Glasgow Math. J. **46** (2004) 311–321. © 2004 Glasgow Mathematical Journal Trust.
DOI: 10.1017/S001708950400179X. Printed in the United Kingdom

INVARIANT CMC SURFACES IN $\mathbb{H}^2 \times \mathbb{R}$

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(Received 25 March, 2003; accepted 12 August, 2003)

Abstract. We explicitly classify helicoidal and translational constant mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

2000 *Mathematics Subject Classification.* 53C42, 53A10.

1. Introduction. Surfaces of constant mean curvature (CMC) play a special role in differential geometry. They arise in a variety of different branches. For example, the boundary of a compact domain Ω, which is a solution of the isoperimetric problem, is a CMC surface. In [**2**] W. T. Hsiang and W. Y. Hsiang studied solutions of the isoperimetric problem in the product of the hyperbolic space with the Euclidean space. In particular, they shown that a solution to the isoperimetric problem in $\mathbb{H}^2 \times \mathbb{R}$ is invariant under the action of an isometry subgroup of the type of $O(2) \times O(1)$ which fixes its centre of gravity. Therefore the boundary yields to a $O(2)$-invariant CMC surface in $\mathbb{H}^2 \times \mathbb{R}$. Due to this property, in [**2**], there is a description of the $O(2)$-invariant CMC surfaces in $\mathbb{H}^2 \times \mathbb{R}$. See also [**5**, Lemma 1.3].

In this note we extend the result of W. T. Hsiang and W. Y. Hsiang to include helicoidal and translational CMC surfaces in $\mathbb{H}^2 \times \mathbb{R}$; that is CMC surfaces which are invariant under the action of a 1-parameter subgroup G of the isometry group $Isom(\mathbb{H}^2 \times \mathbb{R})$ generated by:

- translations along $\mathbb{R}$ (translational surfaces);
- composition of translations along $\mathbb{R}$ and rotations (helicoidal surfaces).

The main ingredient is the Reduction Theorem of M. Do Carmo and W. Hsiang [**1**] which reduces the computation of the mean curvature of a G-invariant surface $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ to that of $\Sigma/G \subset (\mathbb{H}^2 \times \mathbb{R})/G$. Using the Reduction Theorem we find a function J which is constant along the profile curve of a given G-invariant CMC surface. We then give a qualitative description of the G-invariant CMC surfaces by an accurate analysis of the equation $J =$ constant.

2. Preliminaries. Let $G \subset Isom(M)$ be a closed subgroup of the isometry group of a Riemaniann manifold (M, g). The group G is a Lie group which acts on M by

The second author was supported by INdAM.

isometries. A subset $S \subset M$ is called *G-invariant*, and G the *symmetry group* of S, if for any $x \in S$ and $g \in G$, with gx well defined, we have $gx \in S$.

A map $f : M \to N$ between two Riemannian manifolds is a *G-invariant* function if for any $x \in M$ and $g \in G$

$$f(gx) = f(x).$$

A G-invariant function $\zeta : M \to \mathbb{R}$ is simply called an *invariant* of G.

Let now M and N be two Riemannian manifolds with $Isom(M) \subseteq Isom(N)$ and let G be a closed subgroup of $Isom(M)$. Suppose that $f : M \to N$ is a G-equinvariant isometric immersion and suppose that the principal orbit type is the same for both actions. Then f induces an immersion $\tilde{f} : M_r/G \to N_r/G$ between the regular points of the quotient spaces. The space M_r (respect. N_r) can be equipped with a Riemannian metric so that the quotient map $M_r \to M_r/G$ (respect. $N_r \to N_r/G$) is a Riemannian submersion.

From now on, since all will be local, we identify M with its image $f(M) \subset N$. For a given point $x \in M_r \subset N_r$ let $H = G_x$ be the isotropy subgroup of x.

With respect to an Ad_H-invariant metric on the Lie algebra $\mathfrak{g}$ of G we have the following orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{h}^{\perp}$, where $\mathfrak{h}$ is the Lie algebra of H.

Therefore we can define a G-invariant metric on G/H and the space $\mathfrak{h}^{\perp} = T_e(G/H)$ generates $c = \dim(G) - \dim(H)$ linearly independent Killing vector fields $V_1, \ldots, V_c$ which are tangent to the orbit space of $y \in U$, where $U \subset N$ is a neighborhood of x. Let $A(y)$ be a matrix with entries $a_{ij} = g(V_i, V_j)$, and let $\omega(y) = \sqrt{\det(A(y))}$ be the volume function on the orbit $G(y) = \{gy : g \in G\}$. The mean curvature vector of f can be expressed in terms of the mean curvature vector of $\tilde{f}$ and of the function $\omega(y)$ as is shown in the following theorem.

THEOREM 2.1 (Reduction Theorem [**1**]). *Let H and $\tilde{H}$ be the mean curvature vectors of $M_r \subset N_r$ and $M_r/G \subset N_r/G$ respectively. Then*

$$H = \tilde{H} - \operatorname{grad}(\ln \omega).$$

If the group G is compact, the orbits are compact and the Reduction Theorem reads as follows.

COROLLARY 2.2 ([**2**], [**4**]). *Let $V(y)$ be the volume of $G(y)$, $\mathbf{n}$ a horizontal unit normal vector field along M_r and $\tilde{\mathbf{n}}$ the corresponding normal vector field to M_r/G in N_r/G. Then*

$$H(\mathbf{n}) = H(\tilde{\mathbf{n}}) - D_{\tilde{\mathbf{n}}}(\ln V).$$

Let now describe the quotient metric of the regular part of the orbit space N/G.

It is well known (see, for example [7]) that N_r/G can be locally parametrized by the invariant functions of the Killing vector fields of the Lie algebra $\mathfrak{g}$. Let $\{f_1, \ldots, f_d\}$, $d = \dim\big(N_r/G\big)$, be a complete set of invariant functions on a G-invariant subset of N_r. Denote by $\tilde{g}$ the quotient metric in N_r/G and define $h_{ij} = \langle \nabla f_i, \nabla f_j \rangle$, where ∇ is the gradient in (N, g).

THEOREM 2.3 (Quotient Metric Theorem [**2**]). *The quotient metric is given by $\tilde{g}_{ij} = h^{ij}$, or, equivalently, by $d\tilde{s}^2 = \sum_{i,j=1}^{d} h^{ij} df_i \otimes df_j$.*

2.1. The isometry group of $\mathbb{H}^2 \times \mathbb{R}$. Let $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\|^2 < 2\}$ be the disk model of the hyperbolic plane and consider $\mathbb{H}^2 \times \mathbb{R}$ endowed with the metric

$$ds^2 = \frac{dx^2 + dy^2}{F^2} + dz^2$$

where $F = \frac{2-x^2-y^2}{2}$.

PROPOSITION 2.4. *The Lie algebra of the infinitesimal isometries of the product* $(\mathbb{H}^2 \times \mathbb{R}, ds^2)$ *admits the following bases of Killing vector fields*

$$\begin{aligned} X_1 &= (F + y^2)\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y} \\ X_2 &= -xy\frac{\partial}{\partial x} + (F + x^2)\frac{\partial}{\partial y} \\ X_3 &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \\ X_4 &= \frac{\partial}{\partial z}. \end{aligned}$$

Proof. See, for example, **[6]**. □

DEFINITION 2.5. A one-dimensional subgroup G of $Isom(\mathbb{H}^2 \times \mathbb{R})$ is called *helicoidal* if it is generated by linear combinations

$$bX_3 + aX_4 \quad a, b \in \mathbb{R}.$$

In particular, if $b = 0$ the group is *translational*, while if $a = 0$ the group is *rotational*. The surfaces invariant under the action of helicoidal subgroups are called *helicoidal surfaces*.

Let G be a one-dimensional subgroup of $Isom(\mathbb{H}^2 \times \mathbb{R})$ of translational or helicoidal type. Since the action of G on $\mathbb{H}^2 \times \mathbb{R}$ is free then $(\mathbb{H}^2 \times \mathbb{R})_r = \mathbb{H}^2 \times \mathbb{R}$. Let $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ be a G-invariant surface. Then the orbit space $\mathcal{B} = (\mathbb{H}^2 \times \mathbb{R})/G$, can be parametrized by the G-invariant functions u, v and endowed with the quotient metric $d\tilde{s}^2$. The projection of Σ in $\mathcal{B}$ is a curve γ, generally called the *profile curve* of Σ. Parametrising $\gamma(s) = (u(s), v(s))$ by arc length s we can define $\sigma(s)$ as the angle between the tangent vector of γ and the positive direction of the u-axes.

3. CMC surfaces invariant under translations along the z-axes. Let G be the 1-parameter group of isometries generated by translations along the z-axes, that is by the Killing vector field $X_4 = \frac{\partial}{\partial z}$. In this case we can choose the following G-invariant functions:

$$u = x \qquad \text{and} \qquad v = y.$$

In cylindrical coordinates (r, θ, z) the metric of the ambient space takes the form

$$ds^2 = \frac{dr^2 + r^2\, d\theta^2}{F^2} + dz^2,$$

where $F = \frac{2-r^2}{2}$. The matrix h and its inverse take the form

$$(h_{ij}) = \begin{pmatrix} F^2 & 0 \\ 0 & F^2 \end{pmatrix} \quad (h^{ij}) = \begin{pmatrix} \frac{1}{F^2} & 0 \\ 0 & \frac{1}{F^2} \end{pmatrix}.$$

Thus the invariant metric in the orbit space $\mathcal{B} = (\mathbb{H}^2 \times \mathbb{R})/G = \mathbb{H}^2$ is

$$d\widetilde{s}^2 = \frac{du^2 + dv^2}{F^2}.$$

Let $\gamma(s) = (u(s), v(s))$ be a curve parametrized by arc length in $\mathcal{B}$ and let $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ be the surface generated by the action of G on the curve γ. Then the unit tangent vector field to $\gamma(s)$ is

$$\mathbf{t} = (\dot{u}, \dot{v}) = (F\cos\sigma, F\sin\sigma),$$

and the unit normal vector field is

$$\mathbf{n} = (-F\sin\sigma, F\cos\sigma).$$

The geodesic curvature of γ can be expressed as a function of σ by

$$\begin{aligned} k_g &= \frac{1}{2\sqrt{\tilde{g}_{11}\tilde{g}_{22}}}((\tilde{g}_{22})_u\dot{v} - (\tilde{g}_{11})_v\dot{u}) + \dot{\sigma} \\ &= \frac{2(u\dot{v} - v\dot{u})}{(2 - u^2 - v^2)} + \dot{\sigma} \\ &= u\sin\sigma - v\cos\sigma + \dot{\sigma}. \end{aligned}$$

Now, since the volume function of the principal orbit is

$$\omega(\xi) = \sqrt{\langle X_4, X_4\rangle} = \sqrt{\langle E_3, E_3\rangle} = 1,$$

from Theorem 2.1 we have $H = k_g$. Thus γ generates a translational surface if u and v satisfy the following system

$$\begin{cases} \dot{u} = F\cos\sigma \\ \dot{v} = F\sin\sigma \\ \dot{\sigma} = H - u\sin\sigma + v\cos\sigma. \end{cases} \tag{3.1}$$

Proposition 3.1. *If H is constant on the surface Σ, then the function*

$$J(s) = \frac{\dot{\sigma}}{2F}$$

is constant along any curve $\gamma(s)$ which is a solution of system (3.1). *Thus the solutions of* (3.1) *are given by $J(s) = k$, for some $k \in \mathbb{R}$.*

Proof.

$$\dot{J}(s) = \frac{\ddot{\sigma}2F + 2\dot{\sigma}(u\dot{u} + v\dot{v})}{4F^2}$$

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$$\text{(from (3.1))} = \frac{2F(-\dot{u}\sin\sigma + \dot{v}\cos\sigma - u\dot{\sigma}\cos\sigma - v\dot{\sigma}\sin\sigma)}{4F^2} + \frac{2F\dot{\sigma}(u\cos\sigma + v\sin\sigma)}{4F^2}$$
$$\text{(from (3.1))} = \frac{(-\dot{u}\sin\sigma + \dot{v}\cos\sigma)}{2F} = 0.$$

□

THEOREM 3.2. *The CMC surfaces in $\mathbb{H}^2 \times \mathbb{R}$ invariant under the action of the subgroup G generated by the Killing vector field $X_4 = \frac{\partial}{\partial z}$ are:*
(1) *part of minimal vertical planes through the origin (if $k = 0$);*
(2) *part of right cylinders of radius $1/2k$ otherwise.*

Proof. If $k = 0$, from Proposition 3.1, σ is constant and from (3.1) we get that $v = (\tan\sigma)u$ and $H = 0$. The corresponding surface Σ is a part of a minimal vertical plane.

If $k \neq 0$, from Proposition 3.1, $F = \dot{\sigma}/2k$ and from (3.1) we find, after integration, that

$$u = \frac{1}{2k}\sin\sigma + c_1, \qquad \text{and} \qquad v = -\frac{1}{2k}\cos\sigma + c_2, \quad c_1, c_2 \in \mathbb{R}.$$

The corresponding surface Σ is clearly part of a right cylinder of radius $1/2k$. □

4. Helicoidal CMC surfaces in $\mathbb{H}^2 \times \mathbb{R}$. Let G be the subgroup of isometries generated by $X_3 + aX_4 = \frac{\partial}{\partial\theta} + a\frac{\partial}{\partial z}$. The G-invariant functions are the solution of the equation

$$\frac{\partial\zeta}{\partial\theta} + a\frac{\partial\zeta}{\partial z} = 0,$$

that is

$$a\,d\theta = dz,$$

where (r, θ, z) are cylindrical coordinates. Thus the invariant functions are

$$u = r \qquad v = z - a\theta,$$

and the orbit space is $\mathcal{B} = \{(u, v) \in \mathbb{R}^2 : 0 \leq u < \sqrt{2}\}$. The gradient of u and v are

$$\nabla u = F^2\frac{\partial}{\partial r}$$
$$\nabla v = -\frac{F^2 a}{r^2}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial z}$$

with $F = \frac{(2-r^2)}{2}$. Therefore the matrix h (defined in Theorem 2.3) and its inverse take the form

$$(h_{ij}) = \begin{pmatrix} F^2 & 0 \\ 0 & \dfrac{r^2 + F^2a^2}{r^2} \end{pmatrix} \qquad (h^{ij}) = \begin{pmatrix} \dfrac{1}{F^2} & 0 \\ 0 & \dfrac{r^2}{r^2 + F^2a^2} \end{pmatrix}$$

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and the quotient metric is

$$d\tilde{s}^2 = \frac{du^2}{F^2} + \frac{r^2}{r^2 + F^2 a^2}\, dv^2.$$

The tangent and normal unit vector fields to the curve $\gamma(s)$ are

$$\mathbf{t} = \left(F\cos\sigma, \frac{\sqrt{u^2 + F^2 a^2}}{u}\sin\sigma\right)$$
$$\mathbf{n} = \left(-F\sin\sigma, \frac{\sqrt{u^2 + F^2 a^2}}{u}\cos\sigma\right).$$

Then, after calculation of the volume function of a principal orbit

$$\omega(\xi) = \sqrt{\langle X_3 + aX_4, X_3 + aX_4\rangle} = \sqrt{\left(\frac{u^2}{F^2} + a^2\right)},$$

from Theorem 2.1 the mean curvature H of Σ can be written as

$$H = \dot{\sigma} + (2u)^{-1}(u^2 + 2)\sin\sigma.$$

This means that the curve $\gamma(s)$ is a solution of the system

$$\begin{cases} \dot{u} = F\cos\sigma \\ \dot{v} = \dfrac{\sqrt{(u^2 + F^2 a^2)}}{u}\sin\sigma \\ \dot{\sigma} = H - \dfrac{(u^2+2)}{2u}\sin\sigma \end{cases} \tag{4.1}$$

with $F = \frac{2-u^2}{2}$.

REMARK 4.1. Reflection of a solution curve for (4.1) across a line $v = c$ is a solution curve for (4.1).

PROPOSITION 4.2. *If H is constant on the surface Σ, then the function*

$$J(s) = \frac{u\sin\sigma - H}{u^2 - 2}. \tag{4.2}$$

is constant along any solution of (4.1).

Proof. Deriving equation (4.2) and taking into account (4.1) we get immediately $\dot{J}(s) = 0$. □

Thus, the helicoidal CMC surfaces are solutions of the equation

$$\frac{u\sin\sigma - H}{u^2 - 2} = k, \tag{4.3}$$

for all $k \in \mathbb{R}$. Setting $C = k - H/2$, equation (4.3) becomes

$$u\sin\sigma = \left(\frac{H}{2} + C\right)u^2 - 2C. \tag{4.4}$$

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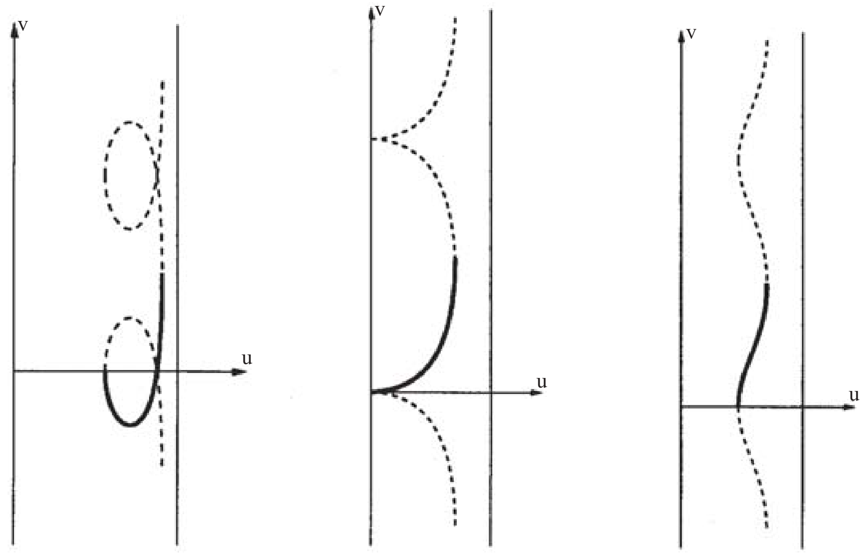
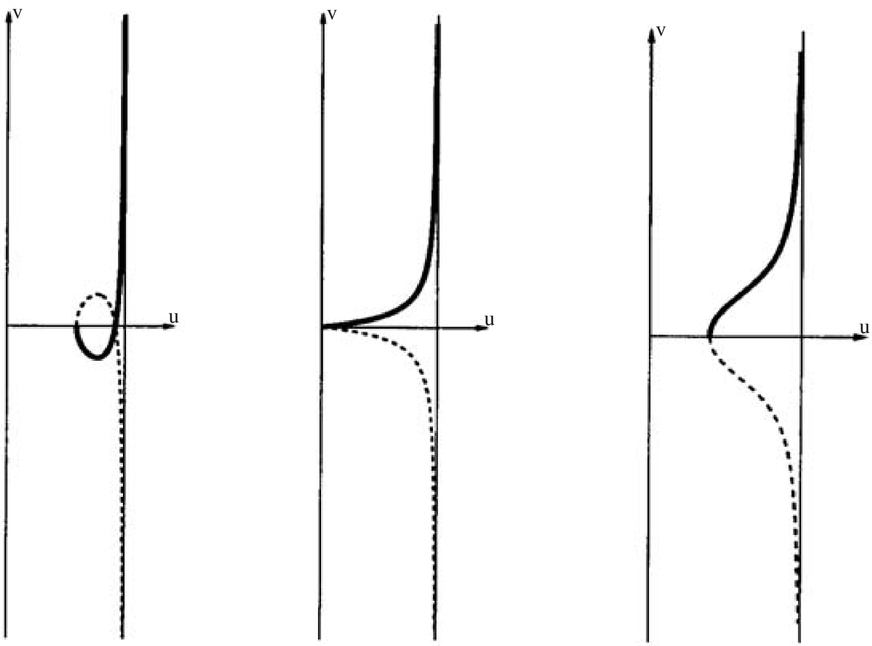


$C > 0$ $C = 0$ $C < 0$

Figure A. Profile curves for $H > \sqrt{2}$



$C > 0$ $C = 0$ $C < 0$

Figure B. Profile curves for $H = \sqrt{2}$

From now on we shall assume that $H \geq 0$, and according to its value the curve γ, which is a solution of (4.1), can be of three types. In fact when $u \to \sqrt{2}$, from (4.4), we have that $\sin\sigma \to \frac{H}{\sqrt{2}}$; this means that:

(I) if $H > \sqrt{2}$ the curve γ does not reach the line $u = \sqrt{2}$;

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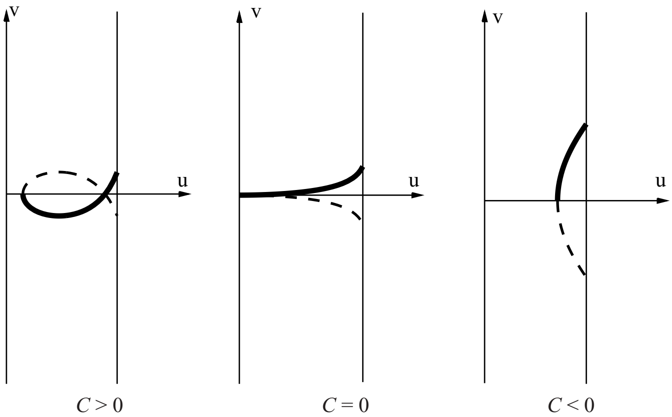

Figure C. Profile curves for $H < \sqrt{2}$

(II) if $H = \sqrt{2}$ the curve γ tends asymptotically to the line $u = \sqrt{2}$;
(III) if $H < \sqrt{2}$ the curve γ tends to the line $u = \sqrt{2}$ with an angle $\sigma < \frac{\pi}{2}$.

We are now ready to prove the main result.

THEOREM 4.3. *Let $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ be a CMC helicoidal surface and let $\gamma = \Sigma/G$ be the profile curve in the orbit space. Then we have the following characterization of γ according to the value of H.*

(I) (**H** $>$ $\sqrt{\mathbf{2}}$) – *The profile curve is of Delaunay type. Moreover if*
- $C > 0$ *it is of* **nodary-type**
- $C = 0$ *it is of* **circle-type**
- $C < 0$ *it is of* **undulary-type** *or a vertical straight line*

(II) (**H** $=$ $\sqrt{\mathbf{2}}$) – *The profile curve is,*
- *for* $C > 0$, *of* **folium-type**
- *for* $C = 0$, *of* **conic-type**
- *for* $C < 0$, *of* **bell-type**

(III) (**H** $<$ $\sqrt{\mathbf{2}}$) – *The profile curve is,*
- *for* $C > 0$, *of* **bounded folium-type**
- *for* $C = 0$, *of* **helicoidal-type** *or a horizontal straight line for* $H = 0$
- *for* $C < 0$, *of* **catenary-type**.

In the Figures (A, B, C) there is a plot of all profiles.

Proof. We shall prove the theorem in three steps: $C > 0$, $C = 0$ and $C < 0$.
Step 1 – $C > 0$. By solving the quadratic equation (4.4) we have

$$u_{1,2} = \frac{\sin\sigma \pm \sqrt{\sin^2\sigma + 4C(2C+H)}}{(2C+H)}. \tag{4.5}$$

In this case it is easy to check that $u_m \leq u \leq u_M$, where

$$u_m = \frac{-1 + \sqrt{1 + 4C(2C+H)}}{(2C+H)}, \qquad \text{and} \qquad u_M = \frac{1 + \sqrt{1 + 4C(2C+H)}}{(2C+H)}.$$

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Choosing initial conditions $u(0) = u_m$ and $v(0) = 0$, we have:

$$\sigma(0) = 3\pi/2, \qquad \dot\sigma(0) = H + \frac{u_m^2 + 2}{2u_m} > 0.$$

Thus the angle $\sigma(s)$ turns in the positive direction. Moreover u_M satisfies the equation $u = (\frac{H}{2} + C)u^2 - 2C$ and, combining with (4.4), we find $\sin\sigma(s_2) = 1$ (i.e. $\sigma(s_2) = \pi/2$), where $u(s_2) = u_M$ for some $s_2 > 0$.

Now, from the third equation of (4.1), $\dot\sigma(s) = 0$ when $\sin\sigma(s) = \frac{2u(s)H}{2+u(s)^2}$, or, using (4.4), when the function $u(s)$ satisfies the equation

$$(H + 2C)u^4 - 2Hu^2 - 8C = 0. \tag{4.6}$$

The latter equation does not admit real solutions in $[0, \sqrt{2})$, thus $\sigma(s)$ is always increasing. This means that there exists an $s_1 \in (0, s_2)$ so that $\sigma(s_1) = 2\pi$. Therefore in $u(s_1) = \sqrt{\frac{4C}{2C+H}}$ there is a local minimum.

Now if $H > \sqrt{2}$, $u_M < \sqrt{2}$ and, according to Remark 4.1, we can reflect the curve infinitely many times. The resulting curve is of *nodary-type*.

If $H = \sqrt{2}$, the profile curve tends asymptotically to the line $u = \sqrt{2}$ and it can be reflected only one time. The curve together with its reflection is called of *folium-type*.

Finally if $H < \sqrt{2}$, $u_M > \sqrt{2}$ and the curve tends, bounded from above, to the line $u = \sqrt{2}$.

Step 2 – $C = 0$. Since $u \sin\sigma = Hu^2/2$, we can choose initial conditions $u(0) = u_m = 0$ and $v(0) = 0$. In $s = 0$ the value of σ is not determined while in $s = s_1$, with $u(s_1) = u_M = 2/H$, $\sigma(s_1) = \pi/2$. Moreover $\dot\sigma(s) > 0$ in $(0, \sqrt{2})$. Then, as in the case $C > 0$, we have the following three subcases.

If $H > \sqrt{2}$, $u_M < \sqrt{2}$ and, according to Remark 4.1, we can reflect the curve infinitely many times. The resulting curve is of *circle-type*.

If $H = \sqrt{2}$, the profile curve tends asymptotically to the line $u = \sqrt{2}$ and it can be reflected only one time. The curve together with its reflection is called of *conic-type*.

Finally if $H < \sqrt{2}$, $u_M > \sqrt{2}$ and the curve tends, bounded from above, to the line $u = \sqrt{2}$. When $H = 0$ we have that $\sigma = 0$ for all u, thus the profile curve is a horizontal line and the resulting helicoidal surface is the standard helicoid. For this reason we shall call this type *helicoidal*.

Step 3 – $C < 0$. Differently from the first two steps in this case we have to check that the discriminant of (4.5) is positive; this is the case when:

(3a) $C \leq -\dfrac{H}{2}$,

(3b) $-\dfrac{H}{2} < C < \dfrac{-H - \sqrt{H^2 - 2}}{4}$, with $H \geq \sqrt{2}$,

(3c) $\dfrac{-H + \sqrt{H^2 - 2}}{4} < C < 0$, with $H \geq \sqrt{2}$,

(3d) $C = \dfrac{-H \pm \sqrt{H^2 - 2}}{4}$, with $H \geq \sqrt{2}$.

In (3a) we have $u_m \leq u \leq u_M$ where:

$$u_m = \frac{1 - \sqrt{1 + 4C(2C + H)}}{(2C + H)}, \qquad \text{and} \qquad u_M = \frac{-1 - \sqrt{1 + 4C(2C + H)}}{(2C + H)}.$$

First note that $u_m < \sqrt{2}$ if and only if $H < \sqrt{2}$.

Choosing initial conditions $u(0) = u_m$ and $v(0) = 0$, and observing that u_m satisfies the equation $u = (\frac{H}{2} + C)u^2 - 2C$, from (4.4) we deduce that $\sin\sigma(0) = 1$ (i.e. $\sigma(0) = \frac{\pi}{2}$). Moreover $\dot\sigma(0) = H - \frac{u_m^2+2}{2u_m} < 0$, thus $\sigma(s)$ turns in the negative direction.

Since the Equation (4.6) does not admit real solution in $(u_m, \sqrt{2})$ the function σ is always decreasing and the curve γ tends to the line $u = \sqrt{2}$ under an angle $H/\sqrt{2}$. The profile curve, after reflection, is then of *catenary-type*.

In (3b) $u_m \leq u \leq u_M$ where

$$u_m = \frac{1 - \sqrt{1 + 4C(2C + H)}}{(2C + H)}, \qquad \text{and} \qquad u_M = \frac{1 + \sqrt{1 + 4C(2C + H)}}{(2C + H)}.$$

An easy computation shows that $u_m > \sqrt{2}$ for all H and C, thus in this case we don't have solution.

In the case (3c) $u_m \leq u \leq u_M$, where

$$u_m = \frac{1 - \sqrt{1 + 4C(2C + H)}}{(2C + H)}, \qquad \text{and} \qquad u_M = \frac{1 + \sqrt{1 + 4C(2C + H)}}{(2C + H)}.$$

Easily we can see that

$$\sin\sigma = \frac{(H + 2C)u^2 - 4C}{2u} > 0,$$

thus $0 < \sigma(s) < \pi$. Choosing initial conditions $u(0) = u_m$ and $v(0) = 0$ we find $\sigma(0) = \pi/2$ and

$$\dot\sigma(0) = \frac{H[1 + 4C(2C + H)] + (4C + H)\sqrt{1 + 4C(2C + H)}}{4C(2C + H)} < 0.$$

This means that the angle $\sigma(s)$ turns in the negative direction. Note that for $u(s_2) = u_M$, $\sigma(s_2) = \pi/2$. Moreover Equation (4.6) admits the real solution $u(s_1) = 2\sqrt{\frac{-C}{2C+H}}$, for some $s_1 \in (0, s_2)$. This implies that in s_1 there is a local minima of $\sigma(s)$ and the curve γ, for $s > s_1$, turns in the positive direction. If $H > \sqrt{2}$ the curve γ can be reflected infinitely many times and is of *undulary-type*. While if $H = \sqrt{2}$ the curve γ tends asymptotically to the line $u = \sqrt{2}$ and can be reflected only one time giving a *bell-type* profile.

In the last case (3d) $\sin^2\sigma = 1$, and from $u = \frac{\sin\sigma}{2C+H}$ and $2C + H > 0$, we must have $\sigma = \pi/2$. Thus we find a vertical straight line that, after the action of the helicoidal group, gives the right cylinder of radius $r = \frac{1}{2C+H}$. Note that $r < \sqrt{2}$ if and only if $C = \frac{-H+\sqrt{H^2-2}}{4} < 0$. □

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