Appendix B
Spherically symmetric solutions and Birkhoff's theorem

We wish to consider Einstein's equations in the case of a spherically symmetric space–time. One might regard the essential feature of a spherically symmetric space–time as the existence of a world-line $\mathscr{L}$ such that the space–time is spherically symmetric about $\mathscr{L}$. Then all points on each spacelike two-sphere $\mathscr{S}_d$ centred on any point p of $\mathscr{L}$, defined by going a constant distance d along all geodesics through p orthogonal to $\mathscr{L}$, are equivalent. If one permutes directions at p by use of the orthogonal group $SO(3)$ leaving $\mathscr{L}$ invariant, the space–time is, by definition, unchanged, and the corresponding points of $\mathscr{S}_d$ are mapped into themselves; so the space–time admits the group $SO(3)$ as a group of isometries, with the orbits of the group the spheres $\mathscr{S}_d$. (There could be particular values of d such that the surface $\mathscr{S}_d$ was just a point p'; then p' would be another centre of symmetry. There can be at most two points (p' and p itself) related in this way.)

However, there might not exist a world-line like $\mathscr{L}$ in some of the space–times one would wish to regard as spherically symmetric. In the Schwarzschild and Reissner–Nordström solutions, for example, space–time is singular at the points for which $r = 0$, which might otherwise have been centres of symmetry. We shall therefore take the existence of the group $SO(3)$ of isometries acting on two-surfaces like $\mathscr{S}_d$ as the characteristic feature of a spherically symmetric space–time. Thus we shall say that space–time is *spherically symmetric* if it admits the group $SO(3)$ as a group of isometries, with the group orbits spacelike two-surfaces. These orbits are then necessarily two-surfaces of constant positive curvature.

For each point q in any orbit $\mathscr{S}(q)$, there is a one-dimensional subgroup I_q of isometries which leaves q invariant (when there is a central axis $\mathscr{L}$, this is the group of rotations about p which leaves the geodesic pq invariant). The set $\mathscr{C}(q)$ of all geodesics orthogonal to $\mathscr{S}(q)$ at q locally form a two-surface left invariant by I_q (since I_q, which permutes directions in $\mathscr{S}(q)$ about q, leaves invariant directions perpendicular to $\mathscr{S}(q)$). At any other point r of $\mathscr{C}(q)$, I_q again permutes directions orthogonal to $\mathscr{C}(q)$, as it leaves $\mathscr{C}(q)$ invariant; since I_q must operate in the group orbit $\mathscr{S}(r)$ through r, this orbit is orthogonal to $\mathscr{C}(q)$. Thus (Schmidt (1967)) the group orbits $\mathscr{S}$ are orthogonal to the surfaces $\mathscr{C}$. Further these surfaces define locally a one–one map between the group orbits, where the image $f(q)$ of q in $\mathscr{S}(r)$ is the intersection of $\mathscr{C}(q)$ and $\mathscr{S}(r)$. Since this map is invariant under I_q, vectors of equal magnitude in $\mathscr{S}(q)$ at q are mapped into vectors of equal magnitude in $\mathscr{S}(r)$ at $f(q)$; and since all the points of $\mathscr{S}(q)$ are equivalent, the same magnitude multiplication factor occurs for the maps of vectors from any point in $\mathscr{S}(q)$ to its image in $\mathscr{S}(r)$. Thus (Schmidt (1967)) the orthogonal surfaces $\mathscr{C}$ map the trajectories $\mathscr{S}$ conformally onto each other.

If one chooses coordinates $\{t, r, \theta, \phi\}$ so that the group orbits $\mathscr{S}$ are the surfaces $\{t, r = \text{constant}\}$ and the orthogonal surfaces $\mathscr{C}$ are the surfaces $\{\theta, \phi = \text{constant}\}$, it now follows that the metric takes the form $\mathrm{d}s^2 = \mathrm{d}\tau^2(t,r) + Y^2(t,r)\,\mathrm{d}\Omega^2(\theta,\phi)$, where $\mathrm{d}\tau^2$ is an indefinite two-surface and $\mathrm{d}\Omega^2$ is a surface of positive constant curvature. If one further chooses the functions t, r so that the curves $\{t = \text{constant}\}$, $\{r = \text{constant}\}$ are orthogonal in the two-surfaces $\mathscr{C}$ (cf. Bergmann, Cahen and Komar (1965)), one can write the metric in the form

$$\mathrm{d}s^2 = \frac{-\mathrm{d}t^2}{F^2(t,r)} + X^2(t,r)\,\mathrm{d}r^2 + Y^2(t,r)\,(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \qquad \text{(A 1)}$$

(Note that this still leaves the freedom to choose arbitrarily either r or t in these surfaces.)

Let an observer moving along the t-lines measure an energy density μ, an isotropic pressure p, an energy flux q, and no anisotropic pressures. Then the field equations for the metric (A 1) may be written in the form

$$-8\pi q = \frac{2X}{F}\left(\frac{Y^{\cdot\prime}}{Y} - \frac{X^{\cdot}Y'}{XY} + \frac{Y^{\cdot}F'}{YF}\right), \qquad \text{(A 2)}$$

$$8\pi\mu = \frac{1}{Y^2} + \frac{2}{X}\left(-\frac{Y'}{XY}\right)' - 3\left(\frac{Y'}{XY}\right)^2 + 2F^2\frac{X^{\cdot}Y^{\cdot}}{XY} + F^2\left(\frac{Y^{\cdot}}{Y}\right)^2, \qquad \text{(A 3)}$$

$$-8\pi p = \frac{1}{Y^2} + 2F\left(F\frac{Y^{\cdot}}{Y}\right)^{\cdot} + 3\left(\frac{Y^{\cdot}}{Y}\right)^2 F^2 + \frac{2}{X^2}\frac{Y'F'}{YF} - \left(\frac{Y'}{XY}\right)^2, \qquad \text{(A 4)}$$

$$4\pi(\mu + 3p) = \frac{1}{X}\left(-\frac{F'}{FX}\right)' - F\left(F\frac{X^{\cdot}}{X}\right)^{\cdot} - 2F\left(F\frac{Y^{\cdot}}{Y}\right)^{\cdot} - F^2\left(\frac{X^{\cdot}}{X}\right)^2$$
$$- 2F^2\left(\frac{Y^{\cdot}}{Y}\right)^2 + \frac{1}{X^2}\left(\frac{F'}{F}\right)^2 - \frac{2}{X^2}\frac{Y'F'}{YF}, \qquad \text{(A 5)}$$

where $'$ denotes $\partial/\partial r$ and $^{\cdot}$ denotes $\partial/\partial t$.

We first consider the *empty space* field equations $R_{ab} = 0$; this means that in (A 2)–(A 5) we must set $\mu = p = q = 0$. The local solution depends on the nature of the surfaces $\{Y = \text{constant}\}$; these surfaces may be timelike, spacelike or null, or they may not be defined (if Y is constant). In the exceptional case when $Y^{;a}Y_{;a} = 0$ on some open set $\mathscr{U}$ (this includes the case when Y is constant),

$$\frac{Y'}{X} = FY^{\cdot} \tag{A 6}$$

holds in $\mathscr{U}$. However when (A 6) holds, the value of $Y^{\cdot\prime}$ determined by (A 2) is inconsistent with (A 3). Thus we may consider some point p where $Y^{;a}Y_{;a} < 0$ or $Y^{;a}Y_{;a} > 0$; the same inequality must hold in some open neighbourhood $\mathscr{U}$ of p.

Consider first the situation when $Y^{;a}Y_{;a} < 0$. Then the surfaces $\{Y = \text{constant}\}$ are timelike in $\mathscr{U}$, and one can choose Y to be the coordinate r. (Then r is an *area coordinate*, as the area of the two-surfaces $\{r, t = \text{constant}\}$ is $4\pi r^2$.) Thus $Y^{\cdot} = 0$, $Y' = 1$ and (A 2) shows that $X^{\cdot} = 0$. Further (A 4) shows that $(F'/F)^{\cdot} = 0$, so one can choose a new time coordinate $t'(t)$ in such a way as to set $F = F(r)$. Then one has $F = F(r)$, $X = X(r)$, $Y = r$; the solution is *necessarily static*. Equation (A 3) now shows $\mathrm{d}(r/X^2)/\mathrm{d}r = 1$, so solutions are of the form $X^2 = (1-2m/r)^{-1}$ where $2m$ is a constant of integration. Equation (A 4) can be integrated, with a suitable choice of a constant of integration, to give $F^2 = X^2$, and then (A 5) is identically satisfied. With these forms of F and X the metric (A 1) becomes

$$\mathrm{d}s^2 = -\left(1-\frac{2m}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\left(1-\dfrac{2m}{r}\right)} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2); \tag{A 7}$$

this is the Schwarzschild metric for $r > 2m$.

Now suppose $Y^{;a}Y_{;a} > 0$. Then the surfaces $\{Y = \text{constant}\}$ are spacelike in $\mathscr{U}$, and one can choose Y to be the coordinate t. Then $Y^{\cdot} = 1$, $Y' = 0$ and (A 2) shows $F' = 0$. One can choose the r-coordinate so that $X = X(t)$; then $F = F(t)$, $X = X(t)$, $Y = t$ and the solution is *spatially homogeneous*. Now (A 4) and (A 5) can be integrated to find the solution

$$\mathrm{d}s^2 = -\frac{\mathrm{d}t^2}{\left(\dfrac{2m}{t}-1\right)} + \left(\frac{2m}{t}-1\right)\mathrm{d}r^2 + t^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{A 8}$$

This is part of the Schwarzschild solution inside the Schwarzschild radius, for the transformation $t \to r'$, $r \to t'$ transforms this metric into the form (A 7) with $r' < 2m$. Finally, if the surfaces $\{Y = \text{constant}\}$ are spacelike in some part of an open set $\mathscr{V}$ and timelike in another part, one can obtain solutions (A 8) and (A 7) in these parts, and then join them together across the surfaces where $Y^{;a}Y_{;a} = 0$ as in §5.5, obtaining a part of the maximal Schwarzschild solution which lies in $\mathscr{V}$. Thus we have proved *Birkhoff's theorem*: any C^2 solution of Einstein's empty space equations which is spherically symmetric in an open set $\mathscr{V}$, is locally equivalent to part of the maximally extended Schwarzschild solution in $\mathscr{V}$. (This is true even if the space is C^0, piecewise C^1; see Bergmann, Cahen and Komar (1965).)

We now consider spherically symmetric *static perfect fluid* solutions. Then one can find coordinates $\{t, r, \theta, \phi\}$ such that the metric has the form (A 1), the fluid moves along the t-lines (so $q = 0$), and $F = F(r)$, $X = X(r)$, $Y = Y(r)$. The field equations (A 3), (A 4) now show that if $Y' = 0$, then $\mu + p = 0$; we exclude this as being unreasonable for a physical fluid, so we assume $Y' \neq 0$. One may therefore again choose Y as the coordinate r; the metric then has the form

$$ds^2 = -\frac{dt^2}{F^2(r)} + X^2(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2). \tag{A 9}$$

The contracted Bianchi identities $T^{ab}{}_{;b} = 0$ now shows

$$p' - (\mu + p)\,F'/F = 0; \tag{A 10}$$

(A 5) is identically satisfied if (A 3), (A 4) and (A 10) are satisfied. Equation (A 3) can be directly integrated to show

$$X^2 = \left(1 - \frac{2\hat{M}}{r}\right)^{-1}, \tag{A 11}$$

where $$\hat{M}(r) \equiv 4\pi \int_0^r \mu r^2\,dr,$$

and the boundary condition $X(0) = 1$ has been used (i.e. the fluid sphere has a regular centre). With (A 10), (A 11), equation (A 4) takes the form

$$\frac{dp}{dr} = -\frac{(\mu + p)(\hat{M} + 4\pi p r^3)}{r(r - 2\hat{M})} \tag{A 12}$$

which determines p as a function of r, if the equation of state is known. Finally (A 10) shows that

$$F(r) = C \exp \int_{p(0)}^{p(r)} \frac{dp}{\mu + p}, \tag{A 13}$$

where C is a constant. Equations (A 11)–(A 13) determine the metric inside the fluid sphere, i.e. up to the value r_0 of r representing the surface of the fluid.