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On the Kodaira Dimension of the Moduli Space of $K3$ Surfaces II

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Abstract. We show that the Kodaira dimension of the moduli space of polarized $K3$ surfaces of degree $2n$ is nonnegative if $n = 42$, 43, 51, 53, 55, 57, 59, 61, 66, 67, 69, 74, 83, 85, 105, 119 or 133. We use an automorphic form associated with the fake monster Lie algebra constructed by Borcherds.

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1. Introduction

Let $\mathcal{K}_{2n}$ be the coarse moduli space of $K3$ surfaces with a primitive polarization of degree $2n$. It follows from the Torelli theorem for $K3$ surfaces that $\mathcal{K}_{2n}$ is described as an arithmetic quotient $\mathcal{D}_{2n}/\Gamma_{2n}$ where $\mathcal{D}_{2n}$ is a 19-dimensional bounded symmetric domain of type IV and Γ_{2n} is an arithmetic subgroup of $\mathrm{O}(2, 19)_{\mathbf{Q}}$. Recently, Borcherds [2] constructed a remarkable automorphic form Φ of weight 12 on a 26-dimensional bounded symmetric domain of type IV. By dividing Φ by a product of linear functions vanishing on $\mathcal{D}_{2n}$ and by restricting it to $\mathcal{D}_{2n}$, we have an automorphic form Φ_{2n} on $\mathcal{D}_{2n}$ with respect to Γ_{2n} (see Borcherds *et al*. [3]). The purpose of this note is to show the following theorem:

THEOREM. *Assume that* $n = 42, 43, 51, 53, 55, 57, 59, 61, 66, 67, 69, 74, 83, 85, 105, 119$ *or* 133. *Then* Φ_{2n} *is a cusp form of weight* 19 *on* $\mathcal{D}_{2n}$ *with respect to* Γ_{2n}.

An automorphic form of weight 19 on $\mathcal{D}_{2n}$ gives a section of the canonical line bundle of the smooth locus of $\mathcal{D}_{2n}/\Gamma_{2n}$. Moreover, it is known that a cusp form of weight 19 gives a global section of the canonical line bundle of a smooth model of a compactification of $\mathcal{D}_{2n}/\Gamma_{2n}$ (Bauermann [1]). Thus we have the corollary:

COROLLARY 1.1. *Assume that n is the same as above. Then the Kodaira dimension of* $\mathcal{K}_{2n} = \mathcal{D}_{2n}/\Gamma_{2n}$ *is nonnegative.*

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If $m = n{\cdot}l^2$ for some natural number l, then $\Gamma_{2n} \supset \Gamma_{2m}$ and we have a dominant map from $\mathcal{D}_{2m}/\Gamma_{2m}$ to $\mathcal{D}_{2n}/\Gamma_{2n}$ (O'Grady [12], Kondō [6], Lemma 3.2). Hence, we have

COROLLARY 1.2. *Assume that n is the same as above and $m = n{\cdot}l^2$ for some l. Then the Kodaira dimension of $\mathcal{K}_{2m}$ is nonnegative.*

We remark that if $1 \leqslant n \leqslant 11, 17$ or 19, then $\mathcal{K}_{2n}$ is unirational and, in particular, its Kodaira dimension is $-\infty$. For $n \leqslant 4$, this is classical and for other n's, this was shown by Mukai [7], [8], [9]. On the other hand, in the paper [6], the author proved that if $n = p^2$ and p is a sufficiently large prime number, then $\mathcal{K}_{2n}$ is of a general type, i.e. the Kodaira dimension of $\mathcal{K}_{2n}$ is equal to dim $\mathcal{K}_{2n}$ (= 19). However, the author did not have an effective estimate of p. Also, Gritsenko [5] proved that the Kodaira dimension of the double cover $\mathcal{D}_{2n}/\bar{\Gamma}_{2n}$ of $\mathcal{D}_{2n}/\Gamma_{2n}$ is nonnegative if n is not perfect square and positive if n is square free and $n > 3$ by using the lifting of Jacobi forms, where $\bar{\Gamma}_{2n} = \Gamma_{2n} \cap \mathrm{SO}(2, 19)_{\mathbf{Q}}$.

2. Preliminaries

A *lattice* L is a free $\mathbf{Z}$-module of finite rank endowed with an integral symmetric bilinear form $\langle\ ,\ \rangle$. If L_1 and L_2 are lattices, then $L_1 \oplus L_2$ denotes the orthogonal direct sum of L_1 and L_2. An isomorphism of lattices preserving the bilinear forms is called an *isometry*. A sublattice S of L is called *primitive* if L/S is torsion free.

A lattice L is *even* if $\langle x, x\rangle$ is even for each $x \in L$. A lattice L is *nondegenerate* if the discriminant $d(L)$ of its bilinear form is nonzero, and *unimodular* if $d(L) = \pm 1$. If L is a nondegenerate lattice, the *signature* of L is a pair (t_+, t_-) where $t_\pm$ denotes the multiplicity of the eigenvalues ± 1 for the quadratic form on $L \otimes \mathbf{R}$.

Let L be a nondegenerate even lattice. The bilinear form of L determines a canonical embedding $L \subset L^* = \mathrm{Hom}(L, \mathbf{Z})$. The factor group L^*/L, which is denoted by A_L, is an Abelian group of order $|d(L)|$. We extend the bilinear form on L to the one on L^*, taking value in $\mathbf{Q}$, and define

$$q_L \colon A_L \to \mathbf{Q}/2\mathbf{Z}, \quad q_L(x + L) = \langle x, x\rangle + 2\mathbf{Z}\,(x \in L^*).$$

We call q_L the *discriminant quadratic form* of L.

We denote by H the hyperbolic lattice defined by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which is an even unimodular lattice of signature (1,1), and by A_m, D_n or E_l an even negative definite lattice associated with the Dynkin diagram of type A_m, D_n or E_l ($m \geqslant 1$, $n \geqslant 4$, $l = 6, 7, 8$). A *root* of a lattice L is a vector r in L with $\langle r, r\rangle = -2$. For an even negative definite lattice L, we denote by $R(L)$ the sublattice of L generated by all roots in L which is called the *root sublattice* of L and isometric to a direct sum of some A_m, D_n, E_l.

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Let X be a $K3$ surface. Put $L = H^2(X, \mathbf{Z})$. Then L admits a canonical structure of a lattice induced from the cup product. It is an even unimodular lattice with signature (3,19) and, hence, isometric to $H \oplus H \oplus H \oplus E_8 \oplus E_8$ (e.g., Nikulin [11], Thm. 1.1.1). Let h be a primitive vector of L with $\langle h, h\rangle = 2n$. Then the orthogonal complement of h in L is isometric to $L_{2n} = H \oplus H \oplus E_8 \oplus E_8 \oplus \langle -2n\rangle$. Put $\Omega_{2n} = \{[\omega] \in \mathbf{P}(L_{2n} \otimes \mathbf{C})\colon \langle \omega, \omega\rangle = 0, \langle \omega, \bar{\omega}\rangle > 0\}$. Then Ω_{2n} consists of two connected components. We denote by $\mathcal{D}_{2n}$ either one connected component, which is a bounded symmetric domain of type IV and of dimension 19. Let Γ_{2n} be the group of isometries of L which fix h and preserve the component $\mathcal{D}_{2n}$. Then Γ_{2n} acts on $\mathcal{D}_{2n}$ properly discontinuously and, hence, by Cartan's theorem $\mathcal{D}_{2n}/\Gamma_{2n}$ has a canonical structure of normal analytic space. We call $\mathcal{D}_{2n}$ the *period space* of $K3$ surfaces with a primitive polarization of degree $2n$. It follows from Torelli theorem for $K3$ surfaces that $\mathcal{D}_{2n}/\Gamma_{2n}$ is a coarse moduli space of $K3$ surface with a primitive polarization of degree $2n$.

A *boundary component* of $\mathcal{D}_{2n}$ is a maximal connected complex analytic subset in $\bar{\mathcal{D}}_{2n} \setminus \mathcal{D}_{2n}$ where $\bar{\mathcal{D}}_{2n}$ is the topological closure of $\mathcal{D}_{2n}$ in $\{[\omega] \in \mathbf{P}(L_{2n} \otimes \mathbf{C})\colon \langle \omega, \omega\rangle = 0\}$. A boundary component is called *rational* if its stabilizer subgroup of $\mathrm{O}(L_{2n} \otimes \mathbf{R})$ is defined over $\mathbf{Q}$. It is known that the set of rational boundary components of $\mathcal{D}_{2n}$ bijectively corresponds to the set of all primitive totally isotropic sublattices of L_{2n} (Scattone [13], (2.1.7)). If E is a totally isotropic sublattice, then the corresponding rational boundary component is defined by $\mathbf{P}(E \otimes \mathbf{C}) \cap \bar{\mathcal{D}}_{2n}$. Since the signature of L_{2n} is (2,19), the dimension of a rational boundary component is either 0 or 1. For simplicity, we now assume that n is square-free. Let F be a rational boundary component and E the corresponding primitive totally isotropic sublattice. If dim $F = 0$, then F is unique modulo Γ_{2n} and if dim $F = 1$, then we have an orthogonal decomposition

$$L_{2n} = H \oplus H \oplus K,$$

where $H \oplus H$ contains E and $K \simeq E^{\perp}/E$ (Scattone [13], Thm. 4.0.1, Lemma 5.2.1). Let $\{e_1, \ldots, e_{21}\}$ be a base of L_{2n} such that

$$(\langle e_i, e_j\rangle)_{1\leqslant i,j\leqslant 21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & Q & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\{e_1, e_2\}$ is a base of E and Q is the intersection matrix of K. In case $\dim F = 0$, F is unique modulo Γ_{2n} and, hence, we may assume that E is generated by e_1. Denote by q_0 (resp. q_1) the symmetric bilinear form associated with the matrix $(\langle e_i, e_j\rangle)_{2\leqslant i,j\leqslant 20}$ (resp. Q). Also denote by L_0 the sublattice $e_1^{\perp}/e_1$

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of L_{2n}. Let $z = \Sigma z_i e_i$ be a homogeneous coordinate of $\mathbf{P}(L_{2n} \otimes \mathbf{C})$. Then we have the following unbounded realization of $\mathcal{D}_{2n}$.

(1.1) In case dim $F = 0$:

$$\mathcal{D}_{2n} \simeq \{z = (z_2, \ldots, z_{20}) \in \mathbf{C}^{19}\colon (\mathrm{Im}(z_i)) \in C(F)\},$$

where $C(F) = \{y = (y_2, \ldots, y_{20}) \in \mathbf{R}^{19}\colon q_0(y, y) > 0, y_2 > 0\}$.

(1.2) In case dim $F = 1$:

$$\mathcal{D}_{2n} \simeq \{(z, w, \tau) = (z_2, z_3, \ldots, z_{19}, z_{20}) \\ \in H^+ \times \mathbf{C}^{17} \times H^+\colon \mathrm{Im}(z) - \mathrm{Re}(h_\tau(w, w)) > 0\},$$

where $h_\tau(w, w') = \{-q_1(w, \bar{w}') + q_1(w, w')\}/4\,\mathrm{Im}\,(\tau)$ (see Kondo [6], Sect. 2).

Let f be an automorphic form on $\mathcal{D}_{2n}$ with respect to Γ_{2n}. Then f has an expansion with respect to (1.1) which is called the *Fourier expansion* of f at F:

$$f = \sum_{\rho \in \bar{C}(F) \cap L_0^*} c_\rho \exp(2\pi\sqrt{-1} \cdot q_0(\rho, z)),$$

where $\bar{C}(F)$ is the closure of $C(F)$. Also with respect to (1.2), f has an expansion which is called the *Fourier–Jacobi expansion* of f at F:

$$f = \sum_{m \geqslant 0} \theta_m(\tau, w) \exp(2\pi\sqrt{-1} mz).$$

We call f a *cusp form* if for any rational boundary components the initial term of the above expansion vanishes, i.e. $c_0 \equiv 0$ if dim $F = 0$ and $\theta_0(\tau, w) \equiv 0$ if dim $F = 0$. Note that $\theta_0(\tau, w)$ does not depend on w. For more details, we refer the reader to Borcherds [2], Gritsenko [5], Kondo [6].

3. Automorphic Forms on the Period Domain of $K3$ Surfaces

In the following, as in Borcherds *et al*. [3], we identify $L_{2n} = H \oplus H \oplus E_8 \oplus E_8 \oplus \langle -2n \rangle$ with the sublattice $H \oplus H \oplus E_8 \oplus E_8 \oplus \mathbf{Z}v$ of $II_{2,26} = H \oplus H \oplus E_8 \oplus E_8 \oplus E_8$, where v is a primitive vector in E_8 with $\langle v, v \rangle = -2n$. Let U be the orthogonal complement of v in E_8. Let $\mathcal{D}$ be the 26-dimensional bounded symmetric domain of type IV associated with $II_{2,26}$. Under the above identification, $\mathcal{D}_{2n} \subset \mathcal{D}$. In the paper [2], Example 2 of section 10, Borcherds constructed an automorphic form Φ of weight 12 on $\mathcal{D}$ with respect to the group of isometries of $II_{2,26}$ which preserve $\mathcal{D}$. The restriction of Φ on $\mathcal{D}_{2n}$ is identically 0 whenever U contains a root. So first divide Φ by a product of linear functions vanishing on the divisors of each of these roots in U, and then restrict it on $\mathcal{D}_{2n}$. Then we have an automorphic form Φ_{2n} on $\mathcal{D}_{2n}$ with respect to Γ_{2n} which has the following properties (Borcherds *et al*. [3], also see Borcherds [2], Thm. 13.1 and its proof):

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(2.1) Φ_{2n} vanishes on the hyperplanes of $\mathcal{D}_{2n}$ each of which is orthogonal to a vector $r' \in L_{2n}^*$ with $-2 \leqslant \langle r', r' \rangle < 0$. Here r' is the projection of a root r of $II_{2,26}$.

(2.2) The weight of Φ_{2n} is equal to the weight $(= 12)$ of Φ plus half the number of roots of U.

Let $\{e_i\}_{1 \leqslant i \leqslant 8}$ be a set of simple roots of E_8 which satisfies the following:

$$\begin{aligned}\langle e_i, e_i \rangle &= -2, \langle e_1, e_2 \rangle = \langle e_2, e_3 \rangle = \langle e_3, e_4 \rangle \\ &= \langle e_3, e_5 \rangle = \langle e_5, e_6 \rangle = \langle e_6, e_7 \rangle = \langle e_7, e_8 \rangle = 1 \quad \text{and} \\ &\langle e_i, e_j \rangle = 0 \quad \text{for other } i, j.\end{aligned}$$

Let R be a root lattice isometric to $A_2 \oplus A_2 \oplus A_1$ or $A_3 \oplus A_1$. Note that the number of roots in R is 14. In the case where $R = A_2 \oplus A_2 \oplus A_1$, we consider R as one of the following sublattices of E_8:

$$\text{(a)}\ \langle e_1, e_2, e_4, e_6, e_7 \rangle, \quad \text{(b)}\ \langle e_1, e_2, e_5, e_6, e_8 \rangle, \quad \text{(c)}\ \langle e_1, e_2, e_5, e_7, e_8 \rangle,$$
$$\text{(d)}\ \langle e_1, e_3, e_4, e_6, e_7 \rangle, \quad \text{(e)}\ \langle e_1, e_3, e_4, e_7, e_8 \rangle.$$

In the case where $R = A_3 \oplus A_1$, we consider R as one of the following sublattices of E_8:

$$\text{(f)}\ \langle e_1, e_2, e_3, e_6 \rangle, \quad \text{(g)}\ \langle e_1, e_2, e_3, e_7 \rangle, \quad \text{(h)}\ \langle e_1, e_2, e_3, e_8 \rangle,$$
$$\text{(i)}\ \langle e_2, e_3, e_4, e_7 \rangle, \quad \text{(j)}\ \langle e_2, e_3, e_5, e_7 \rangle, \quad \text{(k)}\ \langle e_2, e_3, e_5, e_8 \rangle,$$
$$\text{(l)}\ \langle e_3, e_4, e_5, e_7 \rangle, \quad \text{(m)}\ \langle e_1, e_5, e_6, e_7 \rangle, \quad \text{(n)}\ \langle e_4, e_5, e_6, e_7 \rangle,$$
$$\text{(o)}\ \langle e_1, e_6, e_7, e_8 \rangle, \quad \text{(p)}\ \langle e_2, e_6, e_7, e_8 \rangle, \quad \text{(q)}\ \langle e_4, e_6, e_7, e_8 \rangle.$$

If $R = \langle e_\alpha, e_\beta, e_\gamma, e_\delta, e_\kappa \rangle$ (resp. $R = \langle e_\alpha, e_\beta, e_\gamma, e_\delta \rangle$), then we take a vector v as a sum:

$$v = e_\lambda^* + e_\mu^* + e_\nu^* \ (\text{resp. } v = e_\kappa^* + e_\lambda^* + e_\mu^* + e_\nu^*),$$

where $\{\alpha, \beta, \ldots, \nu\} = \{1, 2, \ldots, 8\}$ and e_α^* is the dual of e_α (see Bourbaki [4], Planche VII). Since E_8 is unimodular, $v \in E_8$. A direct calculation shows that the number $-\langle v, v \rangle / 2$ is equal to

(a) 61, (b) 55, (c) 67, (d) 43, (e) 66, (f) 57, (g) 69, (h) 83, (i) 59, (j) 42, (k) 53, (l) 51, (m) 85, (n) 74, (o) 133, (p) 105, (q) 119.

Since $\langle v, v \rangle / 2$ is square free, v is primitive in E_8. Obviously any root in U (= $v^\perp$ in E_8) is contained in R. Hence, the number of roots of U is 14. Thus, we have an automorphic form Φ_{2n} of weight 19 (= 12 + 14/2) by (2.2).

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THEOREM. Φ_{2n} *is a cusp form of weight* 19.

Proof. It suffices to see that Φ_{2n} vanishes on the top dimensional (= 1-dim.) rational boundary components of $\mathcal{D}_{2n}$. Since n is square free, the set of primitive totally isotropic sublattices of rank two corresponds to the set of isomorphy classes of even negative definite lattices K such that $L_{2n} \simeq H \oplus H \oplus K$ (Scattone [13], Thm. 5.0.2). Here totally isotropic sublattice is contained in $H \oplus H$. Since $q_K \simeq q_{L_{2n}} \simeq -q_U$, we have an even negative definite unimodular lattice N of rank 24 such that K and U are primitive sublattices of N, $K^{\perp} \simeq U$ and $U^{\perp} \simeq K$ (Nikulin [11], Cor. 1.6.2). Thus, each K is obtained as the orthogonal complement of a primitive sublattice U of some N. Let $U \subset N$ be a primitive embedding. Since U contains R, N also contains a root. Such lattices N are classified into 23 isomorphism classes which are characterized by its root sublattice $R(N)$ (Niemeier [10]). Since rank $R(N) = 24$, there exists a root r of N whose projection r' into K^* has a negative norm: $-2 \leqslant \langle r', r' \rangle < 0$. Note that the totally isotropic primitive sublattice of rank 2 corresponding to K is contained in the orthogonal complement of r'. It follows from (2.1) that Φ_{2n} vanishes the hyperplane orthogonal to r'. Let

$$\Phi_{2n} = \Sigma\theta_m(\tau, w)\cdot \exp(2\pi\sqrt{-1}mz)$$

be the Fourier–Jacobi expansion of Φ_{2n} at this boundary component. Since

$$\theta_0(\tau, w) = \lim_{\mathrm{Im}(z)\to\infty} \Phi_{2n},$$

and $\theta_0(\tau, w)$ does not depend on w, the above implies that $\theta_0(\tau, w) \equiv 0$. □

Remark. There are another embeddings of R into E_8. In the above, we take embeddings such that the number $\langle v, v\rangle/2$ is square free. Also we can take v as a linear combination of $e^*_\lambda, e^*_\mu, e^*_\nu$ (resp. $e^*_\kappa, e^*_\lambda, e^*_\mu, e^*_\nu$) with positive coefficients. Then Φ_{2n} is a cusp form of weight 19 whenever $\langle v, v\rangle/2$ is square free and Corollaries 1, 2 hold for these n.

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