h^{0,1}\big(H^1(C)\big) = g.$$

In particular, one has the following formulas for the first Betti number of C:

$$b_1(C) = \sum_k \big(r(C,a_k) - 1\big) + 2g = (N-1)(N-2) - \sum_k \mu(C,a_k).$$

Now we consider the case of a curve C having several irreducible components. More precisely, let $C = \bigcup_{j=1}^r C_j$ be the decomposition of C as a union of irreducible components C_j, let $\nu_j : \widetilde{C}_j \to C_j$ be the normalization mappings, and set $g_j = g(\widetilde{C}_j)$. Suppose that the curve C_j has degree N_j, denote by a_k^j for $k = 1,\dots,p_j$ the singular points of C_j, and let $r(C_j, a_k^j)$ be the number of branches of the germ (C_j, a_k^j). Then the formulas (2.1) and (2.2) can be applied to each irreducible curve C_j as well as Proposition 2.1.

Let A be the union of the singular sets of the curves C_j. Let B be the set of points in C sitting on at least two distinct components C_i and C_j. For $b \in B$, let $n(b)$ be the number of irreducible components C_j passing through b. By definition, $n(b) \geq 2$. Moreover, note that the sets A and B are not disjoint in general, and their union is precisely the singular set of C.

Using the additivity of Hodge–Deligne polynomials we get

$$P(C) = P(C_1 \cup \cdots \cup C_r) = \sum_{j=1}^r P(C_j) + \sum_{0 \leq i_1 < \cdots < i_l \leq r} (-1)^{l-1} P(C_{i_1} \cap \cdots \cap C_{i_l}).$$

The first sum is easy to determine using Proposition 2.1:

$$\sum_{j=1}^r P(C_j)(u,v) = ruv - \Big(\sum_{j=1}^r g_j\Big)u - \Big(\sum_{j=1}^r g_j\Big)v + r - \sum_{j,k}\big((r(C_j, a_k^j) - 1\big).$$

Consider now the alternating sum, where $l \geq 2$. The only points of C that give a contribution to this sum are the points in B. Now, for a point $b \in B$, its contribution to the alternating sum is clearly given by

$$c(b) = -\binom{n(b)}{2} + \binom{n(b)}{3} - \cdots + (-1)^{n(b)-1}\binom{n(b)}{n(b)} = -n(b) + 1.$$

Proposition 2.2 *With the above notation and assumptions, we have the following for a reducible plane curve* $C = \bigcup_{j=1}^r C_j$.

(i) *The Hodge-Deligne polynomial of C is given by*

$$P(C)(u,v) = ruv - \Big(\sum_{j=1}^r g_j\Big)u - \Big(\sum_{j=1}^r g_j\Big)v + r - \sum_{j,k}\big((r(C_j, a_k^j) - 1\big) - \sum_{b \in B}\big(n(b) - 1\big).$$

with g_j given by the formula (2.1).

(ii) $H^0(C) = \mathbb{C}$ *is pure of type* $(0,0)$.

(iii) $H^2(C) = \mathbb{C}^r$ *is pure of type* $(1,1)$.

(iv) *The mixed Hodge numbers of the MHS on $H^1(C)$ are given by*

$$h^{0,0}(H^1(C)) = \sum_{j,k}((r(C_j, a_k^j) - 1) + \sum_{b \in B}(n(b) - 1) - r + 1,$$

$$h^{1,0}(H^1(C)) = h^{0,1}(H^1(C)) = \sum_{j=1}^r g_j.$$

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In particular, one has the following formula for the first Betti number of C:

$$b_1(C) = \sum_{j,k}\Big((r(C_j, a_k^j) - 1\Big) + \sum_{b\in B}(n(b) - 1) - r + 1 + 2\sum_{j=1}^{r} g_j.$$

Note that a point in the intersection $A \cap B$ will give a contribution to the last two sums in the above formula for $P(C)$.

Example 2.3 Suppose C is a nodal curve. Then for each singularity $a_k^j \in A$ one has $a_k^j \notin B$ (otherwise we get worse singularities than nodes) and $r(a_k^j) = 2$. Moreover, each point $b \in B$ satisfies $n(b) = 2$. It follows that in this case we get

$$P(C)(u,v) = ruv - \Big(\sum_{j=1}^{r} g_j\Big)u - \Big(\sum_{j=1}^{r} g_j\Big)v + r - n_2,$$

with n_2 the number of nodes of C. More precisely, in this case we have $n_2 = n_2' + n_2''$, where n_2' (resp. n_2'') is the number of nodes of C in A (resp. in B), and one clearly has

$$n_2' = S_1 := \sum_{j,k}\Big((r(C_j, a_k^j) - 1\Big), \quad n_2'' = S_2 := \sum_{b\in B}(n(b) - 1).$$

Example 2.4 Suppose C has only nodes and ordinary triple points as singularities. Then let n_3 be the number of triple points and note that we can write, as above, $n_3 = n_3' + n_3''$, where n_3' (resp. n_3'') is the number of triple points of C in $A_0 = A \smallsetminus B$ (resp. in B). For a point $a \in A_0$, the contribution to the sum S_1 is 2, while the contribution to the sum S_2 is 0.

A point $b \in B$ can be of two types. The first type, corresponding to the partition $3 = 1+1+1$, is when b is the intersection of three components C_j, all smooth at b. The contribution of such a point b is 0 to the sum S_1 and 2 to the sum S_2.

The second type, corresponding to the partition $3 = 2+1$, is when b is the intersection of two components, say C_i and C_j, such that C_i has a node at b, and C_j is smooth at b. The contribution of such a point b is 1 to the sum S_1 and 1 to the sum S_2.

It follows that the contribution of any triple point to the sum $S_1 + S_2$ is equal to 2. Since the double points in C can be treated exactly as in Example 2.3, this yields the following:

$$P(C)(u,v) = ruv - \Big(\sum_{j=1}^{r} g_j\Big)u - \Big(\sum_{j=1}^{r} g_j\Big)v + r - n_2 - 2n_3.$$

When there are only triple points in B of the first type, we obviously have the following additional relations

$$S_1 = n_2' + 2n_3', \quad S_2 = n_2'' + 2n_3''.$$

Example 2.5 Suppose C has only ordinary points of multiplicity 2, 3, and 4 as singularities. Then let n_4 be the number of points of multiplicity 4 and note that we can write, as above, $n_4 = n_4' + n_4''$, where n_4' (resp. n_4'') is the number of points of multiplicity 4 of C in $A_0 = A \smallsetminus B$ (resp. in B). For a point $a \in A_0$ of multiplicity 4, the contribution to the sum S_1 is 3, while the contribution to the sum S_2 is 0.

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A point $b \in B$ can be of 4 types. The first type, corresponding to the partition $4 = 1+1+1+1$, is when b is the intersection of 4 components C_j, all smooth at b. The contribution of such a point b is 0 to the sum S_1 and 3 to the sum S_2.

The second type, corresponding to the partition $4 = 2+1+1$, is when b is the intersection of 3 components, say C_i, C_j, and C_k, such that C_i has a node at b, and C_j and C_k are smooth at b. The contribution of such a point b is 1 to the sum S_1 and 2 to the sum S_2.

The third type, corresponding to the partition $4 = 2+2$, is when b is the intersection of 2 components, say C_i and C_k, such that C_i and C_k have a node at b. The contribution of such a point b is 2 to the sum S_1 and 1 to the sum S_2.

The fourth type, corresponding to the partition $4 = 3+1$, is when b is the intersection of 2 components, say C_i and C_k, such that C_i has a triple point at b, and C_k is smooth at b. The contribution of such a point b is 2 to the sum S_1 and 1 to the sum S_2.

It follows that the contribution of any point of multiplicity 4 to the sum $S_1 + S_2$ is equal to 3. Since the double and triple points in C can be treated exactly as in Example 2.4, this yields the following:

$$P(C)(u,v) = ruv - \Big(\sum_{j=1}^{r} g_j\Big)u - \Big(\sum_{j=1}^{r} g_j\Big)v + r - n_2 - 2n_3 - 3n_4.$$

When there are only points of multiplicity 4 in B of the first type, then we obviously have the following additional relations

$$S_1 = n_2' + 2n_3' + 3n_4'', \quad S_2 = n_2'' + 2n_3'' + 3n_4''.$$

Let us look now at the cohomology of the smooth surface $U = \mathbb{P}^2 \setminus C$. By the additivity we get $P(U) = P(\mathbb{P}^2) - P(C)$, where $P(\mathbb{P}^2) = u^2v^2 + uv + 1$. This yields the following consequence.

Corollary 2.6

$$P(U)(u,v) = u^2v^2 - (r-1)uv + \Big(\sum_{j=1}^{r} g_j\Big)u + \Big(\sum_{j=1}^{r} g_j\Big)v - (r-1) + \sum_{j,k}\Big((r(C_j, a_k^j) - 1\Big) + \sum_{b\in B}(n(b) - 1).$$

The contribution of $H_c^4(U,\mathbb{C})$ to $P(U)$ is the term u^2v^2, and that of $H_c^3(U,\mathbb{C})$ is the term $-(r-1)uv$. Moreover, the dimension $\dim Gr_F^1 H^2(U,\mathbb{C})$ is the number of independent classes of type (1,2), which correspond to classes of type $(1,0)$ in $H_c^2(U)$, and hence to the terms in u in $P(U)$. For both statements see the proof of [1, Theorem 2.1]. This proves the following result.

Theorem 2.7

$$\dim Gr_F^1 H^2(U,\mathbb{C}) = \sum_{j=1}^{r} g_j$$

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and

$$\dim Gr_F^2 H^2(U,\mathbb{C}) = \sum_{j=1}^{r} g_j + \sum_{j,k} \Big(\big(r(C_j, a_k^j) - 1\big) + \sum_{b\in B} (n(b) - 1) - r + 1.$$

In particular, all the components C_j of the curve C are rational if and only if $H^2(U)$ is pure of type $(2,2)$.

Example 2.8 Suppose C has only ordinary points of multiplicity 2, 3, and 4 as singularities. Let n_k be the number of points of multiplicity k, for $k = 2,3,4$; then using Example 2.5, we get the formula

$$\dim Gr_F^2 H^2(U,\mathbb{C}) = \sum_{j=1}^{r} g_j - r + 1 + n_2 + 2n_3 + 3n_4.$$

3 Arrangements of Transversely Intersecting Curves

Recall that $C = \bigcup_{j=1}^{r} C_j$ is the decomposition of C as a union of irreducible components C_j, and the curve C_j has degree N_j. In this section we assume that any curve C_j has only ordinary multiple points as singularities and let $n_k(C_j)$ be the number of ordinary points on C_j of multiplicity k. We also assume that the intersection of any two distinct components C_i and C_j is transverse, *i.e.*, the points in $C_i \cap C_j$ are nodes of the curve $C_i \cup C_j$. This implies in particular that $A \cap B = \varnothing$. The formulas (2.1) and (2.2) yield the equality.

$$g_j = \frac{(N_j-1)(N_j-2)}{2} - \frac{1}{2}\sum_k \big(\mu(C_j, a_k^j) + r(C, a_k^j) - 1 \big).$$

Using this, Theorem 2.7 gives the formula

$$\dim Gr_F^2 H^2(U,\mathbb{C}) = \sum_{j=1}^{r} \frac{(N_j-1)(N_j-2)}{2} - \frac{1}{2}\sum_{j,k} \big(\mu(C_j, a_k^j) - r(C, a_k^j) + 1 \big) + \sum_{b\in B} \big(n(b) - 1 \big) - r + 1.$$

If a_k^j is an ordinary m-multiple point on the curve C_j, one has $\mu(C_j, a_k^j) = (m-1)^2$, and hence

$$\mu(C_j, a_k^j) - r(C, a_k^j) + 1 = (m-1)(m-2).$$

If we denote by n'_m (resp. n''_m) the number of m-multiple points of C coming from just one component C_j (resp. from the intersection of several components C_j), we see that we have

$$\sum_{j,k} \big(\mu(C_j, a_k^j) - r(C, a_k^j) + 1 \big) = \sum_m (m-1)(m-2) n'_m.$$

This equality explains the contribution of the points in A. Now let $b \in B$ such that $n(b) = m$. The number of such points is precisely n''_m. It follows that

$$\sum_{b\in B} \big(n(b) - 1 \big) = \sum_m (m-1) n''_m.$$

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Let $1 \le i < j \le r$ and consider the intersection $C_i \cap C_j$. It contains exactly $N_i N_j$ points, since C_i and C_j intersects transversely. The sum $S = \sum_{1 \le i < j \le r} N_i N_j$ represents the number of all such intersection points. Note that a point $b \in B$ is counted in this sum exactly $\binom{n(b)}{2}$ times. This yields the formula

$$2S = \sum_m m(m-1) n''_m.$$

These formulas give the following result.

Theorem 3.1 *With the above assumptions and notation, one has*

$$\dim Gr_F^2 H^2(U, \mathbb{C}) = \frac{(N-1)(N-2)}{2} - \sum_m \binom{m-1}{2} n_m,$$

with $n_m = n'_m + n''_m$ the number of ordinary m-tuple points of C.

The following consequence of Theorems 2.7 and 3.1 applies in particular to any projective line arrangement.

Corollary 3.2 *Assume that $C = \bigcup_{j=1}^r C_j$ is the decomposition of C as a union of irreducible components C_j, with any curve C_j having only ordinary multiple points as singularities and being rational,* i.e., *$g_j = 0$. If the intersection of any two distinct components C_i and C_j is transverse,* i.e., *the points in $C_i \cap C_j$ are nodes of the curve $C_i \cup C_j$, then one has*

$$\dim H^2(U, \mathbb{C}) = \frac{(N-1)(N-2)}{2} - \sum_m \binom{m-1}{2} n_m,$$

with n_m the number of ordinary m-tuple points of C.

4 Curves with Ordinary Singularities of Multiplicity ≤ 4

Let $C \subset \mathbb{P}^2$ be a curve of degree N having only ordinary singular points of multiplicity at most 4. Set $U = \mathbb{P}^2 \setminus C$, and let $C = \cup_{j=1}^r C_j$ be the decomposition of C in irreducible components. Then

$$P(C) = \sum_{j=1}^r P(C_j) - \sum_{0 \le i < j \le r} P(C_i \cap C_j) + \sum_{0 \le i < j < k \le r} P(C_i \cap C_j \cap C_k) - \sum_{0 \le i < j < k < l \le r} P(C_i \cap C_j \cap C_k \cap C_l).$$

Let a_m^j denote the number of singular points of multiplicity m that belong to the component C_j (note that a point can be singular on two components, being a node on each of them).

Denote by b_3^k (resp. b_4^k) the number of triple points (resp. points of multiplicity 4) of C that are intersection of exactly k components, for $k = 2, 3$ (respectively $k = 3, 4$). Let b_4^2 (resp. $\widetilde{b_4^2}$) be the number of singular points p of multiplicity 4 in C representing the intersection of exactly 2 components, such that one of which has a triple point at

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p (resp. each one has a node at p). Then one has

$$\sum_{0\le i<j\le r} P(C_i \cap C_j) = \sum_{0\le i<j\le r} N_i N_j - b_3^2 - 3\widetilde{b_4^2} - 2b_4^2 - 2b_4^3.$$

Indeed, a point of type b_3^2 (resp. b_4^2, resp. $\widetilde{b_4^2}$) occurs only in one intersection $C_i \cap C_j$ and has the multiplicity 2 (resp. 3, resp. 4) in this intersection. A point of type b_4^3 occurs in 3 intersections $C_i \cap C_j$ with multiplicities $1, 2, 2$, and this accounts for the correction term $-2b_4^3$. Then one has

$$\sum_{0\le i<j<k\le r} P(C_i \cap C_j \cap C_k) = b_3^3 + b_4^3 + \binom{4}{3} b_4^4$$

and

$$\sum_{0\le i<j<k<l\le r} P(C_i \cap C_j \cap C_k \cap C_l) = b_4^4.$$

Hence, by Proposition 2.1, we get the following:

$$\begin{aligned} P(C)(u,v) = ruv - \Big(\sum_{j=1}^{r} g_j\Big) u - \Big(\sum_{j=1}^{r} g_j\Big) v - \sum_{j=1}^{r}(a_2^j + 2a_3^j + 3a_4^j) - \sum N_i N_j \\ + b_3^2 + 3\widetilde{b_4^2} + 2b_4^2 + 3b_4^3 + b_3^3 + 3b_4^4. \end{aligned}$$

Therefore, as above, we obtain

$$\begin{aligned} P(U)(u,v) = u^2v^2 - (r-1)uv + 1 - r + \Big(\sum_{j=1}^{r} g_j\Big) u + \Big(\sum_{j=1}^{r} g_j\Big) v + \sum_{j=1}^{r}(a_2^j + 3a_3^j + 6a_4^j) \\ - \sum_{j=1}^{r}(a_3^j + 3a_4^j) + \sum N_i N_j - b_3^2 - 3\widetilde{b_4^2} - 2b_4^2 - 3b_4^3 - b_3^3 - 3b_4^4. \end{aligned}$$

Finally, we get

$$\begin{aligned} \dim Gr_F^2 H^2(U) &= \sum_{j=1}^{r}(g_j + a_2^j + 3a_3^j + 6a_4^j - 1) + \sum N_i N_j + 1 - (\sum_{j=1}^{r} a_3^j + b_3^2 + b_3^3) \\ &\quad - 3\Big(\sum_{j=1}^{r} a_4^j + \widetilde{b_4^2} + b_4^2 + b_4^3 + b_4^4\Big) + b_4^2 \\ &= \frac{(N-1)(N-2)}{2} - n_3 - 3n_4 + b_4^2, \end{aligned}$$

with n_m the number of ordinary m-tuple points of C.

Theorem 4.1 *Let $C \subset \mathbb{P}^2$ be a curve of degree N having only ordinary singular points of multiplicity at most 4. If $U = \mathbb{P}^2 \setminus C$, then one has*

$$\dim Gr_F^2 H^2(U, \mathbb{C}) = \frac{(N-1)(N-2)}{2} - \sum_{m=3}^{4} \binom{m-1}{2} n_m + b_4^2,$$

with n_m the number of ordinary m-tuple points of C and b_4^2 the number of singular points p of C that are smooth on one component C_i of C and have multiplicity 3 on the other component C_j of C passing through p.

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5 Pole Order Filtration Versus Hodge Filtration for Plane Curve Complements

For any hypersurface V in a projective space $\mathbb{P}^n$, the cohomology groups $H^*(U,\mathbb{C})$ of the complement $U = \mathbb{P}^n \setminus V$ have a pole order filtration P^k; see, for instance, [8]. By the work of Deligne, Dimca [3] and M. Saito [12], one has

$$F^k H^m(U,\mathbb{C}) \subset P^k H^m(U,\mathbb{C})$$

for any k and any m. For $m = 0$ and $m = 1$, the above inclusions are in fact equalities (the case $m = 0$ is obvious and the case $m = 1$ follows from the equality $F^1H^1(U,\mathbb{C}) = H^1(U,\mathbb{C})$). For $m = 2$, we have again that $F^kH^2(U,\mathbb{C}) = P^kH^2(U,\mathbb{C})$ for $k = 0,1$ for obvious reasons, but one can get strict inclusions

$$F^2H^2(U,\mathbb{C}) \neq P^2H^2(U,\mathbb{C})$$

already in the case when $V = C$ is a plane curve; see [5], Remark 2.5, or [4]. However, to give such examples of plane curves was until now rather complicated. We give below a numerical condition that tells us exactly when the above strict inclusion holds.

We first need to recall some basic definitions. Let $S = \oplus_r S_r = \mathbb{C}[x,y,z]$ be the graded ring of polynomials with complex coefficients, where S_r is the vector space of homogeneous polynomials of S of degree r. For a homogeneous polynomial f of degree N, define the Jacobian ideal of f to be the ideal J_f generated in S by the partial derivatives f_x, f_y, f_z of f with respect to x, y, and z. The graded *Milnor algebra* of f is given by

$$M(f) = \bigoplus_r M(f)_r = S/J_f.$$

Note that the dimensions $\dim M(f)_r$ can be easily computed in a given situation using some computer software *e.g.*, Singular.

Let $C \subset \mathbb{P}^2$ be the curve defined by $f = 0$, and suppose that P is a singular point of C with local equation $g = 0$. Define the Tjurina number $\tau(C,P)$ of C at the point P by

$$\tau(C,P) = \dim_{\mathbb{C}} \frac{O_P}{(g,J_g)},$$

where O_p is the local ring of germs of regular functions at P and (g,J_g) is the ideal generated by g and its Jacobian J_g. The Tjurina number $\tau(C)$ of a curve C is given by the sum of the Tjurina numbers of all the singularities of C. Now we can state the main result of this section.

Theorem 5.1 *Let $C : f = 0$ be a reduced curve of degree N in $\mathbb{P}^2$ having only weighted homogeneous singularities and let C_i for $i = 1,\dots,r$ be the irreducible components of C. If $U = \mathbb{P}^2 \setminus C$, then*

$$\dim P^2H^2(U,\mathbb{C}) - \dim F^2H^2(U,\mathbb{C}) = \tau(C) + \sum_{i=1}^{r} g_i - \dim M(f)_{2N-3},$$

where $\tau(C)$ is the global Tjurina number of C and g_i is the genus of the normalization of C_i for $i = 1,\dots,r$.

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In particular we get the following result, which yields a new proof for [7, Theorem 1.3].

Corollary 5.2 *If a reduced plane curve has only nodes as singularities, then one has*

$$\dim M(f)_{2N-3} = \tau(C) + \sum_{i=1}^{r} g_i.$$

Proof Indeed, it is known that for a nodal curve one has the equality $F^2H^2(U,\mathbb{C}) = P^2H^2(U,\mathbb{C})$; see [2] or [12]. ■

Note that we have the following obvious consequence of Theorem 2.7.

Corollary 5.3 *For a reduced plane curve C, one has*

$$\dim P^2H^2(U,\mathbb{C}) - \dim F^2H^2(U,\mathbb{C}) \le \sum_{i=1}^{r} g_i.$$

Proof Indeed, Theorem 2.7 can be restated as

$$\dim H^2(U,\mathbb{C}) - \dim F^2H^2(U,\mathbb{C}) = \sum_{i=1}^{r} g_i$$

in view of the equality $F^1H^2(U,\mathbb{C}) = H^2(U,\mathbb{C})$; see proof of [4, Cor. 1.32, p. 185]. ■

Remark 5.4 If a reduced plane curve C has only rational irreducible components, *i.e.*, $g_i = 0$ for all i, then the above inequality implies $F^2H^2(U,\mathbb{C}) = P^2H^2(U,\mathbb{C})$. This result can be regarded as an improvement of a part of [5, Remark 2.5], where the result is claimed only for curves with nodes and cusps as singularities.

The above discussion also implies the following result, which can be regarded as a generalization of [1, Theorem 4.1 (A)].

Corollary 5.5 *If a reduced plane curve $C : f = 0$ has only weighted homogeneous singularities, then one has*

$$0 \le \dim M(f)_{2N-3} - \tau(C) \le \sum_{i=1}^{r} g_i.$$

In particular, if in addition the curve C has only rational irreducible components, then one has $\dim M(f)_{2N-3} = \tau(C)$.

Now we give the proof of Theorem 5.1. Corollary 1.3 in [8] implies that

$$\dim P^2H^2(U,\mathbb{C}) = \dim H^2(U,\mathbb{C}) + \tau(C) - \dim M(f)_{2N-3}.$$

On the other hand, Theorem 2.7 and the fact $\dim F^1H^2(U,\mathbb{C}) = H^2(U,\mathbb{C})$ yield

$$\dim F^2H^2(U,\mathbb{C}) = \dim H^2(U,\mathbb{C}) - \sum_{i=1}^{r} g_i,$$

which clearly completes the proof of Theorem 5.1. ■

Example 5.6 In this example we present a free divisor $C : f = 0$, whose irreducible components consist of 12 lines and one elliptic curve and where

$$F^2H^2(U,\mathbb{C}) \neq P^2H^2(U,\mathbb{C}).$$

Let $f = xyz(x^3+y^3+z^3)[(x^3+y^3+z^3)^3 - 27x^3y^3z^3]$. If we consider the pencil of cubic curves $(x^3+y^3+z^3, xyz)$, then the curve C contains all the singular fibers of this pencil, and this accounts for the 12 lines given by

$$xyz\big[(x^3+y^3+z^3)^3 - 27x^3y^3z^3\big] = 0$$

and the elliptic curve (hence of genus 1) given by $x^3+y^3+z^3 = 0$. Then C is a free divisor (see [14]) or by a direct computation using Singular, which shows that $I = J_f$, where I is the saturation of the Jacobian ideal J_f; see [6, Remark 4.7]. The direct computation by Singular also yields $\tau(C) = 156$ and $\dim M(f)_{2N-3} = \dim M(f)_{27} = 156$. Moreover, applying [9, Corollary 1.5], we see via a Singular computation that all singularities of the curve C are weighted homogeneous. Alternatively, there are 12 nodes, 3 in each of the 4 singular fibers of the pencils (which are triangles), and the 9 base points of the pencil, each an ordinary point of multiplicity 5. Each of the 12 lines contains exactly 3 of these base points, and they are exactly the intersection of the elliptic curve with the line. This description implies that there are no other singularities in accord with $12 + 9 \times 16 = 156 = \tau(C)$. It follows from Theorem 5.1 that $\dim P^2H^2(U,\mathbb{C}) - \dim F^2H^2(U,\mathbb{C}) = 1$. Hence, the presence of a single irrational component of C leads to $F^2H^2(U,\mathbb{C}) \neq P^2H^2(U,\mathbb{C})$.

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Univ. Nice Sophia Antipolis, CNRS, LJAD, UMR 7351, 06100 Nice, France
e-mail: nancy.n.abdallah@gmail.com