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KRASNOSELSKI–MANN ITERATION FOR HIERARCHICAL FIXED POINTS AND EQUILIBRIUM PROBLEM

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Abstract

We give an explicit Krasnoselski–Mann type method for finding common solutions of the following system of equilibrium and hierarchical fixed points:

$$\begin{cases} G(x^*, y) \geq 0, & \forall y \in C, \\ \text{find } x^* \in \text{Fix}(T) \text{ such that } \langle x^* - f(x^*), x - x^* \rangle \geq 0, & \forall x \in \text{Fix}(T), \end{cases}$$

where C is a closed convex subset of a Hilbert space H, $G : C \times C \to \mathbb{R}$ is an equilibrium function, $T : C \to C$ is a nonexpansive mapping with $\text{Fix}(T)$ its set of fixed points and $f : C \to C$ is a ρ-contraction. Our algorithm is constructed and proved using the idea of the paper of [Y. Yao and Y.-C. Liou, 'Weak and strong convergence of Krasnosel'skiĭ–Mann iteration for hierarchical fixed point problems', *Inverse Problems* **24** (2008), 501–508], in which only the variational inequality problem of finding hierarchically a fixed point of a nonexpansive mapping T with respect to a ρ-contraction f was considered. The paper follows the lines of research of corresponding results of Moudafi and Théra.

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1. Introduction

Let T, V be two nonexpansive mappings from C to C, where C is a closed and convex subset of a Hilbert space H. Consider the variational inequality problem (VIP) of finding hierarchically a fixed point of a nonexpansive mapping T with respect to another nonexpansive mapping V, that is,

$$\text{find } x^* \in \text{Fix}(T) \text{ such that } \langle x^* - Vx^*, y - x^* \rangle \geq 0 \quad y \in \text{Fix}(T). \tag{1.1}$$

(Equivalently, $x^* = P_{\text{Fix}(T)} V x^*$ – that is, x^* is a fixed point of the nonexpansive map $P_{\text{Fix}(T)} V$ – where for K closed convex subset of H, P_K is the metric projection of H on K).

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Of course if $V = I$, the solution set S of (1.1) is just $\mathrm{Fix}(T)$.

The VIP (1.1) covers several topics investigated in literature, among them the following:

(1) (Monotone inclusions) Yamada [32] studies the VIP (1.1) assuming $V = I - \gamma F$, where $\gamma > 0$ is sufficiently small and the operator F is Lipschitzian and strongly monotone.

(2) (Convex optimization [4, 23]) Let φ be a proper lower semicontinuous convex function on H and let ψ be a convex function on H so that $\nabla\psi$ is strongly monotone. Take

$$T = \mathrm{prox}_{\lambda\varphi} := \mathrm{argmin}\left\{\varphi(z) + \frac{1}{2\lambda}\|\cdot - z\|^2\right\}.$$

Then the VIP (1.1) reduces to the hierarchical minimization problem

$$\min_{x \in \ \mathrm{argmin}\ \varphi} \psi(x).$$

(3) (Quadratic minimizations over a fixed point set [14]) If A is a linear bounded strongly positive operator on H, f is a ρ-contraction on H and h is a potential for γf (that is, $h'(x) = \gamma f(x)$) where $\gamma > 0$ is a constant, consider the minimization problem

$$\min_{x \in \mathrm{Fix}(T)} \tfrac{1}{2}\langle Ax, x\rangle - h(x). \tag{1.2}$$

The optimality condition to minimize (1.2) is to find a fixed point of T so that

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0, \quad x \in \mathrm{Fix}(T).$$

Taking $V = I - \lambda(A - \gamma f)$, where $\gamma > 0$ is appropriately chosen so that V is nonexpansive, we find that the previous VIP reduced to (1.1).

(4) Let A be a maximal monotone operator. Take $T = J_\lambda^A := (I + \lambda A)^{-1}$ and $V = I - \gamma\nabla\psi$, where ψ is a convex function such that $\nabla\psi$ is η-Lipschitzian (which is equivalent to the fact that $\nabla\psi$ is η^{-1} co-coercive), with $\gamma \in (0, 2/\eta]$ and $\mathrm{Fix}(J_\lambda^A) = A^{-1}(0)$. So VIP (1.1) reduces to the mathematical program with generalized equation constraint,

$$\min_{0 \in A(x)} \psi(x),$$

considered in [13].

A very particular case of the VIP (1.1) occurs when V is a constant mapping, that is, given $u \in H$,

$$\text{find } x^* \in \mathrm{Fix}(T) \text{ such that } \langle x^* - u, x - x^*\rangle \geq 0, \quad x \in \mathrm{Fix}(T), \tag{1.3}$$

or, equivalently, find the fixed point of T closest to u, that is,

$$x^* = P_{\mathrm{Fix}(T)}u = \mathrm{argmin}_{x \in \mathrm{Fix}(T)} \tfrac{1}{2}\|u - x\|^2.$$

This problem was widely investigated in [2, 9, 12, 22, 26, 28, 29]. The explicit method, initiated by Halpern in [9], generates a sequence $(x_n)_n$ by iterating

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \tag{1.4}$$

where $u, x_0 \in C$ and $(\alpha_n)_n \subset [0, 1]$.

The next result is well known.

THEOREM 1.1 [2, 9, 20, 21, 25–27]. *Assume that* $\mathrm{Fix}(T)$ *is nonempty. Suppose that the sequence* $(\alpha_n)_n$ *satisfies the following:*

(1) $\lim_n \alpha_n = 0$;
(2) $\sum_n \alpha_n = \infty$;
(3) $\sum_n |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_n((\alpha_{n+1} - \alpha_n)/\alpha_n) = 0$.

Then the sequence $(x_n)_n$ *generated by the algorithm (1.4) converges in norm to* $P_{\mathrm{Fix}(T)} u$.

A more general case than V constant is that one $V = f$ with f is a ρ-contraction, that is, $\|f(x) - f(y)\| \le \rho \|x - y\|$, $\rho \in (0, 1)$. In this case we call (1.1) the contractive VIP and the method is also known as viscosity approximation. It was first studied by Moudafi [15] and further developed by Xu [30].

In this method, the explicit scheme (1.4) is replaced by Mann-type scheme

$$x_{n+1} = \lambda_n f(x_n) + (1 - \lambda_n) T x_n \tag{1.5}$$

where $(\lambda_n)_n$ is a sequence in $[0, 1]$.

THEOREM 1.2 [15, 30]. *Assume that* $\mathrm{Fix}(T)$ *is nonempty and let* $(x_n)_n$ *be the sequence by the algorithm (1.5). Assume that:*

(1) $\lim_n \lambda_n = 0$;
(2) $\sum_n \lambda_n = \infty$;
(3) $\sum_n |\lambda_{n+1} - \lambda_n| < \infty$ *or* $\lim_n((\lambda_{n+1} - \lambda_n)/\lambda_n) = 0$.

Then $\lim_n x_n = x^*$ *exists and* x^* *is the unique solution of the variational inequality*

$$\langle (I - f)x^*, x - x^* \rangle \ge 0, \quad x \in \mathrm{Fix}(T).$$

Very recently, Yao and Liou [35] replaced the Mann-type scheme (1.5) with the Krasnoselski–Mann type scheme

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n (\lambda_n f(x_n) + (1 - \lambda_n) T x_n)$$

and proved the following theorem.

THEOREM 1.3 [35]. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* T *be a nonexpansive mapping of* C *into itself such that* $\mathrm{Fix}(T) \neq \emptyset$. *Let* $P : C \to C$ *be a* ρ*-contraction. Let* $(x_n)_n$ *be a sequence generated by*

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n (\sigma_n P x_n + (1 - \sigma_n) T x_n), \quad n \ge 0.$$

Let $(\alpha_n)_n$*,* $(\sigma_n)_n$ *be two real number sequences in* $(0, 1)$ *satisfying the following conditions:*

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$*;*
(ii) $\lim_{n\to\infty} \sigma_n = 0$ *and* $\sum_n \sigma_n = \infty$.

Then:

(1) $(x_n)_n$ *converges strongly to a fixed point of* T*;*
(2) $(x_n)_n$ *is asymptotically regular, namely* $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$*;*
(3) $(x_n)_n$ *converges strongly to a solution of the problem*

$$\textit{find } x^* \in \mathrm{Fix}(T) \textit{ such that } \langle x^* - f(x^*), x - x^*\rangle \ge 0, \quad \forall x \in \mathrm{Fix}(T).$$

The above scheme is a particular case of the Krasnoselski–Mann algorithm

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(\lambda_n V(x_n) + (1 - \lambda_n)T x_n)$$

with V a nonexpansive mapping, introduced by Moudafi [17].

Some algorithms in signal processing and image reconstruction may be written as the well-known Krasnoselski–Mann (K–M) iteration. The main feature of (K–M)-iteration convergence theorems provided a unified framework for analyzing various concrete algorithms. For details, see [3, 5, 31–34].

On the other hand, note that if we put $C = \mathrm{Fix}(T)$ and $G(x, y) := \langle (I - V)x, y - x\rangle$, then the VIP (1.1) can be rewritten as

$$\text{find } x^* \in C \text{ such that } G(x^*, y) \ge 0, \quad y \in C, \tag{1.6}$$

that is, as an equilibrium problem. More generally, following [6], we can have a countable family of bifunctions from $C \times C$ to $\mathbb{R}$. The basic formulation of this class of problems reduces to solving the system of equilibrium problems

$$\text{find } x \in C \text{ such that } G_i(x, y) \ge 0, \quad \forall i \in I, \forall y \in C. \tag{1.7}$$

Blum and Oettli [1, 19] show that, in the case of a single equilibrium problem, the formulation (1.6) covers monotone inclusion problems, saddlepoint problems, VIPs, minimization problems, Nash equilibria in noncooperative games, vector equilibrium problems and certain fixed point problems (see [8]).

It is also worth remarking that, in the case of VIP (1.1), the induced bifunction $G(x, y) := \langle (I - V)x, y - x\rangle$ satisfies the following condition.

CONDITION (1).

(E1) $G(x, x) = 0$ for all $x \in H$.
(E2) $G(x, y) + G(y, x) \le 0$ for all $(x, y) \in H \times H$ (that is, G is monotone).
(E3) For each $x, y, z \in H$,

$$\limsup_{t\to 0} G(tz + (1 - t)x, y) \le G(x, y).$$

(E4) The function $y \to G(x, y)$ is convex and lower semicontinuous for each $x \in H$.

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While many methods have been proposed to solve (1.6) (see [7, 10, 11, 16, 18]), we are not aware of so many results for systems of equilibrium problems. For some partial results on these topics see [6].

Here we study a particular case of a system of two equilibrium functions, one induced by a contractive VIP and one satisfying Condition (1), namely

$$\begin{cases} G(x^*, y) \geq 0, & \forall y \in C, \\ \text{find } x^* \in \mathrm{Fix}(T) \text{ such that } \langle x^* - f(x^*), x - x^* \rangle \geq 0, & \forall x \in \mathrm{Fix}(T). \end{cases} \tag{1.8}$$

Of course such systems include the systems given by a VIP and a contractive VIP.

We show that the following Krasnoselski–Mann-type scheme for the VIP and equilibrium function

$$\begin{cases} x_0 \in C, \\ G(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, & \forall y \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(\lambda_n f(x_n) + (1 - \lambda_n)Tu_n), & n \geq 1, \end{cases} \tag{1.9}$$

solves the system.

2. Preliminaries

We give several known results that are fundamental for our proof.

LEMMA 2.1 [24]. *Let $(x_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ be bounded sequences in a Banach space X and let $(\beta_n)_{n\in\mathbb{N}}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n,$$

for all integers $n \geq 0$, and

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

LEMMA 2.2 [29]. *Assume $(a_n)_n$ is a sequence of nonnegative numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad n \geq 0,$$

where $(\gamma_n)_n$ is a sequence in $(0, 1)$ and $(\delta_n)_n$ is a sequence in $\mathbb{R}$ such that:

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

The next lemmas concern the equilibrium function G and the set of equilibrium points

$$EP(G) = \{x \in C \mid G(x, y) \geq 0, \forall y \in C\}.$$

LEMMA 2.3 [6]. *Let C be a nonempty closed convex subset of H and $G : C \times C \to \mathbb{R}$ satisfy Condition (1). For $x \in C$ and $r > 0$, let $S_r : H \to C$ be the r-resolvent of G,*

$$S_r(x) := \left\{ z \in C \mid G(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}.$$

Then S_r is well defined and the following hold:

(1) *S_r is single-valued;*
(2) *S_r is firmly nonexpansive, that is,*

$$\|S_r x - S_r y\|^2 \leq \langle S_r x - S_r y, x - y\rangle,$$

for all $x, y \in H$;
(3) $\mathrm{Fix}(S_r) = EP(G)$;
(4) *$EP(G)$ is closed and convex.*

LEMMA 2.4 [6]. *Suppose that $G : C \times C \to \mathbb{R}$ is an equilibrium function satisfying Condition (1). Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in H and $(r_n)_{n\in\mathbb{N}}$ a sequence in $(0, +\infty)$. Define, for all $n \in \mathbb{N}$, $u_n := S_{r_n} x_n$ and suppose that $u_n \rightharpoonup p$ and $(x_n - u_n) \to z$. Then $p \in C$ and for all $y \in C$, $G(p, y) + \langle z, p - y\rangle \geq 0$.*

REMARK 2.5. Note that in Lemma 2.4, if $z = 0$, then the weak cluster point p for $(u_n)_{n\in\mathbb{N}}$ is a weak cluster point for $(x_n)_{n\in\mathbb{N}}$ and also an equilibrium point for G.

LEMMA 2.6. *Let $G : C \times C \to \mathbb{R}$ be a bifunction such that Condition (1) holds. Let $(w_n)_n$ be a bounded sequence and $z_n := S_{r_n} w_n$. Let $(r_n)_n$ be a sequence of positive numbers such that $\liminf_n r_n = r > 0$. Then there exists a constant $L > 0$ such that*

$$\|z_{n+1} - z_n\| \leq \|w_{n+1} - w_n\| + L\left|1 - \frac{r_n}{r_{n+1}}\right|. \tag{2.1}$$

PROOF. Since $z_n := S_{r_n} w_n$ and $z_{n+1} := S_{r_{n+1}} w_{n+1}$, we obtain that

$$G(z_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - z_{n+1}, z_{n+1} - w_{n+1}\rangle \geq 0, \quad \forall y \in C,$$

and

$$G(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - w_n\rangle \geq 0, \quad \forall y \in C.$$

In particular,

$$G(z_{n+1}, z_n) + \frac{1}{r_{n+1}}\langle z_n - z_{n+1}, z_{n+1} - w_{n+1}\rangle \geq 0$$

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and

$$G(z_n, z_{n+1}) + \frac{1}{r_n}\langle z_{n+1} - z_n, z_n - w_n\rangle \geq 0.$$

Hence, summing up these two inequalities and using (E2),

$$\frac{1}{r_n}\langle z_{n+1} - z_n, z_n - w_n\rangle + \frac{1}{r_{n+1}}\langle z_n - z_{n+1}, z_{n+1} - w_{n+1}\rangle \geq 0,$$

so it follows that

$$\left\langle z_{n+1} - z_n, \frac{z_n - w_n}{r_n} - \frac{z_{n+1} - w_{n+1}}{r_{n+1}}\right\rangle \geq 0. \tag{2.2}$$

We derive from (2.2) that

$$\begin{aligned}
&\left\langle z_{n+1} - z_n, z_n - w_n - \frac{r_n}{r_{n+1}}(z_{n+1} - w_{n+1})\right\rangle \geq 0\\
&\quad\Rightarrow \left\langle z_{n+1} - z_n, z_n - z_{n+1} - w_n + z_{n+1} - \frac{r_n}{r_{n+1}}(z_{n+1} - w_{n+1})\right\rangle \geq 0\\
&\quad\Rightarrow -\|z_{n+1} - z_n\|^2 + \left\langle z_{n+1} - z_n, (z_{n+1} - w_{n+1})\left(1 - \frac{r_n}{r_{n+1}}\right)\right.\\
&\qquad \left. + (w_{n+1} - w_n)\right\rangle \geq 0.
\end{aligned}$$

Then

$$\begin{aligned}
\|z_{n+1} - z_n\|^2 &\leq \left\langle z_{n+1} - z_n, (z_{n+1} - w_{n+1})\left(1 - \frac{r_n}{r_{n+1}}\right) + (w_{n+1} - w_n)\right\rangle\\
&\leq \|z_{n+1} - z_n\|\left(\|w_{n+1} - w_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\|z_{n+1} - w_{n+1}\|\right),
\end{aligned}$$

and so

$$\|z_{n+1} - z_n\| \leq \|w_{n+1} - w_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\|z_{n+1} - w_{n+1}\|.$$

By hypothesis on $(r_n)_n$, if $L := \sup_n \|z_{n+1} - w_{n+1}\|$, we conclude that

$$\|z_{n+1} - z_n\| \leq \|w_{n+1} - w_n\| + L\left|1 - \frac{r_n}{r_{n+1}}\right|.$$

□

3. Main result

THEOREM 3.1. *Let C be a closed convex subset of a Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \cap EP(G) \neq \emptyset$. Let $f : C \to C$ be a ρ-contraction. Let $(\lambda_n)_n$ be a sequence in $(0, 1)$ such that $\lambda_n \to 0$ and $\sum_n \lambda_n = \infty$. Let $(\alpha_n)_n$ be a sequence in $(0, 1)$ such that $0 < \liminf_n \alpha_n \leq \limsup_n \alpha_n < 1$.*

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Let $(r_n)_n$ *be a sequence of positive real numbers such that* $\liminf_n r_n = r > 0$ *and* $\lim_n |1 - ((r_n)/(r_{n+1}))| = 0$*. Let* $(x_n)_n$*,* $(u_n)_n$ *be the sequences defined by*

$$\begin{cases} x_0 \in C, & \\ G(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n(\lambda_n f(x_n) + (1-\lambda_n)Tu_n), & n \geq 1. \end{cases} \tag{3.1}$$

Then the sequences both converge to a point $z \in \mathrm{Fix}(T) \cap EP(G)$ *which is the unique solution in* $\mathrm{Fix}(T) \cap EP(G)$ *of the variational inequality*

$$\langle z - f(z), z - x\rangle \leq 0, \qquad \forall x \in \mathrm{Fix}(T) \cap EP(G). \tag{3.2}$$

Equivalently, $z = P_{\mathrm{Fix}(T)\cap EP(G)} f z$*.*

PROOF. Since the inequality

$$\|u_n - z\| = \|S_{r_n}x_n - S_{r_n}z\| \leq \|x_n - z\|$$

holds, we only prove that $x_n \to z$. We divide the proof into several steps.

STEP 1. We prove that the sequence $(x_n)_n$ is bounded. Let $v \in \mathrm{Fix}(T) \cap EP(G)$. Then

$$\begin{aligned} \|x_{n+1} - v\| &= \|(1-\alpha_n)(x_n - v) + \alpha_n[\lambda_n(f(x_n) - v) + (1-\lambda_n)(Tu_n - v)]\| \\ &\leq (1-\alpha_n)\|x_n - v\| + \alpha_n[\lambda_n(\|f(x_n) - f(v)\| \\ &\quad + \|f(v) - v\|) + (1-\lambda_n)\|x_n - v\|] \\ &\leq (1-\alpha_n)\|x_n - v\| + \alpha_n\lambda_n\rho\|x_n - v\| \\ &\quad + \alpha_n\lambda_n\|f(v) - v\| + \alpha_n(1-\lambda_n)\|x_n - v\| \\ &= (1-(1-\rho)\lambda_n\alpha_n)\|x_n - v\| + \alpha_n\lambda_n\|f(v) - v\| \\ &\leq \max\left\{\|x_n - v\|, \frac{\|f(v) - v\|}{1-\rho}\right\}. \end{aligned}$$

By induction we obtain that

$$\|x_n - v\| \leq \max\left\{\|x_0 - v\|, \frac{\|f(v) - v\|}{1-\rho}\right\}.$$

STEP 2. We prove that the sequence $(x_n)_n$ is asymptotically regular, that is, $\|x_n - x_{n+1}\| \to 0$, as $n \to \infty$. Set $y_n = \lambda_n f(x_n) + (1-\lambda_n)Tu_n$ and note that

$$\begin{aligned} y_{n+1} - y_n &= \lambda_{n+1}f(x_{n+1}) + (1-\lambda_{n+1})Tu_{n+1} - \lambda_n f(x_n) - (1-\lambda_n)Tu_n \\ &= \lambda_{n+1}(f(x_{n+1}) - f(x_n)) + (\lambda_{n+1} - \lambda_n)f(x_n) \\ &\quad + (1-\lambda_{n+1})(Tu_{n+1} - Tu_n) - (\lambda_{n+1} - \lambda_n)Tu_n. \end{aligned}$$

So, to apply Lemma 2.1 (due to Suzuki), we observe that $x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n$ and

$$\begin{aligned}
&\limsup_n (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \\
&\quad \le \limsup_n [\lambda_{n+1} \|f(x_{n+1}) - f(x_n)\| + |\lambda_{n+1} - \lambda_n| \|f(x_n) - T u_n\| \\
&\qquad + (1-\lambda_{n+1}) \|T u_{n+1} - T u_n\| - \|x_{n+1} - x_n\|] \\
&\quad \le \limsup_n [\lambda_{n+1} \rho \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n| \|f(x_n) - T u_n\| \\
&\qquad + (1-\lambda_{n+1}) \|u_{n+1} - u_n\| - \|x_{n+1} - x_n\|] \\
&\quad \le \limsup_n \bigg[\lambda_{n+1} \|x_{n+1} - x_n\| + (1-\lambda_{n+1}) \bigg(\|x_{n+1} - x_n\| + L \bigg| 1 - \frac{r_n}{r_{n+1}} \bigg| \bigg) \\
&\qquad + |\lambda_{n+1} - \lambda_n| (\|T u_n\| + \|f(x_n)\|) - \|x_{n+1} - x_n\| \bigg],
\end{aligned}$$

where the second inequality holds by (2.1) in Lemma 2.6. By the boundedness of $(x_n)_n$ and the hypotheses on the sequences $(\lambda_n)_n$, $(r_n)_n$ we conclude that

$$\begin{aligned}
&\limsup_n (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \\
&\quad \le \limsup_n \bigg[\|x_{n+1} - x_n\| + L \bigg| 1 - \frac{r_n}{r_{n+1}} \bigg| \\
&\qquad + |\lambda_{n+1} - \lambda_n| (\|T u_n\| + \|f(x_n)\|) - \|x_{n+1} - x_n\| \bigg] \\
&\quad = \limsup_n \bigg[L \bigg| 1 - \frac{r_n}{r_{n+1}} \bigg| + |\lambda_{n+1} - \lambda_n| (\|T u_n\| + \|f(x_n)\|) \bigg] = 0.
\end{aligned}$$

We can apply Lemma 2.1 to derive

$$\lim_n \|x_n - y_n\| = 0. \tag{3.3}$$

On the other hand, a straightforward computation leads to

$$\lim_n \|x_{n+1} - x_n\| = \lim_n \alpha_n \|x_n - y_n\| = 0. \tag{3.4}$$

STEP 3. We prove that $\lim_n \|x_n - u_n\| = 0$. First of all we note that, by the firm nonexpansivity of S_{r_n}, if $p \in EP(G)$, then

$$\begin{aligned}
\|u_n - p\|^2 &= \langle u_n - p, S_{r_n} x_n - S_{r_n} p \rangle \le \langle u_n - p, x_n - p \rangle \\
&= \tfrac{1}{2} (\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2),
\end{aligned}$$

from which

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{3.5}$$

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On the other hand, if $v \in \mathrm{Fix}(T) \cap EP(G)$, then

$$\begin{aligned}
\|x_{n+1} - v\|^2 &= \|(1-\alpha_n)(x_n - v) + \alpha_n(\lambda_n f(x_n) + (1-\lambda_n)Tu_n - v)\|^2 \\
&= \|(1-\alpha_n)(x_n - v) + \alpha_n(Tu_n - v) + \alpha_n\lambda_n(f(x_n) - Tu_n)\|^2 \\
&\le \|(1-\alpha_n)(x_n - v) + \alpha_n(Tu_n - v)\|^2 \\
&\quad + 2\lambda_n\langle f(x_n) - Tu_n, x_{n+1} - v\rangle \\
&\le (1-\alpha_n)\|x_n - v\|^2 + \alpha_n\|Tu_n - Tv\|^2 \\
&\quad + 2\lambda_n\langle f(x_n) - Tu_n, x_{n+1} - v\rangle \\
&\le (1-\alpha_n)\|x_n - v\|^2 + \alpha_n\|u_n - v\|^2 \\
&\quad + 2\lambda_n\langle f(x_n) - Tu_n, x_{n+1} - v\rangle. \qquad (3.6)
\end{aligned}$$

Combining (3.5) with (3.6) and setting

$$z_n = 2\lambda_n\langle f(x_n) - Tu_n, x_{n+1} - v\rangle$$

leads to

$$\begin{aligned}
\|x_{n+1} - v\|^2 &\le (1-\alpha_n)\|x_n - v\|^2 + \alpha_n(\|x_n - v\|^2 - \|x_n - u_n\|^2) + z_n \\
&\le \|x_n - v\|^2 - \alpha_n\|x_n - u_n\|^2 + z_n. \qquad (3.7)
\end{aligned}$$

Thus,

$$\begin{aligned}
\alpha_n\|x_n - u_n\|^2 &\le \|x_n - v\|^2 - \|x_{n+1} - v\|^2 + z_n \\
&\le \|x_n - x_{n+1}\|^2 + 2\|x_n - x_{n+1}\|\|x_{n+1} - v\| + z_n. \qquad (3.8)
\end{aligned}$$

Since $(x_n)_n$ is bounded, $z_n \to 0$. Moreover, by asymptotically regularity of $(x_n)_n$ and by the hypothesis on $(\alpha_n)_n$, from the latter it follows that

$$\lim_n \|x_n - u_n\| = 0, \qquad (3.9)$$

as required.

STEP 4. We now prove that the set of weak cluster points $\omega_w(x_n)$ is a subset of $\mathrm{Fix}(T) \cap EP(G)$. Let $(x_{n_k})_k$ be a subsequence of $(x_n)_n$ weakly converging to a point $p \in C$. Since (3.9) holds, we can apply Lemma 2.4 to ensure that p lies in $EP(G)$.

To show that $p \in \mathrm{Fix}(T)$, we observe that

$$\begin{aligned}
\|x_{n_k} - Tx_{n_k}\| &\le \|x_{n_k+1} - x_{n_k}\| + \|x_{n_k+1} - Tu_{n_k}\| + \|Tu_{n_k} - Tx_{n_k}\| \\
&\le \|x_{n_k+1} - x_{n_k}\| + (1-\alpha_{n_k})\|x_{n_k} - Tu_{n_k}\| \\
&\quad + \alpha_{n_k}\lambda_{n_k}\|f(x_{n_k}) - Tu_{n_k}\| + \|u_{n_k} - x_{n_k}\|,
\end{aligned}$$

thus by hypotheses and by Steps 2 and 3,

$$\begin{aligned}
\lim_k \|x_{n_k} - Tx_{n_k}\| &\le \lim_k \frac{\|x_{n_k+1} - x_{n_k}\| + (2-\alpha_{n_k})\|u_{n_k} - x_{n_k}\|}{\alpha_{n_k}} \\
&\quad + \lambda_{n_k}\|f(x_{n_k}) - Tu_{n_k}\| = 0.
\end{aligned}$$

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Since $x_{n_k} \rightharpoonup p$, by the demiclosedness principle for nonexpansive mappings [2], we have $p \in \mathrm{Fix}(T)$.

REMARK 3.2. Note that from Step 4 it follows that

$$\limsup_n \langle f(z) - z, x_n - z\rangle \le 0, \tag{3.10}$$

where $z \in \mathrm{Fix}(T) \cap EP(G)$ is the unique solution of the variational inequality (3.2). To show this, let $(x_{n_j})_j$ be such that

$$\limsup_n \langle f(z) - z, x_n - z\rangle = \lim_j \langle f(z) - z, x_{n_j} - z\rangle.$$

By eventually passing to subsequences, we may assume that $x_{n_j} \rightharpoonup p$. Then

$$\lim_j \langle f(z) - z, x_{n_j} - z\rangle = \langle f(z) - z, p - z\rangle \le 0$$

since $p \in \mathrm{Fix}(T) \cap EP(G)$.

STEP 5. Finally, we show that $x_n, u_n \to z$, as $n \to \infty$. Since the inequality

$$\|u_n - z\| = \|S_{r_n} x_n - S_{r_n} z\| \le \|x_n - z\|$$

holds, it is enough to prove that $x_n \to z$:

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|(1-\alpha_n)(x_n - z) + \alpha_n(\lambda_n f(x_n) + (1-\lambda_n)Tu_n - z)\|^2 \\
&= \|(1-\alpha_n)(x_n - z) + \alpha_n(1-\lambda_n)(Tu_n - z) + \lambda_n\alpha_n(f(x_n) - z)\|^2 \\
&\le \|(1-\alpha_n)(x_n - z) + \alpha_n(1-\lambda_n)(Tu_n - z)\|^2 \\
&\quad + 2\alpha_n\lambda_n\langle f(x_n) - z, x_{n+1} - z\rangle \\
&\le (1-\alpha_n)\|x_n - z\|^2 + \alpha_n(1-\lambda_n)^2\|x_n - z\|^2 \\
&\quad + 2\alpha_n\lambda_n\langle f(x_n) - z, x_{n+1} - z\rangle \\
&\le (1-\alpha_n + \alpha_n(1-\lambda_n))^2\|x_n - z\|^2 \\
&\quad + 2\alpha_n\lambda_n\langle f(x_n) - f(z), x_{n+1} - z\rangle + 2\alpha_n\lambda_n\langle f(z) - z, x_{n+1} - z\rangle.
\end{aligned}$$

The Cauchy-Schwartz inequality gives

$$\begin{aligned}
2\alpha_n\lambda_n\langle f(x_n) - f(z), x_{n+1} - z\rangle &\le 2\alpha_n\lambda_n\|f(x_n) - f(z)\|\,\|x_{n+1} - z\| \\
&\le \alpha_n\lambda_n[\|f(x_n) - f(z)\|^2 + \|x_{n+1} - z\|^2].
\end{aligned}$$

So,

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\le (1-\alpha_n + \alpha_n(1-\lambda_n)^2)\|x_n - z\|^2 \\
&\quad \alpha_n\lambda_n(\|f(x_n) - f(z)\|^2 + \|x_{n+1} - z\|^2) \\
&\quad + 2\alpha_n\lambda_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\le (1-\alpha_n + \alpha_n(1-\lambda_n)^2 + \alpha_n\lambda_n\rho)\|x_n - z\|^2 \\
&\quad + \alpha_n\lambda_n\|x_{n+1} - z\|^2 + 2\alpha_n\lambda_n\langle f(z) - z, x_{n+1} - z\rangle.
\end{aligned}$$

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We can compute $(1-\alpha_n+\alpha_n(1-\lambda_n)^2+\alpha_n\lambda_n\rho)$ and simplify:

$$\begin{aligned}\|x_{n+1}-z\|^2 &= (1-\alpha_n\lambda_n-\alpha_n\lambda_n(1-\rho-\lambda_n))\|x_n-z\|^2\\ &\quad+\alpha_n\lambda_n\|x_{n+1}-z\|^2+2\alpha_n\lambda_n\langle f(z)-z,x_{n+1}-z\rangle.\end{aligned}$$

Then from the foregoing it follows that

$$\begin{aligned}\|x_{n+1}-z\|^2 \le \bigg(1-\frac{\alpha_n\lambda_n(1-\rho-\lambda_n)}{1-\alpha_n\lambda_n}\bigg)\|x_n-z\|^2\\ +2\frac{\alpha_n\lambda_n}{1-\alpha_n\lambda_n}\langle f(z)-z,x_{n+1}-z\rangle.\end{aligned}\tag{3.11}$$

Putting

$$\begin{aligned}a_n &= \|x_n-z\|^2,\\ \gamma_n &= \frac{\alpha_n\lambda_n(1-\rho-\lambda_n)}{1-\alpha_n\lambda_n}\end{aligned}$$

and

$$\delta_n=2\frac{\alpha_n\lambda_n}{1-\alpha_n\lambda_n}\langle f(z)-z,x_{n+1}-z\rangle,$$

then (3.11) becomes

$$a_{n+1}\le(1-\gamma_n)a_n+\delta_n.$$

Note that $\lim_n\gamma_n=0$ and

$$\limsup_n\frac{\delta_n}{\gamma_n}=\limsup_n 2\frac{\langle f(z)-z,x_{n+1}-z\rangle}{(1-\rho-\lambda_n)}\le 0,$$

by (3.10). Thus we may apply Lemma 2.2 to conclude that

$$\lim_n a_n=\lim_n\|x_n-z\|=0. \qquad \square$$

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