Can. J. Math., Vol. XXIII, No. 4, 1971, pp. 627–643

SUBGROUPS OF *HNN* GROUPS AND GROUPS WITH ONE DEFINING RELATION

A. KARRASS AND D. SOLITAR

1. Introduction. *HNN* groups have appeared in several papers, e.g., [**3**; **4**; **5**; **6**; **8**]. In this paper we use the results in [**6**] to obtain a structure theorem for the subgroups of an *HNN* group and give several applications.

We shall use the terminology and notation of [**6**]. In particular, if K is a group and $\{\varphi_i\}$ is a collection of isomorphisms of subgroups $\{L_i\}$ into K, then we call the group

$$(1)\quad G = \langle t_1, t_2, \ldots, K; \text{rel } K, t_1L_1t_1^{-1} = \varphi_1(L_1), t_2L_2t_2^{-1} = \varphi_2(L_2), \ldots\rangle$$

the *HNN group with base* K, *associated subgroups* $\{L_i, \varphi_i(L_i)\}$ and *free part* the group generated by $t_1, t_2, \ldots$. (We usually denote $\varphi_i(L_i)$ by M_i or L_{-i}.) The notion of a *tree product* as defined in [**6**] will also be needed.

Let H *be a subgroup of* (1). Then we prove (in Theorem 1) that H *is itself an HNN group* (*with possibly trivial free part*); *its base is a tree product* S *with vertices of the form* $gKg^{-1} \cap H$ *and amalgamated subgroups either trivial or conjugates of the* L_i *intersected with* H; *each of its non-trivial associated subgroups is contained in a vertex of* S *and either equals this containing vertex or is a conjugate of an* L_i *intersected with* H; *moreover, every conjugate of* K *intersected with* H *is either trivial or is conjugate in* H *to a vertex of* S.

Some of the results we derive from this theorem are the following:

If H *has trivial intersection with each conjugate of* L_i, *then* H *is a free product of a free group and groups of the type* $gKg^{-1} \cap H$; *in particular, if* H *has trivial intersection with each conjugate of* K, *then* H *is free* (see Theorem 6).

If K *is a locally indicable group and each* L_i *is cyclic, then* G *is locally indicable* (see Theorem 2)*; if* K *is a finitely generated torsion-free nilpotent group then* G *is locally indicable.*

Suppose that the base K *has the property that its finitely generated subgroups are finitely related and that* L_i *has the property that all of its subgroups are finitely generated* (*or more generally that the intersection of a finitely generated subgroup of* G *with finitely many conjugates of the* L_i *is again finitely generated*), *then* G *has the property that all of its finitely generated subgroups are finitely related* (see Theorem 8).

We also determine the structure of a subgroup H which satisfies a non-trivial law (see Theorem 4); in particular, *if the* L_i *are free and* K *is torsion free, then* H

Received October 9, 1970. This research was supported by NRC Grants A5614 and A5602.

is either conjugate to a subgroup of K, a countable ascending union of cyclic groups, or a group with presentation

$$\langle \tau, \alpha; \tau\alpha\tau^{-1} = \alpha^n \rangle$$

(see Corollary to Theorem 4).

If H is a finitely generated normal subgroup of G, and H is not contained in the intersection of all L_i, then HK is of finite index in G (see Theorem 9).

If the base K is finite (*or more generally, the subgroup generated by the associated subgroups L_i, M_i is finite*), *then G has the finitely generated intersection property*, i.e., *the intersection of two finitely generated subgroups is finitely generated* (see Theorem 10).

Results about *HNN* groups can be applied to groups with one defining relation. For, as observed in [**8**] every infinite group G with one defining relator is an *HNN* group whose free part is infinite cyclic, whose base K is a group with one defining relator, and whose associated subgroups are free. Moreover, by the standard Magnus embedding (see, for example, [**7**, § 4.4]), the group G with one defining relation can be embedded in such an *HNN* group in which the base K has a shorter relator than that of G (unless the defining relator of G consists of a single syllable). For example, if

$$G = \langle a, b; R \rangle$$

and R involves both a and b and has zero-exponent sum on a, then

$$G = \langle t, K; R_0, tLt^{-1} = M \rangle,$$

where $t = a$, R_0 is the relation obtained from R by rewriting it in terms of the conjugates $b_i = a^i b a^{-i}$, K is the group with the single defining relation R_0 and with generators b_i where i ranges between the minimum subscript λ and maximum subscript μ occurring on b in R_0 and L is the free group on $b_\lambda, \ldots, b_{\mu-1}$.

Thus, theoretically we can describe the subgroups of a group with one defining relator in terms of the subgroups of another group with one defining relator of shorter length.

A direct consequence of this observation is that *every finitely generated torsion-free group with one defining relation can be obtained from an infinite cyclic group by applying finitely often the operations of forming an amalgamated product of two factors already obtained and taking a subgroup of a group already obtained* (this result does not hold for every finitely generated torsion-free group, see e.g., [**2**]).

Using this point of view we also prove the following: *suppose that G is a group with one defining relation*

$$G = \langle a, b, c, \ldots; R \rangle \tag{2}$$

and that H is a subgroup of G satisfying a non-trivial law. If G is torsion-free, then H is either locally cyclic or is metabelian with presentation

$$\langle \tau, \alpha; \tau\alpha\tau^{-1} = \alpha^n \rangle,$$

where n is some integer; if G has torsion, then H is cyclic or infinite dihedral (see Theorem 5).

(This last theorem generalizes some results of [**8**; **9**]. In [**8**; **9**] it is proved that an abelian subgroup of a group G with one defining relation is either locally cyclic or free abelian of rank two. In [**9**] it is proved also that if G has elements of finite order then any solvable subgroup of G is cyclic or infinite dihedral.)

As to the other subgroups of (2) we establish: *in a group with one defining relation any subgroup not satisfying any non-trivial law must contain a free subgroup of rank two* (see Theorem 3).

2. The subgroup theorem and some applications. The method of proof of the subgroup theorem for *HNN* groups uses the standard embedding (of [**5**]) of the *HNN* group G given by (1) in the amalgamated product

$$E = (A * B;\, U) = X * G = Y * G, \tag{3}$$

where

$A = X * K, B = Y * K,\ U = K * \ldots * x_i L_i x_i^{-1} * \ldots = K * \ldots * y_i M_i y_i^{-1} * \ldots,$ and X, Y are the free groups on x_i, y_i respectively, and $t_i = y_i^{-1}x_i$ for $i = 1, 2, \ldots n$.

A simple application of this embedding of an *HNN* group in an amalgamated product and the observation in [**8**] mentioned in the introduction is given by the following result: *every finitely generated torsion-free group with one defining relator can be obtained from an infinite cyclic group by applying finitely often the operations of forming an amalgamated product of two factors already obtained and taking a subgroup of a group already obtained.*

The proof is by induction on the length of the defining relator. If the length of the defining relator is one, then the group is a finitely generated free group and so is obtainable from an infinite cyclic group using the allowable operations. Clearly, if the base of an *HNN* group with finitely generated free part is obtainable, then by the above embedding, the *HNN* group itself is obtainable. Since a group with one defining relator is a subgroup of an *HNN* group with base a group with one defining relator of shorter length, we have the result.

Similarly, *a finitely generated group with one defining relator having torsion can be obtained by starting the above process with a finite cyclic group instead of an infinite one.*

LEMMA 1. *Let E be the free product of any two groups X and G, let H, K be subgroups of G and let p be an element of E having normal form* (*in E*)

$$p = g_1 w_1 g_2 w_2 \ldots g_r w_r,$$

where $g_j \in G$, $w_j \in X$ and each g_j, $w_j \neq 1$ except possibly g_1 or w_r. If

$$p(X * K)p^{-1} \cap H \neq 1,$$

https://doi.org/10.4153/CJM-1971-070-x Published online by Cambridge University Press

then $p(X * K)p^{-1} \cap H = g_1Kg_1^{-1} \cap H$ and $g_2, \ldots, g_r$ are in K.

Moreover, let $L_0, \ldots, L_n$ be subgroups of G, let $x_0, \ldots, x_n$ be distinct elements of X and let $U = x_0L_0x_0^{-1} * \ldots * x_nL_nx_n^{-1}$. If $pUp^{-1} \cap H \neq 1$, then $pUp^{-1} \cap H = g_1L_{j_1}g_1^{-1} \cap H$ and

$$p = g_1x_{j_1}^{-1}x_{j_2}g_2x_{j_2}^{-1}x_{j_3}g_3 \ldots g_rx_{j_r}^{-1},$$

where $0 \leqq j_i \leqq n$ and $g_i \in L_{j_i}$ for $2 \leqq i \leqq r$.

Proof. Let $Q = p(X * K)p^{-1} \cap H \neq 1$. Then

$$p^{-1}Qp = (X * K) \cap w_r^{-1}g_r^{-1} \ldots w_1^{-1}(g_1^{-1}Hg_1)w_1 \ldots g_rw_r.$$

Now an element ($\neq 1$) in the right hand side must have the form

$$w_r^{-1}g_r^{-1} \ldots w_1^{-1}g_1'w_1 \ldots g_rw_r$$

where $g_1', g_2, \ldots, g_r$ are in K; hence $Q = g_1Kg_1^{-1} \cap H$ and $g_2, \ldots, g_r$ are in K.

Next suppose that $R = pUp^{-1} \cap H \neq 1$. Then

$$p^{-1}Rp = U \cap w_r^{-1}g_r^{-1} \ldots w_1^{-1}(g_1^{-1}Hg_1)w_1 \ldots g_rw_r.$$

Now the normal subgroup N generated by G in E is the free product of the conjugates of G by distinct elements of X; moreover, U is the free product of subgroups from distinct factors of N. An element ($\neq 1$) in $p^{-1}Rp$ has the normal form in (the free product) N given by

$$g_r^{-w_r^{-1}}g_{r-1}^{-w_r^{-1}w_{r-1}^{-1}} \ldots (g_1^{-1}hg_1)^{w_r^{-1}\ldots w_1^{-1}} \ldots g_{r-1}^{w_r^{-1}w_{r-1}^{-1}}g_r^{w_r^{-1}}.$$

Therefore the product $w_s \ldots w_r$ must be $x_{j_s}^{-1}$; moreover, g_s is in L_{j_s}, $2 \leqq s \leqq r$, and $g_1^{-1}hg_1$ is in L_{j_1}. Hence $R = g_1L_{j_1}g_1^{-1} \cap H$ and p has the form asserted.

THEOREM 1. *Let G be the HNN group given by* (1) *and let H be a subgroup of G. Then H is an HNN group (with possibly trivial free part) whose base is a tree product S with vertices of the form $gKg^{-1} \cap H$, where neighboring vertices are joined by the identity subgroup, or have the form $gKg^{-1} \cap H$ and $gt_i^{-1}Kt_ig^{-1} \cap H$ and are joined by the amalgamated subgroup $gL_ig^{-1} \cap H = gt_i^{-1}M_it_ig^{-1} \cap H$; each of the non-trivial associated subgroups is contained in a vertex of S and equals this containing vertex or has the form $\gamma L_i\gamma^{-1} \cap H$.*

Proof. Embed G in the amalgamated product E described in (3). Applying the subgroup theorem [**6**, Theorem 5] to E, we have that H is an HNN group whose base is a tree product S with vertices of the form $D(X * K)D^{-1} \cap H$ or $D(Y * K)D^{-1} \cap H$. Employing Lemma 1 with X equal to X or Y, we obtain that each vertex ($\neq 1$) of S is of the form $gKg^{-1} \cap H$ where $g \in G$. Moreover, neighboring vertices of S have the form $D(X * K)D^{-1} \cap H, D(Y * K)D^{-1} \cap H$ with amalgamated subgroup $DUD^{-1} \cap H$. Again employing Lemma 1 with $X = X$, $x_0 = 1$ and $L_0 = K$, we have that if this amalgamated subgroup is different from 1, then D has the form

$$g_1x_{j_1}^{-1}x_{j_2}g_2x_{j_2}^{-1}x_{j_3} \ldots g_rx_{j_r}^{-1}, \tag{4}$$

where $g_s \in L_{j_s}$, $2 \leqq s \leqq r$.

Replacing each x_i in (4) by y_it_i (where $y_0 = t_0 = 1$), and again applying Lemma 1, it follows that if $j_1 = 0$, $D(X * K)D^{-1} \cap H = D(Y * K)D^{-1} \cap H = g_1Kg_1^{-1} \cap H$; and if $j_1 \neq 0$, then $D(X * K)D^{-1} \cap H = g_1Kg_1^{-1} \cap H$, $D(Y * K)D^{-1} \cap H = g_1t_{j_1}^{-1}Kt_{j_1}g_1^{-1} \cap H$ and $DUD^{-1} \cap H = g_1L_{j_1}g_1^{-1} \cap \mathrm{H} = g_1t_{j_1}^{-1}M_{j_1}t_{j_1}g_1^{-1} \cap H$.

Clearly in any tree product a subtree consisting of equal vertices may be replaced by a single vertex without altering the resulting group. Hence we may write S as a tree product in which neighboring vertices are as asserted in the theorem.

Moreover, by [**6**, Theorem 5], a pair of associated subgroups of H has the form $\delta U\delta^{-1} \cap H$ (which is in $\delta(X * K)\delta^{-1} \cap H$, a vertex of S) and $\delta' U(\delta')^{-1} \cap H$ (which is in $\delta'(Y * K)(\delta')^{-1} \cap H$, a vertex of S). Hence if $\delta U\delta^{-1} \cap H \neq 1$ and if δ has the form (4), then $\delta U\delta^{-1} \cap H = \delta(X * K)\delta^{-1} \cap H = g_1Kg_1^{-1} \cap H$ if $j_1 = 0$ and $\delta U\delta^{-1} \cap H = g_1L_{j_1}g_1^{-1} \cap H$ if $j_1 \neq 0$. Similarly, $\delta' U(\delta')^{-1} \cap H = \delta'(Y * K)(\delta')^{-1} \cap H$ or $\delta' U(\delta')^{-1} \cap H = g_1'M_{j_1}(g_1')^{-1} \cap H = g_1't_{j_1}L_{j_1}t_{j_1}^{-1}(g_1')^{-1} \cap H$. This completes the proof of Theorem **1**.

COROLLARY 1. *Any subgroup of an HNN group* (1) *having trivial intersection with each conjugate of the base K is a free group.*

COROLLARY 2. *If a subgroup H of* (1) *is generated by its intersections with conjugates of K, then H is the tree product S described in* Theorem 1.

Proof. Since H is generated by its intersections with conjugates of K, H is generated by its intersections with conjugates of $X * K$ and $Y * K$, and so (by [**6**, Theorem 5, Corollary 1]) the free part of H is trivial. Therefore H is the tree product S.

COROLLARY 3. *Under the same hypotheses as in Theorem* 1, *every subgroup $gKg^{-1} \cap H$ is either trivial or conjugate in H to a vertex of S.*

Proof. By Lemma 1, $gKg^{-1} \cap H = g(X * K)g^{-1} \cap H$ which is

$$hD_\alpha(X * K)D_\alpha^{-1}h^{-1} \cap H$$

(where $h \in H$ and D_α is a double coset representative $\bmod(H, X * K)$), which in turn is a conjugate in H of a vertex of S.

THEOREM 2. *Let K be a locally indicable group and let the associated subgroups L_i be cyclic. Then the HNN group G given in* (1) *is locally indicable.*

Proof. The proof is just like that of [**6**, Theorem 9].

Theorem 1 also allows us to prove (as in the proof of [**6**, Theorem 8]) that if the base K has all its subgroups finitely presented, then all finitely generated subgroups of the HNN group (1) are finitely related. This will be strengthened in Theorem 8. For example, *a group $A = \langle a, b; a^nb^ra^{-n} = b^s\rangle$ with $rs \neq 0$ is*

https://doi.org/10.4153/CJM-1971-070-x Published online by Cambridge University Press

locally indicable and every finitely generated subgroup is finitely related. For, the group $B = (\langle t \rangle * A; t = a^n)$ equals $(\langle a \rangle * \langle t, b; tb^r t^{-1} = b^s \rangle; a^n = t)$, and the result follows from the above and [**6**, Theorems 8 and 9].

COROLLARY. *Let G be an HNN group as in* (1). *If the base K is a finitely generated torsion-free nilpotent group, then G is locally indicable.*

Proof. The proof is just like that of [**6**, Theorem 9] if one uses the following result of [**1**]: the free product with amalgamated subgroup of a locally indicable group and a finitely generated torsion-free nilpotent group is locally indicable.

The following theorem places a restriction on groups which can be embedded in one defining relator groups.

THEOREM 3. *Every subgroup of a group with one defining relation either contains a free subgroup of rank two or is solvable.*

Proof. We first observe the following: *if H is an HNN group whose base is a tree product S and H contains no free subgroup of rank two, then* (a) *H is a vertex of S, or* (b) *H is an ascending union of amalgamated subgroups of S, or* (c) *H has the form $(A * B; U)$ where A, B are vertices of S and U is an amalgamated subgroup of S of index two in A and B, or* (d) *H has a presentation*

$$\langle t, S'; \text{rel } S', tS't^{-1} = S'' \rangle,$$

where $S'' < S'$ and S', S'' are a pair of associated subgroups and t generates the free part of H. This is proved by the same argument as that for [**6**, Theorem 7].

We next show that if the base K of an HNN group has the property that there exists an integer s such that every subgroup of K is solvable of length $\leqq s$ or has a free subgroup of rank two, then the HNN group G defined by (1) has the same property except that s is replaced by $s + 2$.

For, consider a subgroup H of the HNN group G which does not contain a free subgroup of rank two. Then by Theorem 1, H is itself an HNN group with base S a tree product whose vertices are conjugates of subgroups of K and with amalgamated subgroups which are subgroups of conjugates of K. We show that H is solvable of length $\leqq s + 2$. For, by the above observation and Theorem 1, if possibilities (a) or (b) hold, then H is solvable of length $\leqq s$. If possibility (c) holds, then $H/U \simeq Z_2 * Z_2$, which is metabelian; hence H is solvable of length $\leqq s + 2$. Finally, if possibility (d) holds, then the normal subgroup N of H generated by S' is an ascending union of subgroups contained in conjugates of K and hence is solvable of length $\leqq s$; since H/N is cyclic, H is solvable of length $\leqq s + 1$.

Returning to groups with one defining relation, we show by induction on the length λ of the relator that every subgroup is solvable of length $\leqq 2\lambda$ or contains a free subgroup of rank two. As mentioned in the introduction, a group with one defining relator of length $\lambda > 1$ can be embedded as a subgroup of an HNN group G' whose base K is a group with one defining relator of length

https://doi.org/10.4153/CJM-1971-070-x Published online by Cambridge University Press

$\leqq \lambda - 1$. Hence by inductive hypothesis and the preceding argument every subgroup of the *HNN* group G' is solvable of length $\leqq 2(\lambda - 1) + 2 = 2\lambda$ or contains a free subgroup of rank two.

COROLLARY 1. *Every subgroup of a group with one definition relation having torsion is cyclic, infinite dihedral, or contains a free subgroup of rank two.*

Proof. This follows from Theorem 3 by using the following result of [**9**]: every solvable subgroup of a group with one defining relation having torsion is cyclic or infinite dihedral.

We shall also sharpen Theorem 3 for torsion-free one relator groups by showing that a solvable subgroup of such a group is metabelian of a very special type (see Theorem 5).

COROLLARY 2. *Let the base K of an HNN group have the property that there exists a non-trivial law such that every subgroup of K either satisfies this law or contains a free subgroup of rank two. Then the HNN group also has this property; the law in the case of the HNN group is obtained from the law associated with K by replacing each variable X in the law by the corresponding commutator*

$$[[X_1, X_2], [X_3, X_4]].$$

3. Subgroups satisfying a law.

LEMMA 2. *Let A be the free product $X * K$ of two groups X and K and let $x_0 = 1, x_1, \ldots, x_n$ be distinct elements of X such that $x_i x_j^{-1} \neq x_p x_q^{-1}$ unless $i = j$ or $i = p$, where $0 \leqq i, j, p, q, \leqq n$. Suppose that $L_0 = K, L_1, \ldots, L_n$ are subgroups of K, and that $U = x_0 L_0 x_0^{-1} * \ldots * x_n L_n x_n^{-1}$, and let $a_1 \in A - U$. If $a_1 U a_1^{-1} \cap U \neq 1$, then*

$$a_1 U a_1^{-1} \cap U = a x_i L_i x_i^{-1} a^{-1} \cap U = u x_j (L_j \cap k L_i k^{-1}) x_j^{-1} u^{-1}$$

for some $0 \leqq i, j \leqq n$, and $a_1 = a u_1$, u and u_1 are in U, and $k \in K$; moreover, $a = u x_j k x_i^{-1}$, $k \neq 1$ or $k \notin L_j$, and $i \neq 0$ or $j \neq 0$.

Proof. We first note that the given condition on the $\{x_i\}$ is equivalent to requiring that if $1 \neq x \in X$, then $\{x_i\} \cap \{x x_i\}$ contains at most one element.

It is convenient to denote an element of X by x_σ where σ ranges over some index set containing $\{0, 1, \ldots, n\}$. Using this notation, the normal subgroup N of A generated by K is the free product of the conjugates K^{x_σ}. Moreover, the element a_1 may be written in the form

$$a_1 = x_\sigma . k_1^{x_{\sigma 1}} \ldots k_r^{x_{\sigma r}} . u_1 = a u_1, \tag{5}$$

where $x_\sigma, x_{\sigma_i} \in X, k_i \in K, \sigma_i \neq \sigma_{i+1}, u_1 \in U$, and if $\sigma_r \in \{0, 1, \ldots, n\}$ then $k_r \notin L_{\sigma_r}$. Clearly, $a_1 U a_1^{-1} \cap U = a U a^{-1} \cap U$.

Now an element $d (\neq 1)$ of $D = U \cap a^{-1} U a$ can be written in the form

$$d = c_1^{x_{\tau 1}} \ldots c_s^{x_{\tau s}},$$

where $\tau_i \in \{0, 1, \ldots, n\}$, $c_i \in L_{\tau_i}$ and $\tau_i \neq \tau_{i+1}$. We first show that $s = 1$. For, suppose that $s \geqq 2$. Then

$$y \quad ada^{-1} = k_1^{x_\sigma x_{\sigma_1}} \ldots k_r^{x_\sigma x_{\sigma_r}} c_1^{x_\sigma x_{\tau_1}} \ldots c_s^{x_\sigma x_{\tau_s}} k_r^{-x_\sigma x_{\sigma_r}} \ldots k_1^{-x_\sigma x_{\sigma_1}}$$

is in U. It is easy to see that $x_\sigma x_{\tau_1}, \ldots, x_\sigma x_{\tau_s}$ will occur as exponents when ada^{-1} is written in reduced form as an element of N. Since $s \geqq 2$ and all exponents on elements in U are in $\{x_0, \ldots, x_n\}$, it follows (from the given condition on the $\{x_i\}$) that $x_\sigma = 1$ and so $r \geqq 1$. But this implies that $ada^{-1} \notin U$; for, the reduced form (as an element of N) of ada^{-1} will contain the factor $k_r^{x_{\sigma_r}}$ if $\sigma_r \neq \tau_1$ and the factor $(k_r c_1)^{x_{\tau_1}}$ if $\sigma_r = \tau_1$. Hence $s = 1$ and each element $d (\neq 1)$ of D must have the form

$$d = c^{x_i},$$

where $i \in \{0, 1, \ldots, n\}$ and $c \in L_i$; clearly (since D is a subgroup), all elements of D must have the same exponent x_i. Hence

$$aUa^{-1} \cap U = aDa^{-1} \cap U = ax_i L_i x_i^{-1} a^{-1} \cap U.$$

Moreover, a can also be written in the form

$$a = uk_1^{x_{\sigma_1}} \ldots k_r^{x_{\sigma_r}} . x_\sigma, \tag{6}$$

where $u \in U$, x_σ, $x_{\sigma_i} \in X$, $k_i \in K$, $\sigma_i \neq \sigma_{i+1}$, and if $\sigma_1 \in \{0, \ldots, n\}$ then $k_1 \notin L_{\sigma_1}$(σ, σ_i, k_i, r in (6) need not be the same as those in (5)). Now if $1 \neq c \in L_i$ and

$$ac^{x_i}a^{-1} = uk_1^{x_{\sigma_1}} \ldots k_r^{x_{\sigma_r}} . c^{x_\sigma x_i} . k_r^{-x_{\sigma_r}} \ldots k_1^{-x_{\sigma_1}} u^{-1}$$

is in U, then $x_\sigma x_i = x_j$ where $j \in \{0, \ldots, n\}$; moreover, since $k_1^{x_{\sigma_1}} \notin U$, either $r = 0$, or $r = 1$ and $\sigma_1 \neq 0$. If $r = 0$, then $a = ux_j x_i^{-1}$, $j \neq i$, and so

$$ax_i L_i x_i^{-1} a^{-1} \cap U = ux_j (L_j \cap L_i) x_j^{-1} u^{-1}.$$

If $r = 1$, then $x_{\sigma_1} = x_\sigma x_i = x_j$ and $a = ux_j k_1 x_i^{-1}$ and so $ax_i L_i x_i^{-1} a^{-1} \cap U = ux_j (L_j \cap k_1 L_i k_1^{-1}) x_j^{-1} u^{-1}$. This completes the proof of Lemma 2.

LEMMA 3. *Suppose that G is the HNN group* (1); *embed G in the amalgamated product E defined by* (3). *Let $x_0 = y_0 = 1$ and let $L_0 = M_0 = K$. If $p \in E - U$, then $pUp^{-1} \cap U$ is trivial or the intersection of finitely many conjugates of the associated subgroups $L_i (i \neq 0)$; finally, if $p \in E - K$ and $H < G$, then $pKp^{-1} \cap K \cap H$ is trivial or the intersection of H with finitely many conjugates of the associated subgroups $L_i (i \neq 0)$.*

Proof. If $p \notin U$, then p has a reduced form in $(A * B; U)$, say,

$$p = b_1 a_1 \ldots b_r a_r,$$

where the factors alternate from A and B and are not in U. Now if $pUp^{-1} \cap U \neq 1$, then

$$pUp^{-1} \cap U = b_1 \ldots b_r (a_r U a_r^{-1} \cap U) b_r^{-1} \ldots b_1^{-1} \cap U.$$

By Lemma 2,

$$a_r U a_r^{-1} \cap U = a_{r_1} x_i L_i x_i^{-1} a_{r_1}^{-1} \cap U,$$

where $a_r = a_{r_1} . u_1$, $u_1 \in U$. Hence

$$pUp^{-1} \cap U = p_1 x_i L_i x_i^{-1} p_1^{-1} \cap U, \tag{7}$$

where $p_1 = pu_1^{-1}$.

Now we prove by induction on the syllable length of p_1 that the right hand side of (7) is the intersection of finitely many conjugates of the associated subgroups. If p_1 has syllable length one, say, $p = a_1 \in A$, then by Lemma 2,

$$a_1 x_i L_i x_i^{-1} a_1^{-1} \cap U = ux_j(L_j \cap kL_i k^{-1})x_j^{-1}u^{-1}, \tag{8}$$

where $u \in U$, $k \notin L_j$ unless $k = 1$ and $i \neq 0$ or $j \neq 0$. If $i = 0$, then the right hand side of (8) reduces to $ux_j L_j x_j^{-1} u^{-1}$, $j \neq 0$; if $j = 0$, then the right hand side of (8) reduces to $uL_i u^{-1}$, $i \neq 0$.

If p has the form qa where q has shorter syllable length than p and, say, $a \in A$, then by Lemma 2

$$qax_i L_i x_i^{-1} a^{-1} q^{-1} \cap U = qux_j(L_j \cap kL_i k^{-1})x_j^{-1}u^{-1}q^{-1} \cap U \tag{9}$$

which is $qux_j L_j x_j^{-1}(qu)^{-1} \cap U$ if $i = 0$, and is

$$qux_j L_j x_j^{-1}(qu)^{-1} \cap U \cap qux_j k L_i k^{-1} x_j^{-1} u^{-1} q^{-1}$$

if $i \neq 0$. Using the inductive hypothesis, we have that the left hand side of (9) is the intersection of finitely many conjugates of the associated subgroups.

Finally, if $p \notin K$ and $pKp^{-1} \cap K \cap H \neq 1$, then $p \notin U$. Now $pKp^{-1} \cap K \cap H = px_0 L_0 x_0^{-1} p^{-1} \cap U \cap H$, which by the preceding result is the intersection of H with finitely many conjugates of the associated subgroups. This completes the proof of Lemma 3.

COROLLARY. *Under the same hypotheses as in Lemma* 3, *if* $r, s \in E$, $s^{-1}r \notin U$, *and* $1 \neq rUr^{-1} \cap H < sUs^{-1} \cap H$, *then* $rUr^{-1} \cap H$ *is the intersection of* H *with finitely many conjugates* (*by elements of* G) *of the associated subgroups* $L_i (i \neq 0)$.

Proof. By Lemma 3, $s^{-1}rUr^{-1}s \cap U$ (and therefore $rUr^{-1} \cap sUs^{-1}$) is the intersection of finitely many conjugates (by elements of E) of the associated subgroups $L_i(i \neq 0)$. Moreover, since $E = X * G$ and $H, L_i < G$, we have that $cL_i c^{-1} \cap H \neq 1$ implies that c is in G.

THEOREM 4. *Let G be the HNN group* (1). *Then any subgroup H of G which satisfies a non-trivial law is one of the following:*

(10) *a subgroup of a conjugate of K;*

(11) *a countable ascending union of subgroups of conjugates of the L_i;*

(12) *an HNN group with presentation*

$$\langle t, S'; \text{rel } S', tS't^{-1} = S'' \rangle,$$

where $S'' < S'$ and S' is the intersection of H with finitely many conjugates of the associated subgroups L_i;

(13) *an amalgamated product $(C * D;\ V)$ where C, D are each the intersection of H with a conjugate of K, V is of index two in C and D, and V is the intersection of H with a conjugate of some L_i or $V = 1$.*

Proof. Embed G in the amalgamated product E defined by (3). According to [**6**, Theorem 7], H is one of the following:

(10′) a subgroup of a conjugate of $X * K$ or $Y * K$ and therefore, by Lemma 1, a subgroup of gKg^{-1} for some $g \in G$;

(11′) a countable ascending union $D_iUD_i^{-1} \cap H$ where $HD_iU \neq HD_{i+1}U$, and hence $D_iUD_i^{-1} \cap H$ is contained in a conjugate of L_{j_i} (by the Corollary to Lemma 3); thus H is a countable ascending union of subgroups of conjugates of the L_i;

(12′) an HNN group with presentation

$$\langle t, S';\ \text{rel}\ S',\ tU_H{}^{\delta}t^{-1} = U_H{}^{\delta'}\rangle,$$

where $S' = gp(U_H{}^{\delta}, U_H{}^{\delta'})$, δ ends in an α-symbol which is not in U, δ' ends in a β-symbol not in U, and $U_H{}^{\delta} < U_H{}^{\delta'}$ or vice versa. Since $\delta U \neq \delta' U$, $U_H{}^{\delta'}$ (as well as $U_H{}^{\delta}$) is the intersection of H with finitely many conjugates of the associated subgroups L_i;

(13′) an amalgamated product $(C * D;\ V)$ where V is of index two in C and D. Moreover, the proof of [**6**, Theorem 7] shows that C and D are vertices in the tree product S given in Theorem 1 and that V is an amalgamated subgroup of S; hence, C and D are conjugates of K intersected with H and V is a conjugate of some L_i intersected with H or $V = 1$.

COROLLARY 1. *The conclusions of Theorem* 4 *hold if in the hypothesis we replace "satisfies a non-trivial law" by "contains no free subgroup of rank two".*

Proof. See the first remark in the proof of Theorem 3.

COROLLARY 2. *Let G be an HNN group given by* (1). *Suppose that each L_i has the property that the only subgroups satisfying a non-trivial law are cyclic. Let H be a subgroup of G satisfying a non-trivial law. If H is torsion-free then H is one of the following:*

(14) *a subgroup of a conjugate of K;*

(15) *a countable ascending union of cyclic groups;*

(16) *a group with presentation $\langle \tau, \alpha;\ \tau\alpha\tau^{-1} = \alpha^n\rangle$, where n is an integer.*

If H is allowed to have torsion, then one has the additional possibilities given by

(16′) *$H = \langle \tau, \alpha;\ \alpha^r, \tau\alpha\tau^{-1} = \alpha^n\rangle$ where r,n are integers;*

(17) *$H = (C * D;\ V)$ where V is cyclic and of index two in both C and D, and C, D are subgroups of conjugates of K.*

Proof. In the face of the additional hypothesis on L_i, (10), (11), (12), and (13) reduce to (14), (15), (16′), and (17) respectively.

Suppose further that H is torsion-free; then (16′) clearly reduces to (16). Moreover, in (17), $V = gp(v)$ is infinite cyclic. Since V is of index two in C,

$$C = \langle \gamma, v; \gamma^2 = v^s, \gamma v \gamma^{-1} = v^n \rangle.$$

Therefore $v^s = \gamma v^s \gamma^{-1} = v^{sn}$ and hence, since $s \neq 0$, $n = 1$ and so C is infinite cyclic; similarly for D. Thus,

$$H = \langle c, d; c^2 = d^2 \rangle = \langle \tau, \alpha; \tau\alpha\tau^{-1} = \alpha^{-1} \rangle$$

with $c = \tau$, $d = \alpha\tau$; this group is included under (16).

Remark. The following remark (suggested by a result of [**9**]) further restricts the groups that can occur in (15) and (16) above: *let G be an HNN group given by* (1). *Suppose that for some prime p, each $L_i (i > 0$ or $i < 0)$ is p-pure in K (i.e., $k^{p^r} \in L_i$ implies that $k^{p^r} = k_1{}^{p^r}$ with $k_1 \in L_i$) and that each element $k(\neq 1)$ of K is divisible by only finitely many powers of p (i.e., $x^{p^r} = k$ has a solution x in K for only finitely many r). Then each element $g(\neq 1)$ of G is divisible by only finitely many powers of p.*

For, embed G in the amalgamated product E described in (3). We first show that U is p-pure in A (and B). Since $A = X * K$ and $U < K^A$ where K^A is the normal closure of K in A, any root of an element of U is in K^A; it therefore suffices to show that U is p-pure in K^A. But K^A is the free product of the conjugates xKx^{-1} where x ranges over X and U is a free product of subgroups of such conjugates which are p-pure in their respective factors; it easily follows then that U is p-pure in K^A. Consequently, A, B, and hence E have the property that each of its elements $(\neq 1)$ is divisible by only finitely many powers of p (see [**9**, Lemma 1.13]).

For example, the group $G = \langle t, a; ta^r t^{-1} = a^s \rangle$ cannot contain a copy of the additive subgroup of rationals whose denominators are powers of a prime p where $p \nmid rs$.

THEOREM 5. *Let G be a group with one defining relator R, and let H be a subgroup satisfying a non-trivial law. If G is torsion-free, then H is metabelian and is either locally cyclic or has the presentation*

$$H = \langle \tau, \alpha; \tau\alpha\tau^{-1} = \alpha^n \rangle.$$

If G has elements of finite order, then H is either cyclic or infinite dihedral.

Proof. Unless R has only one syllable (in which case the result follows easily from the Kurosh subgroup theorem), G can be embedded in an *HNN* group with cyclic free part; its base K is a group on one defining relator whose length is shorter than that of R; K is torsion-free if and only if G is torsion-free; and each of the pair of associated subgroups is free. Hence by induction and the Corollary to Theorem 4, it follows that if G is torsion-free then H is as asserted.

If G has torsion, the result follows directly from Corollary 1 of Theorem 3. This completes the proof of Theorem 5.

Thus, in particular, *any subgroup H of a group G with one defining relation must intersect non-trivially every verbal subgroup* ($\neq 1$) *of G unless H is locally cyclic, infinite dihedral, or metabelian with a presentation* $\langle \tau, \alpha; \tau\alpha\tau^{-1} = \alpha^n \rangle$.

4. Subgroups which are free products.

THEOREM 6. *Let G be the HNN group* (1). *If H is a subgroup of G with trivial intersection with the conjugates of each L_i, then H is the free product of a free group and the intersections of H with certain conjugates of K.*

Proof. By Theorem 1 (since the intersections of H with the conjugates of each L_i is trivial), H is an *HNN* group whose base S is a free product of vertices ($\neq 1$) of the type $gKg^{-1} \cap H$. Moreover, two subgroups which are associated are either both trivial or both vertices. Let $\tau_1, \tau_2, \ldots$ be the free generators of the free part of H. Those τ_i whose associated subgroups are trivial generate a free group which is a free factor of H; we may factor this out of H.

We next construct a graph whose vertices are the vertices of S; the edge τ_j joins the vertex V_{j_1} to the vertex V_{j_2} if $\tau_j V_{j_1} \tau_j^{-1} = V_{j_2}$. Now a connected component of this graph must be a tree. For, a simple closed path in the graph corresponds to a freely reduced word $\tau(\neq 1)$ in the τ_i which conjugates a vertex $gKg^{-1} \cap H$ ($\neq 1$) back into itself. But by Lemma 3, since τ cannot be in a vertex, $\tau gKg^{-1}\tau^{-1} \cap gKg^{-1} \cap H$ is contained in the intersection of a conjugate of some L_i with H and hence is trivial.

If we choose one vertex from each component of the graph just constructed, H will be the free product of these vertices and the free group on $\tau_1, \tau_2, \ldots$. Indeed, if $V_1, V_2, \ldots$ are the vertices in a given component, then

$$V_j = \delta_j V_1 \delta_j^{-1}, \tag{18}$$

where δ_j is a freely reduced word in the τ_i. Hence, if in the relation $\tau_k V_{j_1} \tau_k^{-1} = V_{j_2}$ (involving vertices in the component of V_1) we make the substitution (18) for V_{j_1} and V_{j_2} we obtain,

$$(\delta_{j_2}^{-1}\tau_k\delta_{j_1}) V_1 (\delta_{j_2}^{-1}\tau_k\delta_{j_1})^{-1} = V_1;$$

hence $\delta_{j_2}^{-1}\tau_k\delta_{j_1}$ must be freely equal to 1 in the τ_i. This completes the proof of Theorem 6.

5. Finitely generated subgroups.

THEOREM 7. *Let G be the HNN group* (1), *and let H be a finitely generated subgroup whose free part (according to the description of Theorem* 1*) has rank n. Then H can be presented by*

$$H = \langle \tau_1, \ldots, \tau_n, S'; \operatorname{rel} S', \tau_1 L_1' \tau_1^{-1} = M_1', \ldots, \tau_n L_n' \tau_n^{-1} = M_n' \rangle, \tag{19}$$

where S' is a tree product whose vertices are conjugates of K intersected with H and each of whose amalgamated subgroups is trivial or the intersection of H with a *conjugate of an associated subgroup L_i; moreover, L_j' and M_j' are in the subgroup H_{j-1} of H with presentation*

(20) $H_{j-1} = \langle \tau_1, \ldots, \tau_{j-1}, S'; \tau_1 L_1' \tau_1^{-1} = M_1', \ldots, \tau_{j-1} L_{j-1}' \tau_{j-1}^{-1} = M_{j-1}' \rangle$

and are both either trivial, or the intersection of finitely many subgroups of the form

(21) $$g_i L_{j_i} g_i^{-1} \cap H,$$

or the subgroup generated by finitely many of the subgroups in (21)*; $\tau_1, \ldots, \tau_n$ freely generate the free part of H; finally, every subgroup $gKg^{-1} \cap H$ is conjugate within H to a vertex of S'.*

Proof. By Theorem 1, H is an HNN group with free part finitely generated by, say, $\tau_1', \ldots, \tau_n'$ (possibly empty) with base S a tree product of finitely many vertices ($\neq 1$), say, $V_1, \ldots, V_r$ (by [**6**, Lemma 3]), each V_i being of the form $gKg^{-1} \cap H$ and whose amalgamated subgroups ($\neq 1$) are of the form $gL_i g^{-1} \cap H$. Moreover, a pair of associated subgroups are either both conjugates of some L_i intersected with H or are both vertices of S. Let $\tau'_{p+1}, \ldots, \tau_n'$ be those τ_j' whose associated subgroups are the intersections of H with finitely many conjugates of the L_i. Then clearly H can be regarded as an HNN group with free part generated by $\tau_{p+1} = \tau'_{p+1}, \ldots, \tau_n = \tau_n'$ and base H_p which is the HNN subgroup of H generated by $\tau_1', \ldots, \tau_p'$ and S.

Now the associated subgroups of H_p are those vertices of S which are associated subgroups of H but are not intersections of H with finitely many conjugates of the L_i. It follows from Lemma 3 (since the vertices of S are of the form $gKg^{-1} \cap H$) that the normalizer of any associated subgroup of H_p has trivial intersection with the free part of H_p.

We next show the following: suppose that H_p is an HNN group with free part generated by $\tau_1', \ldots, \tau_p'$, whose base S is a tree product of finitely many vertices $V_1, \ldots, V_r (\neq 1)$; suppose that the associated subgroups of H_p are certain vertices of S; and suppose that the normalizer of each of these associated subgroups has trivial intersection with the free part of H_p. Then H_p has a presentation

(22) $H_p = \langle \tau_1, \ldots, \tau_p, S'; \mathrm{rel}\, S', \tau_1 L_1' \tau_1^{-1} = M_1', \ldots, \tau_p L_p' \tau_p^{-1} = M_p' \rangle,$

where S' is a tree product whose vertices are certain vertices of S; two vertices of S' which are neighbors in S have the same amalgamated subgroup joining them as in S and otherwise two neighboring vertices of S' are joined by the identity subgroup; moreover, both L_j' and M_j' are subgroups generated by finitely many conjugates of the amalgamated subgroups of S and are both in the subgroup H_{j-1} of H_p defined by (20); $\tau_1, \ldots, \tau_p$ freely generate the free part of H_p; and every vertex of S is conjugate in H to a vertex of S'.

We prove this result by induction on p, the rank of the free part of H_p. If $p = 0$, then the free part of H_p is trivial, and we may take $S' = S$. Otherwise, renumbering the vertices (if necessary) we may assume that V_1, V_2 are associated with $\tau_p = \tau_p'$ and that

$$V_1 = \tau_p V_2 \tau_p^{-1}.$$

Using this relation we may eliminate V_1 from the presentation for H_p as an HNN group with base S. This replacement will only affect those relators which involve generators from V_1. The relators in V_1 alone become conjugates by τ_p of relators in V_2 alone and hence may be deleted. The relators of the form $V_1 = \tau_{1j} V_j \tau_{1j}^{-1}$ (where τ_{1j} is either a τ_i' or its inverse) become

$$V_2 = \tau_p^{-1}\tau_{1j} V_j \tau_{1j}^{-1}\tau_p;$$

if for each $j \neq 2$ we replace $\tau_p^{-1}\tau_{1j}$ by τ_{2j} (this is just a Nielsen transformation on the free part of H_p), we obtain the relators

$$V_2 = \tau_{2j} V_j \tau_{2j}^{-1}, \quad j \neq 2.$$

The relators $U_{1i} = U_{i1}$ amalgamating a subgroup of V_1 with a subgroup of V_i become

$$\tau_p U_{1i}' \tau_p^{-1} = U_{i1},$$

where U_{1i}' is the subgroup $\tau_p^{-1} U_{1i} \tau_p$ of V_2. Thus, if L_p' is the subgroup of S generated by all the U_{1i}', and M_p' is the subgroup of S generated by all U_{i1}, then H_p is an HNN group

$$H_p = \langle \tau_p, H_{p-1}; \text{rel } H_{p-1}, \tau_p L_p' \tau_p^{-1} = M_p' \rangle$$

with free part generated by τ_p, with associated subgroups L_p', M_p'. Moreover, the base H_{p-1} of this HNN representation for H_p is an HNN group whose free part is a free factor of rank $p - 1$ of the free part of H_p; the base of H_{p-1} is a tree product whose vertices are $V_2, \ldots, V_r$ and whose amalgamated subgroups are trivial or amalgamated subgroups of S; and the associated subgroups of H_{p-1} are among $V_2, \ldots, V_r$. Thus, H_{p-1} satisfies the same conditions as H_p except that its free part has smaller rank. Hence by inductive hypothesis, H_{p-1} and therefore H_p has the form asserted in (22). Moreover, each vertex $V_1, \ldots, V_r$ of S is conjugate in H_p to a vertex in H_{p-1}, and hence to a vertex in S'.

Consequently, H has the presentation asserted in the theorem. Finally, $gKg^{-1} \cap H$ is trivial, or conjugate to a vertex of S (Corollary 3 to Theorem 1) and therefore to a vertex of S'.

COROLLARY. *Let H be a finitely generated subgroup of the HNN group G given by* (1) *and suppose that the intersection of H with finitely many conjugates of the associated subgroups L_i is finitely generated. Then $gKg^{-1} \cap H$ is finitely generated for each $g \in G$.*

Proof. By the above theorem $gKg^{-1} \cap H$ is conjugate to some vertex of S'. Moreover, each of the groups L_j' is finitely generated. Now H_j is an HNN

group with base H_{j-1} and associated subgroups L_j', M_j'; hence by [**6**, Lemma 3] it follows that $H_{n-1}, H_{n-2}, \ldots, H_1, H_0 = S'$ are finitely generated and the vertices of S' are finitely generated (by [**6**, Theorem 4]).

THEOREM 8. *Let G be the HNN group defined by* (1). *Suppose that K has the property that all its finitely generated subgroups are finitely related and each* L_i *has the property that all of its subgroups are finitely generated. Then G has the property that all of its finitely generated subgroups are finitely related.*

Proof. Let H be a finitely generated subgroup of G. Then in the *HNN* representation of H given by Theorem 1, the free part of H is finitely generated and the tree product S has finitely many vertices, each of which is finitely generated (by the preceding corollary); hence these vertices are finitely related. Moreover, the amalgamated subgroups of S and the associated subgroups of H are finitely generated. Consequently, H is finitely related.

COROLLARY. *Let G be the HNN group defined by* (1). *Suppose that each finitely generated subgroup of the base K is finitely related and that each associated subgroup* L_i *is finitely generated. Let H be a finitely generated subgroup of G. If G has the finitely generated intersection property (or, more generally, if the intersection of H with finitely many conjugates of the* L_i *is finitely generated), then H is finitely related.*

THEOREM 9. *Let G be the HNN group defined by* (1) *and let H be a finitely generated subgroup containing a normal subgroup N of G where N is not contained in the intersection of all the associated subgroups* L_i, M_i. *Then the double coset index of* (H, K) *in G is finite; in particular, H is of finite index in G if and only if the intersection of K with each conjugate of H is of finite index in K.*

Proof. Now $N \not< K$; for otherwise, $N < K \cap t_i^{-\epsilon} K t_i^{\epsilon} = L_{\epsilon i}$ for each i. If the free part of G has rank two or more, G may be written as a proper amalgamated product of two factors with K as an amalgamated subgroup (see the remark preceding Lemma 2 in [**6**]). Hence it follows from [**6**, Theorem 10] that (H, K) is of finite index in G.

We may therefore assume that the free part of G is infinite cyclic generated by t. Embed G in the amalgamated product E described in (3); we show that every coset of (H, U) in E is double ended (and therefore as in the proof of [**6**, Theorem 10] that (H, U) is of finite index in E) or (H, K) has finite index in G.

To show that every coset of (H, U) is double ended it suffices (as in the proof of [**6**, Theorem 10]) to show that N contains an element of E which begins and ends with an α-syllable (not in U) and one which begins and ends with a β-syllable (not in U).

Now an element g of G which is not in K has the form

$$g = k_1 t^{\sigma_1} k_2 t^{\sigma_2} \ldots k_r t^{\sigma_r} k_{r+1}, \tag{23}$$

where $k_i \in K, \sigma_i \neq 0$, and if $\operatorname{sgn} \sigma_{i-1} = \epsilon$ and $\operatorname{sgn} \sigma_i = -\epsilon$ then $k_i \notin L_\epsilon$.

https://doi.org/10.4153/CJM-1971-070-x Published online by Cambridge University Press

Moreover, if g is written in a reduced form as an element of the amalgamated product E, then the initial syllable of G will be k_1y^{-1} if $\sigma_1 > 0$ and will be k_1x^{-1} if $\sigma_1 < 0$; in a similar manner we can describe the terminal syllable of g in E.

Let g as in (23) be an element of $N - K$ such that the sum of the absolute values of the σ_i is minimum. Since N is normal in G, we may assume that $k_{r+1} = 1$ and that if $\operatorname{sgn} \sigma_r = \epsilon$ and $\operatorname{sgn} \sigma_1 = -\epsilon$ then $k_1 \notin L_\epsilon$.

Suppose that in g there exist i, j such that $\sigma_i > 0$ and $\sigma_j < 0$. Then there exist p, q such that $\sigma_p > 0$, $\sigma_{p+1} < 0$, $\sigma_q < 0$ and $\sigma_{q+1} > 0$ (where the subscript $r + 1$ is replaced by 1); hence

$$k_{p+1}t^{\sigma_{p+1}} \ldots k_pt^{\sigma_p}$$

and

$$k_{q+1}t^{\sigma_{q+1}} \ldots k_qt^{\sigma_q}$$

are in N and begin and end in an α-syllable and in a β-syllable, respectively.

We may therefore assume that all σ_i have the same sign and in fact all are positive. If $K \neq L$, then choosing k so that $k_rk^{-1} \notin L$, we have that N contains

$$tkt^{\sigma_r}k_1t^{\sigma_1} \ldots k_rk^{-1}t^{-1},$$

which begins and ends in a β-syllable. Similarly, if $K \neq M$, then N contains

$$t^{-1}kk_1t^{\sigma_1} \ldots k_rt^{\sigma_r}k^{-1}t,$$

which begins and ends in an α-syllable.

Suppose then that K equals one of the associated subgroups, say, $K = L$. Then letting $\sigma = \sigma_1 + \ldots + \sigma_r$ we have that $t^\sigma gt^{-\sigma}$ is in N and equals $t^\sigma k'$, $k' \in K$. Hence N contains $t^{i\sigma}k_i'$, where i is any integer and $k_i' \in K$. Thus NK contains $t^{i\sigma}Kt^{-i\sigma}$ and the union of these over all negative i yields K^G, the normal subgroup generated by K in G. Hence if $g' \in G$, then $Hg'K = Hg'NK = Ht^\lambda NK = Ht^\lambda K$, where $0 \leqq \lambda < \sigma$; thus, (H, K) is of finite index in G.

We may therefore assume that (H, U) has finite index in E. Let $Hg_1U, \ldots, Hg_sU$ be the (H, U) cosets containing elements of G, and let $g \in G$. Then $g = hg_iu$, $h \in H$, $u \in U$. Hence $u \in G \cap U = K$. Consequently, $g \in Hg_iK$ and so (H, K) is of finite index in G.

THEOREM 10. *Let G be an HNN group as in* (1) *and suppose that its base K has the finitely generated intersection property. If the subgroup K' of K generated by all the associated subgroups L_i, M_i is finite, then G has the finitely generated intersection property. In particular, if the base K of an HNN group is finite, then the HNN group satisfies the finitely generated intersection property.*

Proof. Let H_1, H_2 be finitely generated subgroups of G, and suppose that K is finite. Embed G in the amalgamated product E described in (3). Since H_1, H_2 are finitely generated, [**6**, Lemma 8, Corollary] implies that the number of double ended (H_1, U) and (H_2, U) cosets is finite. Moreover, it follows as in the proof of [**6**, Theorem 11] that if the number of double ended $(H_1 \cap H_2, U)$ cosets is finite, then $H_1 \cap H_2$ is finitely generated.

We show that there are only finitely many double ended $(H_1 \cap H_2, U)$ cosets by showing that the intersection of an (H_1, U) coset and an (H_2, U) coset contains only finitely many $(H_1 \cap H_2, U)$ cosets. Suppose then that $(H_1 \cap H_2)qU < H_1pU \cap H_2pU$ where p is a fixed element. We show that $(H_1 \cap H_2)qU = (H_1 \cap H_2)gpU$ where g is one of a finite set of elements (which depends upon p). Indeed, $q = h_1pu_1 = h_2pu_2, h_i \in H_i, u_1, u_2 \in U$. If $u_1 = u_2$, then $h_1 = h_2$ and we may choose $g = 1$. Otherwise, $1 \neq h_2^{-1}h_1 = pu_2u_1^{-1}p^{-1} \in G \cap pUp^{-1} = g_1L_{j_1}g_1^{-1}$ (by Lemma 1 with $H = G$), $0 \leqq j_1 \leqq n$, $L_0 = K$. Since K is finite, $g' = pu_2u_1^{-1}p^{-1}$ ranges over a finite set. Moreover, $pu_2 = g'pu_1$ so $q(pu_1)^{-1} \in H_1 \cap (H_2g') = (H_1 \cap H_2)g$; since g' ranges over a finite set, we may restrict g to range over a finite set. Thus $(H_1 \cap H_2)q = (H_1 \cap H_2)gpu_1$ and $(H_1 \cap H_2)qU = (H_1 \cap H_2)gpU$, where g ranges over a finite set. This completes the argument for the case when K is finite.

If now we merely assume that K has the finitely generated intersection property and that K' is finite, then G is an amalgamated product of K and an *HNN* group with base K' and the same associated subgroups as G, with the subgroup K' amalgamated. Since each of the factors has the finitely generated intersection property and K' is finite, [**6**, Theorem 11] implies that G has the finitely generated intersection property.

A similar argument shows that it suffices to assume that each pair L_i, M_i generates a finite group.

References

1. G. Baumslag, *On generalized free products of torsion-free nilpotent groups.* I (to appear in Illinois J. Math.).

2. G. Baumslag, A. Karrass and D. Solitar, *Torsion-free groups and amalgamated products*, Proc. Amer. Math. Soc. *24* (1970), 688–690.

3. J. L. Britton, *The word problem*, Ann. of Math. *34* (1963), 16–32.

4. G. Higman, *Subgroups of finitely presented groups*, Proc. Roy. Soc. London Ser. A *262* (1961), 455–475.

5. G. Higman, B. H. Neumann and H. Neumann, *Embedding theorems for groups*, J. London Math. Soc. *24* (1949), 247–254.

6. A. Karrass and D. Solitar, *The subgroups of a free product of two groups with an amalgamated subgroup*, Trans. Amer. Math. Soc. *149* (1970), 227–255.

7. W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory*, Pure and Appl. Math., Vol. 13 (Interscience, New York 1966).

8. D. I. Moldavanski, *Certain subgroups of groups with one defining relation*, Siberian Math. J. *8* (1967), 1370–1384.

9. B. B. Newman, *Some aspects of one relator groups* (to appear).

10. B. H. Neumann, *An essay on free products of groups with amalgamations*, Philos. Trans. Roy. Soc. London Ser. A *246* (1954), 503–554.

York University,
Toronto, Ontario