Proceedings of the Edinburgh Mathematical Society (2011) **54**, 799–807
DOI:10.1017/S0013091509001667

WEAKLY s-SUPPLEMENTALLY EMBEDDED MINIMAL SUBGROUPS OF FINITE GROUPS

TAO ZHAO, XIANHUA LI AND YONG XU

School of Mathematical Science, Soochow University, Suzhou, Jiangsu 215006, People's Republic of China (xhli@suda.edu.cn)

(Received 17 November 2009)

Abstract Suppose that G is a finite group and H is a subgroup of G. We call H a weakly s-supplementally embedded subgroup of G if there exist a subgroup T of G and an s-quasinormally embedded subgroup H_{se} of G contained in H such that $G = HT$ and $H \cap T \leqslant H_{se}$. We investigate the influence of the weakly s-supplementally embedded property of some minimal subgroups on the structure of finite groups. As an application of our results, some earlier results are generalized.

Keywords: weakly s-supplementally embedded subgroup; supersolvable group; nilpotent group

2010 *Mathematics subject classification:* Primary 20D10; 20D20

1. Introduction and notation

All groups considered in this paper are finite. We use conventional notions and notation, as in [**11**]. $Z_\infty(G)$ denotes the hypercentre of G, $\mathcal{F}$ stands for a formation, $\mathcal{U}$ and $\mathcal{N}$ denote the classes of all supersolvable groups and nilpotent groups, respectively, $G^{\mathcal{F}}$ denotes the $\mathcal{F}$-residual and $Z_{\mathcal{F}}(G)$ denotes the $\mathcal{F}$-hypercentre of G.

A number of authors have investigated the structure of a group G under the assumption that some minimal subgroups of G satisfy some condition in G. For example, Buckley [**6**] proved that if G is a group of odd order and all minimal subgroups of G are normal in G, then G is supersolvable. Shaalan [**16**] proved that if G is a group and every cyclic subgroup of prime order or order 4 is s-quasinormal in G, then G is supersolvable. Meanwhile, some authors have also considered how minimal subgroups can be embedded in a (p-)nilpotent group. Ito [**11**, Chapter III, Theorem 5.3] has proved that if G is a group of odd order and all minimal subgroups of G lie in the centre of G, then G is nilpotent. Recently, many extensions have been made by using formation theory, such as in [**2**, **7**]. In this paper, we give an extension of the results mentioned above by the weakly s-supplementally embedded property of some minimal subgroups. As an application of our results, some recent results are generalized.

2. Basic definitions and preliminary results

Following Kegel [**12**], a subgroup H of a group G is said to be *s-quasinormal* in G if $HP = PH$ for every Sylow subgroup P of G. Recently, Ballester-Bolinches and Pedraza-Aguilera [**3**] generalized the notion of s-quasinormal subgroup to the s-quasinormally embedded subgroup. A subgroup H of G is said to be *s-quasinormally embedded* in G provided every Sylow subgroup of H is a Sylow subgroup of some s-quasinormal subgroup of G. We give the following concept.

Definition 2.1. A subgroup H of a group G is said to be *weakly s-supplementally embedded* in G if there exists a subgroup T of G such that $G = HT$ and $H \cap T \leqslant H_{se}$, where H_{se} is an s-quasinormally embedded subgroup of G contained in H.

Lemma 2.2 (Kegel [12]). *Let H be a subgroup of a group G.*

(i) *If H is s-quasinormal in G, then H is subnormal in G.*

(ii) *Let $N \trianglelefteq G$. If H is s-quasinormal in G, then HN/N is s-quasinormal in G/N.*

(iii) *If H is an s-quasinormal p-subgroup of G for some prime p, then $N_G(H) \geqslant O^p(G)$.*

Lemma 2.3 (Ballester-Bolinches and Pedraza-Aguilera [3, Lemma 1]). *Suppose that U is s-quasinormally embedded in a group G, $H \leqslant G$ and $N \trianglelefteq G$.*

(i) *If $U \leqslant H$, then U is s-quasinormally embedded in H.*

(ii) *UN is s-quasinormally embedded in G and UN/N is s-quasinormally embedded in G/N.*

Lemma 2.4 (Li *et al.* [15, Lemma 2.4]). *Let G be a group and let P be a subgroup of G contained in $O_p(G)$. If P is s-quasinormally embedded in G, then P is s-quasinormal in G.*

Lemma 2.5 (Li and Wang [14, Lemma 2.8]). *Suppose that G is a group and P is a normal p-subgroup of G contained in $Z_\infty(G)$; then $C_G(P) \geqslant O^p(G)$.*

Now we give some basic properties of weakly s-supplementally embedded subgroups.

Lemma 2.6. *Let U be a weakly s-supplementally embedded subgroup and N a normal subgroup of G. Then we have the following.*

(i) *If $U \leqslant H \leqslant G$, then U is weakly s-supplementally embedded in H.*

(ii) *If $N \leqslant U$, then U/N is weakly s-supplementally embedded in G/N.*

(iii) *If $(|U|, |N|) = 1$, then UN/N is weakly s-supplementally embedded in G/N.*

Proof. By the hypothesis, there exist a subgroup T of G and an s-quasinormally embedded subgroup U_{se} of G contained in U such that $G = UT$ and $U \cap T \leqslant U_{se}$.

(i) $H = U(H \cap T)$ and $U \cap (H \cap T) = U \cap T \leqslant U_{se}$. By Lemma 2.3 (i), U_{se} is s-quasinormally embedded in H. Hence, U is weakly s-supplementally embedded in H.

(ii) $G/N = (U/N)(TN/N)$ and $(U/N) \cap (TN/N) = (U \cap TN)/N = (U \cap T)N/N \leqslant U_{se}N/N$. By Lemma 2.3 (ii), $U_{se}N/N$ is s-quasinormally embedded in G/N. Hence, U/N is weakly s-supplementally embedded in G/N.

(iii) It is easy to see that $N \leqslant T$ and $G/N = (UN/N)(T/N)$. Since $(UN/N) \cap (T/N) = (U \cap T)N/N \leqslant U_{se}N/N$, $U_{se}N/N$ is s-quasinormally embedded in G/N by Lemma 2.3 (ii). Hence, UN/N is weakly s-supplementally embedded in G/N. □

Lemma 2.7 (Huppert [11, Chapter III, Theorem 5.2]). *Suppose that G is a group which is not nilpotent but whose proper subgroups are all nilpotent. Then*

(i) *G has a normal Sylow p-subgroup P and $G = PQ$, where Q is a non-normal cyclic Sylow q-subgroup for some prime $q \neq p$,*

(ii) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$,*

(iii) *the exponent of P is p or 4.*

Lemma 2.8. *Let $\mathcal{F}$ be a saturated formation. Assume that G is a group such that G does not belong to $\mathcal{F}$ and there exists a maximal subgroup M of G such that $M \in \mathcal{F}$ and $G = MF(G)$, where $F(G)$ is the Fitting subgroup of G. Then*

(i) *$G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G,*

(ii) *$G^{\mathcal{F}}$ is a p-subgroup for some prime p,*

(iii) *$G^{\mathcal{F}}$ has exponent p if $p > 2$ and exponent at most 4 if $p = 2$,*

(iv) *$G^{\mathcal{F}}$ is either elementary abelian or $(G^{\mathcal{F}})' = Z(G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$ is an elementary abelian group.*

Proof. By [**2**, Proposition 1], (ii)–(iv) hold.

Now we prove statement (i). Since $G = MF(G)$, we have $G/\Phi(G) = M/\Phi(G) \cdot F(G)/\Phi(G)$. By [**11**, Chapter III, Theorem 4.5], $F(G)/\Phi(G)$ is the product of all solvable minimal normal subgroups of $G/\Phi(G)$. Thus, there exists a minimal normal subgroup $H/\Phi(G)$ of $G/\Phi(G)$ such that $H/\Phi(G) \nleq M/\Phi(G)$. The maximality of M in G implies that $G = MH$. Since $G/H \cong M/(M \cap H) \in \mathcal{F}$, we have $G^{\mathcal{F}} \leqslant H$. Since $\mathcal{F}$ is a saturated formation and G does not belong to $\mathcal{F}$, we have $G^{\mathcal{F}} \nleq \Phi(G)$ and hence $\Phi(G) < G^{\mathcal{F}}\Phi(G) \leqslant H$. But $G^{\mathcal{F}}\Phi(G)/\Phi(G) \lhd G/\Phi(G)$, so by the minimality of $H/\Phi(G)$ in $G/\Phi(G)$, we have $G^{\mathcal{F}}\Phi(G) = H$. Hence, $G = MH = MG^{\mathcal{F}}\Phi(G) = MG^{\mathcal{F}}$. Since $\Phi(G^{\mathcal{F}}) \leqslant \Phi(G) \leqslant M$, without loss of generality we may assume that $\Phi(G^{\mathcal{F}}) = 1$. Then $G^{\mathcal{F}}$ is elementary abelian. By the Krull–Schmidt Theorem, it is easy to obtain that $G^{\mathcal{F}}$ is a direct product of some minimal normal subgroups of G. Since $G^{\mathcal{F}} \nleq M$, there exists a minimal normal subgroup N of G such that $N \leqslant G^{\mathcal{F}}$ and $N \nleq M$. By the maximality of M in G, we get $G = MN$. Since $G/N \cong (M/(M \cap N) \in \mathcal{F}$, we have $G^{\mathcal{F}} \leqslant N$ and so $G^{\mathcal{F}} = N$; hence, (i) holds. □

Lemma 2.9 (Skiba [17, Lemma 2.16]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G be a group with a normal subgroup E such that $G/E \in \mathcal{F}$. If E is cyclic, then $G \in \mathcal{F}$.*

3. Main results

Theorem 3.1. *Let H be a normal subgroup of G such that G/H is supersolvable. If every cyclic subgroup $\langle x\rangle$ of any non-cyclic Sylow subgroup of H with prime order or order 4 (if the Sylow 2-subgroup of H is non-abelian) not having a supersolvable supplement in G is weakly s-supplementally embedded in G, then G is supersolvable.*

Proof. Assume that the result is false and consider a counter-example (G, H) for which $|G| + |H|$ is minimal. Then we have the following.

Step 1 (every proper subgroup of *G* is supersolvable). Let K be a proper subgroup of G and let $\langle x\rangle$ be a cyclic subgroup of any non-cyclic Sylow subgroup of $H \cap K$ with prime order. It is clear that $\langle x\rangle$ is also a cyclic subgroup of a non-cyclic Sylow subgroup of H with prime order. By the hypothesis, $\langle x\rangle$ either is weakly s-supplementally embedded or has a supersolvable supplement in G. If $\langle x\rangle$ has a supersolvable supplement T in G, then $\langle x\rangle$ has a supersolvable supplement $K \cap T$ in K. If $\langle x\rangle$ is weakly s-supplementally embedded in G, then it is weakly s-supplementally embedded in K by Lemma 2.6. If the Sylow 2-subgroups of $H \cap K$ are non-abelian, let $\langle y\rangle$ be a cyclic subgroup of $H \cap K$ with order 4. It is clear that at this time the Sylow 2-subgroups of H are also non-abelian and $\langle y\rangle$ is a cyclic subgroup of H with order 4. Then, by the hypothesis, $\langle y\rangle$ either is weakly s-supplementally embedded or has a supersolvable supplement in G. With an argument similar to that above, we also have that $\langle y\rangle$ either is weakly s-supplementally embedded or has a supersolvable supplement in K. Hence, the hypothesis holds for $(K, H \cap K)$. The minimal choice of G implies that K is supersolvable. Thus, we have proved that G is not supersolvable but every proper subgroup of G is supersolvable. A well-known result of Doerk [**8**] implies that there exists a normal Sylow p-subgroup P of G such that $G = PM$, where M is a supersolvable maximal subgroup of G, and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$. Moreover, the exponent of P is p if $p > 2$ and the exponent of P is at most 4 if $p = 2$.

Step 2 ($P = H$ is not cyclic). Now G/P is a homomorphic image of M, and therefore supersolvable. By the hypothesis, G/H is supersolvable, so $G/(P \cap H)$ is supersolvable. It is clear that $(G, P \cap H)$ satisfies the hypothesis of the theorem. If $P \cap H < H$, then G would be supersolvable by the choice of the pair (G, H). Hence, $P \cap H = H$, i.e. H is a p-group. Since $H \trianglelefteq G$ and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, it follows that either $H\Phi(P) = \Phi(P)$ or $H\Phi(P) = P$. In the former case, $H \leqslant \Phi(P) \leqslant \Phi(G)$, so $G/\Phi(G)$ and consequently also G are supersolvable: a contradiction. So $H\Phi(P) = P$, which yields that $H = P$. Recall that G/P is supersolvable; if P is cyclic, then G would be supersolvable: a contradiction.

Step 3 ($\langle x\rangle$ is *s*-quasinormal in *G* for any element $x \in P$). Let $1 \neq x \in P$; then $\langle x\rangle$ is a cyclic group with prime order or order 4 by Step 1. Let T be any supplement of $\langle x\rangle$ in G. Then $G = \langle x\rangle T$ and $P = P \cap G = P \cap \langle x\rangle T = \langle x\rangle(P \cap T)$. Since $P/\Phi(P)$ is abelian, $(P \cap T)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$, and hence $(P \cap T)\Phi(P) \trianglelefteq G$. Since $P/\Phi(P)$ is a chief factor of G, $P \cap T \leqslant \Phi(P)$ or $P \cap T = P$. If $P \cap T \leqslant \Phi(P)$ for some supplement T, then $P = \langle x\rangle$ is cyclic, contradicting Step 2. Now assume that $P \cap T = P$ for every supplement T. Then $T = G$ is the unique supplement of $\langle x \rangle$ in G. Since G is not supersolvable, by the hypothesis, $\langle x \rangle$ is weakly s-supplementally embedded in G. Thus, $\langle x \rangle = \langle x \rangle \cap T = \langle x \rangle_{se}$ is s-quasinormally embedded in G. Since $\langle x \rangle \leqslant P \leqslant O_p(G)$, we have that $\langle x \rangle$ is s-quasinormal in G by Lemma 2.4.

Step 4 (the final contradiction). Assume that $|P/\Phi(P)| \neq p$ and let $T/\Phi(P)$ be any non-trivial cyclic subgroup of $P/\Phi(P)$. Let $x \in T \setminus \Phi(P)$ such that $T = \langle x \rangle \Phi(P)$. Since $\langle x \rangle$ is s-quasinormal in G by Step 3, $T/\Phi(P)$ is s-quasinormal in $G/\Phi(P)$ by Lemma 2.2. It follows from [**17**, Lemma 2.11] that $P/\Phi(P)$ has a maximal subgroup that is normal in $G/\Phi(P)$. But this is impossible since $P/\Phi(P)$ is a chief factor of G. Thus, $|P/\Phi(P)| = p$ and P is cyclic: the final contradiction. This contradiction completes the proof of the theorem. □

Theorem 3.2. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that H is a normal subgroup of G such that $G/H \in \mathcal{F}$. If every cyclic subgroup $\langle x \rangle$ of any non-cyclic Sylow subgroup of H with prime order or order 4 (if the Sylow 2-subgroup of H is non-abelian) not having a supersolvable supplement in G is weakly s-supplementally embedded in G, then $G \in \mathcal{F}$.*

Proof. Assume that the result is false and let G be a counter-example of minimal order. Then we have the following.

Step 1 (*H* is supersolvable). By the hypothesis and Lemma 2.6, we have every cyclic subgroup $\langle x \rangle$ of any non-cyclic Sylow subgroup of H with prime order or order 4 (if the Sylow 2-subgroup of H is non-abelian) not having a supersolvable supplement in H is weakly s-supplementally embedded in H. So H is supersolvable by Theorem 3.1.

Step 2. $G^{\mathcal{F}}$ is a p-group for some prime p and $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G, $G^{\mathcal{F}}$ has exponent p if $p > 2$ and exponent at most 4 if $p = 2$.

Let $p = \max \pi(H)$ and $P \in \mathrm{Syl}_p(H)$. Since H is supersolvable, we have $P \,\mathrm{char}\, H \trianglelefteq G$, so $P \trianglelefteq G$. Consider G/P. From Lemma 2.6 we know the hypothesis holds for $(G/P, H/P)$. Then the minimal choice of G implies that $G/P \in \mathcal{F}$; thus, $G^{\mathcal{F}} \leqslant P$ is a p-group. Since $\mathcal{F}$ is a saturated formation and $G \notin \mathcal{F}$, $G^{\mathcal{F}} \nleqslant \Phi(G)$. Let M be a maximal subgroup of G such that $G^{\mathcal{F}} \nleqslant M$; then $G = MG^{\mathcal{F}} = MF(G)$. Since $M/(M \cap H) \cong MH/H = G/H \in \mathcal{F}$, a trivial argument shows that the hypothesis holds for $(M, M \cap H)$. The minimal choice of G implies that $M \in \mathcal{F}$. Now, by Lemma 2.8, we have that $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G, and $G^{\mathcal{F}}$ has exponent p when $p > 2$ and exponent at most 4 when $p = 2$.

With an argument similar to the one in the proof of Theorem 3.1 (Step 3), we have the following.

Step 3. $\langle x \rangle$ is s-quasinormal in G for any element $x \in G^{\mathcal{F}}$.

Step 4 (the final contradiction). Let $T/\Phi(G^{\mathcal{F}})$ be any non-trivial cyclic subgroup of $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ and $x \in T \setminus \Phi(G^{\mathcal{F}})$. Then $T = \langle x \rangle \Phi(G^{\mathcal{F}})$. Since $\langle x \rangle$ is s-quasinormal in G by Step 3, $T/\Phi(G^{\mathcal{F}})$ is s-quasinormal in $G/\Phi(G^{\mathcal{F}})$ by Lemma 2.2. It follows from [**17**, Lemma 2.11] that $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ has a maximal subgroup which is normal in $G/\Phi(G^{\mathcal{F}})$.

Since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G, we have $|G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})| = p$ and $G^{\mathcal{F}}$ is cyclic. So $G \in \mathcal{F}$ by Lemma 2.9: the final contradiction. This contradiction completes the proof of the theorem. □

Theorem 3.3. *Let N be a normal subgroup of a group G such that G/N is nilpotent. If every cyclic subgroup of N with prime order is contained in $Z_\infty(G)$ and every cyclic subgroup of N with order 4 not having a supersolvable supplement in G is weakly s-supplementally embedded in G, then G is nilpotent.*

Proof. Assume that the result is false and let G be a counter-example of minimal order. Then we have the following.

Step 1 (every proper subgroup of G is nilpotent). Let H be a proper subgroup of G. Since G/N is nilpotent, $H/(H \cap N) \cong HN/N \leqslant G/N$ is nilpotent. Every subgroup of $H \cap N$ of prime order is contained in $Z_\infty(G) \cap H \leqslant Z_\infty(H)$. On the other hand, for every cyclic subgroup K of order 4 of $H \cap N$, if K does not have a supersolvable supplement in H, then K does not have a supersolvable supplement in G. Thus, K is weakly s-supplementally embedded in G by the hypothesis and then it is weakly s-supplementally embedded in H by Lemma 2.6. Therefore, $(H, H \cap N)$ satisfies the hypothesis of the theorem, and the minimal choice of G shows H is nilpotent. Thus, G is a group which is not nilpotent but whose proper subgroups are all nilpotent. Then by Lemma 2.7, $G = PQ$, where P is a normal Sylow p-subgroup and Q a non-normal cyclic Sylow q-subgroup of G for some prime $q \neq p$, $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, and $\exp(P) = p$ when $p > 2$, while $\exp(P)$ is at most 4 when $p = 2$.

Step 2 ($P \leqslant N$, $p = 2$ and $\exp P = 4$). Since both G/N and G/P are nilpotent, $G/(P \cap N) \lesssim G/P \times G/N$ is nilpotent. If $P \nleqslant N$, then $P \cap N < P$ and $Q(P \cap N) < G$. Thus, $Q(P \cap N)$ is nilpotent by Step 1; then $Q(P \cap N) = Q \times (P \cap N)$ and $Q \operatorname{char} Q(P \cap N)$. On the other hand,

$$G/(P \cap N) = (P/(P \cap N))(Q(P \cap N)/(P \cap N))$$

implies that

$$Q(P \cap N)/(P \cap N) \trianglelefteq G/(P \cap N) \quad \text{and} \quad Q(P \cap N) \trianglelefteq G.$$

Therefore, $Q \trianglelefteq G$: a contradiction. Thus, we have $P \leqslant N$. If $\exp P = p$, then $P = P \cap N \leqslant Z_\infty(G)$. Lemma 2.5 implies that $G = P \times Q$: a contradiction. Thus, we have $p = 2$ and $\exp P = 4$.

Step 3 (for every $x \in P \setminus \Phi(P)$, we have $o(x) = 4$). Suppose there exists an $x \in P \setminus \Phi(P)$ such that $o(x) = 2$. Let $M = \langle x \rangle^G$. Then $M \leqslant P$ and $M\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$, so we have $P = M\Phi(P) = M \leqslant Z_\infty(G)$ as $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$: a contradiction.

Step 4 (the final contradiction). By Step 3, every element of $P \setminus \Phi(P)$ is of order 4. Let $x \in P \setminus \Phi(P)$ and let T be a supplement of $\langle x \rangle$ in G. Then $P = P \cap \langle x \rangle T = \langle x \rangle (P \cap T)$. Since $P/\Phi(P)$ is abelian, $(P \cap T)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$ and hence $(P \cap T)\Phi(P) \trianglelefteq G$.

Since $P/\Phi(P)$ is a chief factor of G, $P \cap T \leqslant \Phi(P)$ or $P \cap T = P$. If $P \cap T \leqslant \Phi(P)$ for some supplement T, then $P = \langle x \rangle$ is cyclic, so G is nilpotent by [**11**, Chapter IV, Theorem 2.8]: a contradiction. Now assume that $P \cap T = P$ for every supplement T. Then $T = G$ is the unique supplement of $\langle x \rangle$ in G. If G is supersolvable, then $Q \trianglelefteq G$ since $q > p = 2$. Thus, $G = P \times Q$ is nilpotent: a contradiction. So, by the hypothesis, $\langle x \rangle$ is weakly s-supplementally embedded in G. Thus, $\langle x \rangle = \langle x \rangle \cap T = \langle x \rangle_{se}$ is s-quasinormally embedded in G. Since $\langle x \rangle \leqslant P \leqslant O_p(G)$, $\langle x \rangle$ is s-quasinormal in G. Thus, $\langle x \rangle Q$ is a subgroup of G and by [**11**, Chapter IV, Theorem 2.8], we may assume that $\langle x \rangle Q < G$. So $\langle x \rangle Q$ is nilpotent and $\langle x \rangle Q = \langle x \rangle \times Q$. Therefore, $x \in N_G(Q)$; it follows that $P \leqslant N_G(Q)$ and $G = P \times Q$: the final contradiction. This contradiction completes the proof of the theorem. □

Theorem 3.4. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{N}$. If every cyclic subgroup of $G^{\mathcal{F}}$ with order 4 is weakly s-supplementally embedded in G, then $G \in \mathcal{F}$ if and only if every cyclic subgroup of $G^{\mathcal{F}}$ of prime order lies in the $\mathcal{F}$-hypercentre $Z_{\mathcal{F}}(G)$ of G.*

Proof. We need to prove only the sufficiency. Assume that the result is false and let G be a counter-example of minimal order. Then $G \notin \mathcal{F}$. Let $\langle x \rangle$ be a subgroup of $G^{\mathcal{F}}$ of prime order. Then $\langle x \rangle \leqslant Z_{\mathcal{F}}(G) \cap G^{\mathcal{F}}$, so $\langle x \rangle$ is contained in $Z(G^{\mathcal{F}})$ by [**9**, Chapter IV, Theorem 6.10]. By Lemma 2.6, every cyclic subgroup of $G^{\mathcal{F}}$ of order 4 is weakly s-supplementally embedded in $G^{\mathcal{F}}$. Theorem 3.3 implies that $G^{\mathcal{F}}$ is nilpotent and so solvable. If $G^{\mathcal{F}} \leqslant \Phi(G)$, then $G/\Phi(G) \in \mathcal{F}$; hence, $G \in \mathcal{F}$. This is a contradiction. So there exists a maximal subgroup M of G such that $G = MG^{\mathcal{F}} = MF(G)$. By [**1**, Theorem 3.5], we may choose M to be an $\mathcal{F}$-critical maximal subgroup and $G/M_G \notin \mathcal{F}$.

Since $M/(M \cap G^{\mathcal{F}}) \cong G/G^{\mathcal{F}} \in \mathcal{F}$, we have $M^{\mathcal{F}} \leqslant M \cap G^{\mathcal{F}}$ and so $M^{\mathcal{F}} \leqslant G^{\mathcal{F}}$. Let $1 = N_0 \leqslant N_1 \leqslant \cdots \leqslant N_t = Z_{\mathcal{F}}(G) \leqslant \cdots \leqslant G$ be a chief series of G through $Z_{\mathcal{F}}(G)$. Then $1 = N_0 \cap M \leqslant N_1 \cap M \leqslant \cdots \leqslant N_t \cap M = Z_{\mathcal{F}}(G) \cap M \leqslant \cdots \leqslant M$ is a normal series of M through $Z_{\mathcal{F}}(G) \cap M$. Let f be the canonical definition of $\mathcal{F}$. Then, for any chief factor N_i/N_{i-1}, $1 \leqslant i \leqslant t$, of G and any prime p dividing $|N_i/N_{i-1}|$, we have $G/C_G(N_i/N_{i-1}) \in f(p)$. Since $F(G) \leqslant C_G(N_i/N_{i-1})$ by [**11**, Chapter III, Theorem 4.3], we know $G = MC_G(N_i/N_{i-1})$. Then

$$M/C_M(N_i/N_{i-1}) = M/(M \cap C_G(N_i/N_{i-1})) \cong G/C_G(N_i/N_{i-1}) \in f(p).$$

Since $C_M(N_i/N_{i-1}) \leqslant C_M(N_i \cap M/N_{i-1} \cap M)$, we have $M/C_M(N_i \cap M/N_{i-1} \cap M) \in f(p)$ for any prime p dividing $|(N_i \cap M)/(N_{i-1} \cap M)|$. Refining the above normal series of M to a chief series of M, we obtain $Z_{\mathcal{F}}(G) \cap M \leqslant Z_{\mathcal{F}}(M)$. So every subgroup of $M^{\mathcal{F}}$ of prime order is contained in $Z_{\mathcal{F}}(M)$, and every cyclic subgroup of $M^{\mathcal{F}}$ of order 4 is weakly s-supplementally embedded in M by Lemma 2.6. Hence, M satisfies the hypothesis of the theorem. The minimal choice of G implies that $M \in \mathcal{F}$. By Lemma 2.8, $G^{\mathcal{F}}$ is a p-group for some prime p, and $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a minimal normal subgroup of $G/\Phi(G^{\mathcal{F}})$. Moreover, $G^{\mathcal{F}}$ has exponent p if $p > 2$ and exponent at most 4 if $p = 2$.

If $\exp G^{\mathcal{F}} = p$, then $G^{\mathcal{F}} = \Omega_1(G^{\mathcal{F}}) \leqslant Z_{\mathcal{F}}(G)$ by the hypothesis; this would imply $G \in \mathcal{F}$: a contradiction. So we have $p = 2$ and $\exp G^{\mathcal{F}} = 4$. If there exists an $x \in G^{\mathcal{F}} \setminus \Phi(G^{\mathcal{F}})$ and $o(x) = 2$, denote $H = \langle x \rangle^G$; then $H \trianglelefteq G$ and $H \leqslant \Omega_1(G^{\mathcal{F}}) \leqslant Z_{\mathcal{F}}(G)$. On the other hand, $G^{\mathcal{F}} = H\Phi(G^{\mathcal{F}}) = H$ since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a minimal normal subgroup of $G/\Phi(G^{\mathcal{F}})$; this is a contradiction. So, for any $x \in G^{\mathcal{F}} \setminus \Phi(G^{\mathcal{F}})$, we have $o(x) = 4$. Then $\langle x\rangle$ is weakly s-supplementally embedded in G by the hypothesis.

Let T be any supplement of $\langle x\rangle$ in G. Then $G^{\mathcal{F}} = G^{\mathcal{F}} \cap \langle x\rangle T = \langle x\rangle(G^{\mathcal{F}} \cap T)$. Since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is abelian, $(G^{\mathcal{F}} \cap T)\Phi(G^{\mathcal{F}})/\Phi(G^{\mathcal{F}}) \trianglelefteq G/\Phi(G^{\mathcal{F}})$ and hence $(G^{\mathcal{F}} \cap T)\Phi(G^{\mathcal{F}}) \trianglelefteq G$. Since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G, $G^{\mathcal{F}} \cap T \leqslant \Phi(G^{\mathcal{F}})$ or $G^{\mathcal{F}} \cap T = G^{\mathcal{F}}$.

If $G^{\mathcal{F}} \cap T \leqslant \Phi(G^{\mathcal{F}})$ for some supplement T, then $\langle x\rangle = G^{\mathcal{F}}$ is s-quasinormal in G. If $G^{\mathcal{F}} \cap T = G^{\mathcal{F}}$ for every supplement T, then $T = G$ is the unique supplement of $\langle x\rangle$ in G. So $\langle x\rangle = \langle x\rangle \cap T = \langle x\rangle_{se}$ is s-quasinormally embedded in G. Since $\langle x\rangle \leqslant G^{\mathcal{F}} \leqslant O_p(G)$, we also have $\langle x\rangle$ is s-quasinormal in G by Lemma 2.4.

Thus, for any $q \in \pi(G)$, $q \neq 2$, $\langle x\rangle$ is normalized by every Sylow q-subgroup Q of M. So Q acts on $\langle x\rangle$ by conjugation. But the automorphism group of the cyclic group of order 4 is the cyclic group of order 2, so Q acts trivially on $\langle x\rangle$ and Q centralizes $\langle x\rangle$. Thus, $\langle x\rangle$ is centralized by $O^2(M)$; this implies that $G^{\mathcal{F}}$ is centralized by $O^2(M)$. Hence, $O^2(M) \trianglelefteq G$ as $G = MG^{\mathcal{F}}$. Thus, it follows that G/M_G is a 2-group. Therefore, $G/M_G \in \mathcal{F}$ since $\mathcal{N} \subseteq \mathcal{F}$: the final contradiction. This contradiction completes the proof of the theorem. □

4. Some applications

Following [**17**], a subgroup H of a group G is *weakly s-supplemented* in G if G has a subgroup T such that $HT = G$ and $H \cap T \leqslant H_{sG}$, where H_{sG} is the largest s-quasinormal subgroup of G contained in H. From the definition, we know that every weakly s-supplemented subgroup is a weakly s-supplementally embedded subgroup. Furthermore, all subgroups, including normal subgroups, quasinormal (permutable) subgroups, s-quasinormal subgroups, s-quasinormally embedded subgroups, c-normal subgroups, c-supplemented subgroups, Q-supplemented subgroups and c^*-normal subgroups, are weakly s-supplementally embedded subgroups. Hence, Theorems 3.2 and 3.4 generalize many earlier results. For example, [**2**, Theorem 2], [**16**, Theorem 3.4], [**18**, Theorem 4.2], [**5**, Theorem 4.1] and [**13**, Theorem 3.4] are the corollaries of Theorem 3.2; [**14**, Theorem 4.2], [**5**, Theorem 4.3], [**4**, Theorem 3.1] and [**10**, Theorem B] are the corollaries of Theorem 3.4.

Acknowledgements. This work was supported by the National Natural Science Foundation of China (Grant 10871032) and the Natural Science Foundation of Jiangsu Province (Grant BK2008156). The authors are very grateful for the helpful suggestions of the referee and editor.

References

1. A. Ballester-Bolinches, $\mathcal{H}$-normalizers and local definitions of saturated formations of finite groups, *Israel J. Math.* **67** (1989), 312–326.
2. A. Ballester-Bolinches and M. C. Pedraza-Aguilera, On minimal subgroups of finite groups, *Acta Math. Hungar.* **73**(4) (1996), 335–342.
3. A. Ballester-Bolinches and M. C. Pedraza-Aguilera, Sufficient conditions for supersolvability of finite groups, *J. Pure Appl. Alg.* **127** (1998), 113–118.

4. A. Ballester-Bolinches and Y. Wang, Finite groups with some c-normal minimal subgroups, *J. Pure Appl. Alg.* **153** (2000), 121–127.
5. A. Ballester-Bolinches, Y. Wang and X. Guo, C-supplemented subgroups of finite groups, *Glasgow Math. J.* **42** (2000), 383–389.
6. J. Buckley, Finite groups whose minimal subgroups are normal, *Math. Z.* **116** (1970), 15–17.
7. J. B. Derr, W. E. Deskins and N. P. Mukherjee, The influence of minimal p-subgroups on the structure of finite groups, *Arch. Math.* **45** (1985), 1–4.
8. K. Doerk, Minimal nicht uberauflosbare, endliche Gruppen, *Math. Z.* **91** (1966), 198–205.
9. K. Doerk and T. Hawkes, *Finite soluble groups* (Walter de Gruyter, Berlin, 1992).
10. W. Guo, F. Xie and B. Li, Some open questions in the theory of generalized permutable subgroups, *Sci. China* A **52** (2009), 1–13.
11. B. Huppert, *Endliche Gruppen*, Volume I (Springer, 1967).
12. O. H. Kegel, Sylow-Gruppen und abnormalteiler endlicher Gruppen, *Math. Z.* **78** (1962), 205–221.
13. Y. Li, G-covering systems of subgroups for the class of supersolvable groups, *Sb. Math. J.* **46**(3) (2006), 474–480.
14. Y. Li and Y. Wang, On π-quasinormally embedded subgroups of finite group, *J. Alg.* **281** (2004), 109–123.
15. Y. Li, Y. Wang and H. Wei, On p-nilpotency of finite groups with some subgroups π-quasinormally embedded, *Acta Math. Hungar.* **108**(4) (2005), 283–298.
16. A. Shaalan, The influence of s-quasinormality of some subgroups on the structure of a finite group, *Acta Math. Hungar.* **56** (1990), 287–293.
17. A. N. Skiba, On weakly s-permutable subgroups of finite groups, *J. Alg.* **315** (2007), 192–209.
18. Y. Wang, On c-normality and its properties, *J. Alg.* **180** (1996), 954–965.