4
Scalar fields

The simplest quantum field theory is that of a free scalar particle. On a lattice this becomes the Gaussian model of statistical mechanics. Here we will solve this system exactly to introduce lattice field theory. As with the conventional continuum theory, Fourier transform techniques are the key to this solution. We conclude this chapter with some general remarks on interacting scalar fields.

We begin with the standard Lagrangian density for a self-conjugate free field

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 + \tfrac{1}{2}m^2\phi^2. \tag{4.1}$$

Here $\phi(x)$ is a real function of the four space-time coordinates x_μ. The discussion here is easily generalized to an arbitrary number of dimensions and complex fields. The Greek indices denoting vector quantities run from one to four. A repeated index, as implied in eq. (4.1), is understood to be summed; however, as we work in Euclidian space, no metric tensor is implied. To every field configuration corresponds an action

$$S = \int \mathrm{d}^4x\, \mathscr{L}. \tag{4.2}$$

The Feynman path integral is a sum over all configurations

$$Z = \int [\mathrm{d}\phi]\, \mathrm{e}^{-S}, \tag{4.3}$$

where, as in the previous chapter, the integration measure needs definition.

We proceed directly to a four-dimensional hypercubic lattice. Thus we restrict our coordinates to the form

$$x_\mu = an_\mu, \tag{4.4}$$

where a is the lattice spacing and n_μ has four integer components. As an infrared cutoff, we allow the individual components of n to assume only a finite number N of independent values

$$-N/2 < n_\mu \leqslant N/2. \tag{4.5}$$

Outside this range we assume the lattice is periodic; we identify n with $n+N$. Thus our lattice has N^4 sites. We now replace the derivatives of ϕ with nearest neighbor differences

$$\partial_\mu \phi(x_\nu) \to (\phi_{n_\nu+\delta_{\nu\mu}} - \phi_{n_\nu})/a, \tag{4.6}$$

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where the Kronecker function is defined

$$\delta_{\mu\nu} = \begin{cases} 1, & \mu = \nu \\ 0, & \mu \neq \nu. \end{cases} \tag{4.7}$$

The action is a sum

$$S = a^4 [\sum_{\{m,n\}} (\phi_m - \phi_n)^2/(2a^2) + \sum_n m^2\phi_n^2/2], \tag{4.8}$$

where $\{m, n\}$ represents the set of all nearest-neighbor pairs of lattice sites. The path integration measure is now simply defined as an ordinary integral over each of the lattice fields

$$Z = \int (\prod_n \mathrm{d}\phi_n) \,\mathrm{e}^{-S}. \tag{4.9}$$

At this point we observe that the action is a quadratic form in the field variables

$$S = \tfrac{1}{2}\phi_m M_{mn} \phi_n, \tag{4.10}$$

where M is an N^4-dimensional square matrix and we adopt the usual summation convention on repeated indices. The integral in eq. (4.9) is of the standard Gaussian form and has the value

$$Z = |M/2\pi|^{-\frac{1}{2}}, \tag{4.11}$$

where the vertical bars denote the determinant of the enclosed matrix. We will now introduce a Fourier transform on the lattice. This will diagonalize M and make the determinant trivial.

Let f_n be an arbitrary complex function on the lattice sites. Its Fourier transform is defined

$$\tilde{f}_k = F_{kn} f_n = \sum_n f_n \,\mathrm{e}^{2\pi i k_\mu n_\mu/N}. \tag{4.12}$$

The index k also carries four integer valued components, each in the range of eq. (4.5). This linear transform is easily inverted with the identity

$$\sum_k \mathrm{e}^{-2\pi i k\cdot n/N} = N^4 \prod_\mu \delta_{n_\mu, 0} \equiv N^4 \delta^4_{n,0}. \tag{4.13}$$

Thus we have

$$(F^{-1})_{nk} = N^{-4}\,\mathrm{e}^{-2\pi i k\cdot n/N} = N^{-4} F^*_{kn} \tag{4.14}$$

or

$$f_n = N^{-4} \sum_k \tilde{f}_k \,\mathrm{e}^{-2\pi i k\cdot n/N}. \tag{4.15}$$

The utility of the Fourier series appears when we consider sums of local quadratic forms, such as appear in our lattice action. In particular, the useful identities

$$\sum_n f^*_n g_n = N^{-4} \sum_k \tilde{f}^*_k \tilde{g}_k \tag{4.16}$$

and

$$\sum_n f^*_{n_\mu + \delta_{\mu\nu}} g_n = N^{-4} \sum_k \tilde{f}^*_k \tilde{g}_k \,\mathrm{e}^{2\pi i k_\nu/N} \tag{4.17}$$

reduce the action to

$$S = a^4 N^{-4} \sum_k \tfrac{1}{2}\tilde{M}_k |\tilde{\phi}_k|^2, \tag{4.18}$$

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where $$\tilde{M}_k = m^2 + 2a^{-2}\sum_\mu (1-\cos(2\pi k_\mu/N)). \tag{4.19}$$

The Fourier transform has diagonalized M

$$M_{mn} = a^4 N^{-4} \sum_k F^*_{mk} F_{nk} \tilde{M}_k. \tag{4.20}$$

To evaluate the determinant of this matrix, first note that eq. (4.14) implies

$$|N^{-4}F^*| = |F|^{-1}. \tag{4.21}$$

Thus we have the exact expression for our path integral

$$Z = |M/2\pi|^{-\frac{1}{2}} = \prod_k (a^4\tilde{M}_k/2\pi)^{-\frac{1}{2}}. \tag{4.22}$$

This equation is not very useful as it stands. To obtain Green's functions, we consider external sources J_n on the lattice sites and coupled to the field ϕ. Consequently we generalize our action to

$$S(J) = \tfrac{1}{2}\phi_m M_{mn}\phi_n - J_n\phi_n. \tag{4.23}$$

The partition function now depends on the sources

$$Z(J) = \int [\mathrm{d}\phi]\,\mathrm{e}^{-S(J)}. \tag{4.24}$$

This quantity is a generating function for the Green's functions, which follow from differentiation with respect to the sources

$$\begin{aligned}\langle\phi_{n_1}\cdots\phi_{n_j}\rangle &= Z^{-1}\int[\mathrm{d}\phi]\,\mathrm{e}^{-S}\phi_{n_1}\cdots\phi_{n_j}\Big|_{J=0}\\ &= Z^{-1}\left\{\frac{\mathrm{d}}{\mathrm{d}J_{n_1}}\cdots\frac{\mathrm{d}}{\mathrm{d}J_{n_j}}Z(J)\right\}\Big|_{J=0}\end{aligned} \tag{4.25}$$

Completing the square in eq. (4.23) and shifting the integration in eq. (4.24) gives the exact expression for this free-field generating function

$$Z(J) = Z(0)\exp(\tfrac{1}{2}J_m(M^{-1})_{mn}J_n), \tag{4.26}$$

where $Z(0)$ is given in eq. (4.22). From this we see that the propagator or two-point function is simply the inverse of the matrix M

$$\langle\phi_m\phi_n\rangle = (M^{-1})_{mn}. \tag{4.27}$$

Momentum space makes this inversion trivial

$$\langle\phi_m\phi_n\rangle = a^{-4}N^{-4}\sum_k \tilde{M}_k^{-1}\,\mathrm{e}^{2\pi i k\cdot(m-n)/N}. \tag{4.28}$$

To put this expression into a more familiar form, we first take N to infinity and change the momentum sum into an integral with the replacements

$$q_\mu = 2\pi k_\mu/(Na), \tag{4.29}$$

$$a^{-4}N^{-4}\sum_k \to \int\frac{\mathrm{d}^4q}{(2\pi)^4}. \tag{4.30}$$

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Here each component of q runs over the finite range

$$-\pi/a < q_\mu \leqslant \pi/a. \tag{4.31}$$

This explicitly shows the momentum space effect of the lattice cutoff. The propagator now assumes the form

$$\langle \phi_m \phi_n \rangle = \int_{-\pi/a}^{\pi/a} \frac{d^4 q}{(2\pi)^4} \frac{e^{-iq\cdot x}}{m^2 + 2a^{-2} \sum_\mu [1 - \cos(aq_\mu)]}, \tag{4.32}$$

where

$$x_\mu = -a(m_\mu - n_\mu). \tag{4.33}$$

For the continuum limit $a \to 0$ we expand the cosine

$$2a^{-2} \sum_\mu (1 - \cos(aq_\mu)) = q^2 + O(a^2) \tag{4.34}$$

and obtain

$$\langle \phi_m \phi_n \rangle = \int \frac{d^4 q}{(2\pi)^4} \frac{e^{-iq\cdot x}}{m^2 + q^2} + O(a^2). \tag{4.35}$$

This is the familiar Feynman propagator function in Euclidian space.

Up to this point we have been considering a free field. Now we add an interaction term to our action

$$S = \tfrac{1}{2}\phi_m M_{mn} \phi_n - J_n \phi_n + \sum_n V_I(\phi_n). \tag{4.36}$$

The full potential felt by the field ϕ includes the mass term from eq. (4.8)

$$V(\phi) = \tfrac{1}{2}m^2\phi^2 + V_I(\phi). \tag{4.37}$$

The minima of this function form the basis for semiclassical treatments, with which we will not concern ourselves here. As a concrete example, the usual ϕ^4 theory takes

$$V_I(\phi) = g_0 \phi^4. \tag{4.38}$$

Here g_0 is the bare coupling with the lattice cutoff in place. The full generating function of the interacting theory is still

$$Z(J) = \int [d\phi] \, e^{-S(J)}. \tag{4.39}$$

Note that the potential $V(\phi)$ must be bounded below if this integral is to make any sense. In particular, the ϕ^4 theory with negative coupling is sick, and therefore we do not expect analyticity at vanishing g_0. Perturbation in g_0 yields at best an asymptotic series (Dyson, 1952).

The usual perturbation expansion follows from a formal exploitation of eq. (4.25) to give

$$Z(J) = \exp\left(\sum_n V_I(d/dJ_n)\right) Z_0(J), \tag{4.40}$$

where $Z_0(J)$ is the free-field generating function from eq. (4.26). An expansion of the exponent in this equation gives the Feynman series in terms of vertices from the interaction term and propagators from $Z_0(J)$.

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The Green's functions, which follow by differentiating Z with respect to the sources, are the full n-point functions and include, in general, disconnected pieces. In particular, if ϕ has a vacuum expectation value, one might prefer to subtract this and study the connected propagator

$$\langle \phi_m \phi_n \rangle_c = \langle \phi_m \phi_n \rangle - \langle \phi_m \rangle \langle \phi_n \rangle. \tag{4.41}$$

A general connected Green's function is defined through the corresponding generating function, which is simply the logarithm of Z

$$F(J) = \ln(Z(J)), \tag{4.42}$$

$$\langle \phi_{n_1} \cdots \phi_{n_j} \rangle_c = \left(\frac{\mathrm{d}}{\mathrm{d}J_{n_1}} \cdots \frac{\mathrm{d}}{\mathrm{d}J_{n_j}} F(J) \right) \bigg|_{J=0}. \tag{4.43}$$

Note that in the statistical mechanical analog $F(J)$ is proportional to the free energy.

We conclude this chapter with some brief remarks on the strong coupling expansion for this scalar theory. Considering the ϕ^4 theory of eq. (4.38), we change integration variables in the path integral from ϕ to $g^{\frac{1}{4}}\phi$, and we perform a similar formal manipulation to that giving eq. (4.40). Thus we find

$$\begin{aligned} Z(g^{\frac{1}{4}}J) &= g_0^{-N^4/4} \int [\mathrm{d}\phi]\, \mathrm{e}^{-\frac{1}{2}g_0^{-\frac{1}{2}}\phi M \phi}\, \mathrm{e}^{-\Sigma_n(\phi_n^4 - J_n \phi_n)} \\ &= g_0^{-N^4/4} \exp\left(-\tfrac{1}{2} g_0^{-\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}J} M \frac{\mathrm{d}}{\mathrm{d}J} \right) \prod_n f(J_n), \end{aligned} \tag{4.44}$$

where $f(J)$ is an ordinary one-dimensional integral

$$f(J) = \int_{-\infty}^{\infty} \mathrm{d}\phi\, \mathrm{e}^{-(\phi^4 - J\phi)}. \tag{4.45}$$

An expansion of the exponential on the right hand side of eq. (4.44) forms the basis for a strong coupling expansion in powers of $g_0^{-\frac{1}{2}}$. Unfortunately, in the continuum limit the matrix M grows, and therefore for fixed coupling we are no longer expanding in a small quantity. As we are more interested in gauge theories, we will not discuss here the techniques invented in attempts to overcome this problem. We only wish to emphasize that a strong coupling series is quite natural when the lattice is in place (Baker and Kincaid, 1979; Bender *et al.*, 1981).

Problems

1. Verify equation (4.19).
2. Show that a rescaling of the field normalization puts the action in the form

$$S = \sum_m \phi_m^2 + K \sum_{\{mn\}} \phi_m \phi_n.$$

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Show that in the continuum limit the 'hopping constant' K goes to unity at a rate dependent on the mass.

3. One might consider as a non-perturbative cutoff disregarding a field's Fourier components which carry momentum larger than some cutoff parameter. How does this compare to the lattice cutoff in real space?

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