Proceedings of the Edinburgh Mathematical Society (2014) **57**, 367–376
DOI:10.1017/S0013091513000436

INTEGER POINTS OF MEROMORPHIC FUNCTIONS

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(Received 19 January 2012)

Abstract Working from a half-plane result of Fletcher and Langley, we show that if f is an integer-valued function on some subset of the natural numbers of positive lower density and is meromorphic of sufficiently small exponential type in the plane, then f is a polynomial.

Keywords: integer-valued functions; meromorphic functions; Nevanlinna theory; value distribution

2010 *Mathematics subject classification:* Primary 30D30; 30D35

1. Introduction

An integer-valued function is one such that $f(\mathbb{Z}) \subseteq \mathbb{Z}$, a simple example being a polynomial with integer coefficients, or $\sin(\pi z)$. Research in this field generally focuses on functions that are integer valued on some subset of $\mathbb{Z}$. Pólya proved an early result in this field.

Proposition 1.1 (see [10]). *Let f be entire, taking integer values on $\mathbb{N} \cup \{0\}$, and suppose that*

$$\limsup_{r\to\infty} \frac{M(r,f)}{2^r} < 1,$$

where $M(r,f)$ is the maximum modulus function of f. Then, f is a polynomial.

Langley, in [7], showed that the lim sup cannot be replaced by a lim inf. A corollary to Pólya's result is that 2^z is the slowest growing transcendental entire function to take integer values on the non-negative integers. Pólya further showed the following.

Proposition 1.2 (see [10]). *Let f be an entire function such that $f(n) \in \mathbb{Z}$ for $n = 0, 1, 2, \dots$ and*

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{r} \leqslant \alpha \leqslant \log 2.$$

Then, there exist polynomials $P_j(z)$ such that

$$f(z) = P_1(z)2^z + P_2(z).$$

This was later improved to $\alpha \leqslant \log 2 + \frac{1}{1500}$ by Selberg in [**12**], then further improved by Pisot in [**9**].

Fletcher and Langley proved the following half-plane analogue to Pólya's result [**3**].

Proposition 1.3 (see [3]). *Let d, J and λ satisfy*

$$0 < d < 1, \quad J \in \mathbb{N}, \quad \lambda > 0, \quad \frac{16}{J}\left(1 + \log\left(1 + \frac{J}{2}\right)\right) + 8(J-1)\lambda < d^2.$$

Let $E \subset \mathbb{N}$ have lower density

$$\underline{D}(E) = \liminf_{n\to\infty} \frac{|E \cap \{1, \dots, n\}|}{n} > d,$$

let f be analytic of exponential type less than λ in the closed right half-plane, and assume that $f(n) \in \mathbb{Z}$ for every $n \in E$. Then, f is a polynomial.

Further related work may be found in [**1**, **2**, **8**, **11**, **14**], among others. However, there does not appear to have been any research into whether an analogue of Pólya's result can be obtained for meromorphic functions. In this paper, we generalize the result of Fletcher and Langley to meromorphic functions, following the general method of their proof, which was in turn based on a method of Waldschmidt [**13**]. Our result is restricted to functions that are meromorphic in the whole plane rather than a half-plane, mainly due to the Poisson–Jensen formula being significantly easier to use in the whole plane. We use the standard Nevanlinna theory terminology of [**4**] throughout.

Theorem 1.4. *Given $d \in (0, 1)$, there exists some $\lambda = \lambda(d) > 0$ with the following property. Let f be meromorphic in the plane, taking integer values on some set $E \subseteq \mathbb{N}$ of lower density $d_0 > d$, with $T(r, f) \leqslant \lambda r$ for all $r \geqslant r_0$. Then, f is a polynomial.*

In the appendix, we will calculate how small λ needs to be.

2. Lemmas

We begin with some lemmas. The first is an elementary result comparing the integrated and unintegrated counting functions.

Lemma 2.1. *Let $0 < s < S$, and let h be a meromorphic function on the set $|z| \leqslant S$. Then,*

$$N(S, h) \geqslant n(s, h) \log \frac{S}{s} + n(0, h) \log s.$$

This is a well-known result, and so we omit the proof. The next lemma is found in many texts, including [**5**], where it is presented as a mass distribution result. A more elementary proof can be found in [**6**].

Lemma 2.2 (the (Boutroux–)Cartan lemma). *Let $z_1, \dots, z_n \in \mathbb{C}$, and $\gamma > 0$. Then,*

$$V(z) = \sum_{j=1}^{n} \log|z - z_j| > n \log \gamma \tag{2.1}$$

for all z outside a union U of open discs of total radius at most 6γ.

https://doi.org/10.1017/S0013091513000436 Published online by Cambridge University Press

Remark 2.3. We may assume that the discs are disjoint, since if some point z_0 is within two discs, of radius r_1 and r_2, respectively, we may choose a new disc of radius $r_3 < r_1 + r_2$ that encloses both original discs. We may also assume that each disc contains at least one z_j, as otherwise (2.1) applies on the boundary of that disc, and, since V is harmonic inside the disc, we may extend (2.1) to the interior. We may, therefore, assume that there are at most n discs.

We now apply the Boutroux–Cartan lemma to give a bound on the logarithm of the modulus of a function in terms of its Nevanlinna characteristic.

Lemma 2.4. *Let $m \geqslant 0$, $s \geqslant 1$, $0 < \varepsilon \leqslant 1$, and let h be meromorphic on the set $|z| \leqslant 8s$ with at least m distinct zeros in $|z| \leqslant s$. Then,*

$$\log|h(z)| \leqslant \left(6 - \frac{\log\varepsilon}{\log 2}\right)T(8s, h) + m\log\tfrac{6}{7} \tag{2.2}$$

for all $|z| \leqslant 2s$ lying outside a union U of at most $n(4s, h)$ open discs of total radius at most $24\varepsilon s$.

Remark 2.5. A disc of radius $s > 0$ contains at most $1 + 2s$ distinct integers, and, thus, the number of integers in U is at most the number of discs plus double the total radius.

Proof. Let $S = 4s$ and let $n = n(4s, h)$, and further let $b_1, \dots, b_n$ be the poles of h in $|z| \leqslant S$, repeated according to multiplicity. If $m > 0$, let $a_1, \dots, a_m$ be distinct zeros of h in $|z| \leqslant s$. Finally, define the function g by

$$g(z) = h(z)\prod_{j=1}^{m}\frac{S^2 - \bar{a}_j z}{S(z - a_j)}\prod_{k=1}^{n}\frac{z - b_k}{S},$$

where an empty product is taken as 1. Thus, g is analytic on $|z| \leqslant S$. Also, for $|z| = S$, we have that

$$\left|\frac{S^2 - \bar{a}_j z}{S(z - a_j)}\right| = 1 \quad \text{and} \quad \left|\frac{z - b_k}{S}\right| \leqslant 2,$$

and therefore

$$T(S, g) = m(S, g) \leqslant m(S, h) + n(S, h)\log 2.$$

Since $S > 1$, we have, by Lemma 2.1, that

$$N(2S, h) \geqslant n(S, h)\log 2,$$

and, consequently,

$$T(S, g) \leqslant m(S, h) + N(2S, h) \leqslant 2T(2S, h).$$

Thus, by the standard comparison between the maximum modulus and characteristic functions for functions analytic on a disc centred at the origin, we have, for $|z| \leqslant 2s = S/2$, that

$$\log|g(z)| \leqslant \frac{S + S/2}{S - S/2}T(S, g) = 3T(S, g) \leqslant 6T(2S, h) = 6T(8s, h).$$

Also in this region we have that $|z-a_j| \leqslant 3s$ and $|S^2-\bar{a}_j z| \geqslant 14s^2$, and, therefore,

$$\left|\frac{S(z-a_j)}{S^2-\bar{a}_j z}\right| \leqslant \frac{4s3s}{14s^2} = \frac{6}{7}.$$

We apply the Boutroux–Cartan lemma with $\gamma = \varepsilon S$ to find that, outside a union U of at most n open discs of total radius at most $24\varepsilon s$,

$$\sum_{k=1}^{n} \log|z-b_k| \geqslant n \log 4\varepsilon s.$$

Thus, for $|z| \leqslant 2s$, $z \notin U$,

$$\begin{aligned}\log|h(z)| &= \log|g(z)| + \sum_{j=1}^{m} \log\left|\frac{S(z-a_j)}{S^2-\bar{a}_j z}\right| + \sum_{k=1}^{n} \log S - \sum_{k=1}^{n} \log|z-b_k| \\ &\leqslant 6T(8s,h) + m\log\tfrac{6}{7} - n\log 4\varepsilon s + n\log 4s \\ &= 6T(8s,h) + m\log\tfrac{6}{7} - n\log\varepsilon,\end{aligned}$$

where, by Lemma 2.1,

$$n = n(4s,h) \leqslant \frac{N(8s,h)}{\log 2} \leqslant \frac{T(8s,h)}{\log 2},$$

from which the result follows. □

The following lemma allows us to say that if a function has some zeros in a certain segment of the real line, then it has more zeros in a larger segment. Repeated application of this allows us to cover the entire range $[1,\infty)$.

Lemma 2.6. *Given $d \in (0,1)$, there exists $\vartheta = \vartheta(d) > 0$ with the following property. Let $R \geqslant 1$, let $E \subseteq \mathbb{N}$ be such that $|E \cap [1,r]| \geqslant dr$ for all $r \geqslant R$, let $F(E) \subseteq \mathbb{Z}$, where F is meromorphic in $\mathbb{C}$ and has at least $dR/2$ distinct zeros in $E \cap [1,R]$, and let $T(r,F) \leqslant \vartheta r$ for all $r \geqslant R$. Then, F has at least dR distinct zeros in $E \cap [1,2R]$.*

Proof. Let $\varepsilon = d/96$, and let m be the least integer such that $m \geqslant dR/2$. We apply Lemma 2.4 with $h = F$ and $s = R$ to give, for $|z| \leqslant 2R$ outside some union U of at most $n(4R,F)$ open discs of total radius at most $dR/4$, that

$$\log|F(z)| \leqslant \left(6 - \frac{\log\varepsilon}{\log 2}\right)8\vartheta R + \frac{dR}{2}\log\frac{6}{7}. \tag{2.3}$$

It is easy to check that, with small enough ϑ, this gives that $\log|F(z)| < 0$. Furthermore, by our earlier remark on Lemma 2.4, U encloses at most

$$n(4R,F) + 48\varepsilon R \leqslant \frac{T(8R,F)}{\log 2} + 48\varepsilon R \leqslant \left(\frac{8\vartheta}{\log 2} + \frac{d}{2}\right)R \tag{2.4}$$

integers. Given that $|E \cap [1,2R]| \geqslant 2dR$, it is clear that if ϑ is small enough, then after removing any points of $E \cap [1,2R] \cap U$ we are left with at least dR integers in $(E \cap [1,2R]) \setminus U$, which, since $F(E) \subseteq \mathbb{Z}$ and $|F(z)| < 1$ at these points, must be zeros of F. □

We now consider several lemmas from [**3**], which form the main structure of the proof. We first create a sequence of polynomials, then look at an application of linear forms, and finally note that if a function is algebraic on a half-plane and takes integer values, then it is a polynomial.

Lemma 2.7 (see [3]). *Define the polynomials* $p_0, p_1, \dots$ *by*

$$p_0(z) = 1, \qquad p_1(z) = z, \qquad p_h(z) = \frac{z(z-1)\cdots(z-h+1)}{h!}, \quad h = 2, 3, \dots.$$

Then, for $R > 0$, $H \in \mathbb{N}$, $0 \leqslant h \leqslant H$ *and* $|z| \leqslant R$, *we have* $p_h(\mathbb{Z}) \subseteq \mathbb{Z}$ *and*

$$|p_h(z)| \leqslant \mathrm{e}^H \left(\frac{R}{H} + 1\right)^H.$$

Proof. It is easy to see that $p_h(\mathbb{Z}) \subseteq \mathbb{Z}$. For the inequality, we write

$$|p_h(z)| \leqslant \frac{(R+H)^h}{h!} \leqslant \frac{H^h}{h!}\left(\frac{R}{H} + 1\right)^H \leqslant \mathrm{e}^H \left(\frac{R}{H} + 1\right)^H.$$

□

Lemma 2.8 (see [3]). *Let* $B \geqslant 1$ *and* $N \geqslant 2$ *be integers. Suppose that* $L_1, \dots, L_m$ *are linear forms in the* n *variables* $x_1, \dots, x_n$, *with real coefficients* $a_{j,k}$ *for* $j = 1, \dots, m$ *and* $k = 1, \dots, n$, *such that* $L_j = a_{j,1}x_1 + \cdots + a_{j,n}x_n$. *Suppose further that* $n > m$ *and*

$$\max_{j,k} |a_{j,k}| \leqslant B.$$

Then, there exist integers $x_1, \dots, x_n$, *not all zero, such that, for* $j = 1, \dots, m$ *and* $k = 1, \dots, n$,

$$|L_j| \leqslant \frac{1}{N} \quad \text{and} \quad |x_k| \leqslant 2(2nBN)^{m/(n-m)}.$$

Lemma 2.9 (see [3]). *Let the algebraic function* f *be analytic on the half-plane* $\mathrm{Re}(z) \geqslant 0$ *and satisfy* $f(E) \subseteq \mathbb{Z}$ *for some set* $E \subseteq \mathbb{N}$ *of positive lower density. Then,* f *is a polynomial.*

3. Proof of Theorem 1.4

Fix a large positive integer J, and, given J, let R be a large positive integer. How large J must be will be determined later.

Apply Lemma 2.4 with $h = f$, $m = 0$, $s = R/2$ and $\varepsilon = d/96$ to give that, for $|z| \leqslant R$ outside a union U of open discs of total radius at most $dR/8$,

$$\log|f(z)| \leqslant \left(6 - \frac{\log(d/96)}{\log 2}\right) 4\lambda R = \Lambda R. \tag{3.1}$$

By (2.4), replacing R with $R/2$,

$$|\mathbb{Z} \cap U| \leqslant \frac{T(4R, f)}{\log 2} + 24\varepsilon R \leqslant \left(\frac{4\lambda}{\log 2} + \frac{d}{4}\right) R < \frac{dR}{3} \tag{3.2}$$

for small enough λ. Since R is large, we therefore have $m \geqslant dR/2$ distinct integers $\alpha_1, \dots, \alpha_m \in E \cap [1, R]$, where $m/J \in \mathbb{N}$, for which $f(\alpha_j) \in \mathbb{Z}$ and (3.1) is satisfied.

Now, set $n = 2m$, $H = n/J \in \mathbb{N}$, and form $n = HJ$ functions

$$g_k(z) = p_{\mu(k)}(z) f(z)^{\nu(k)} \tag{3.3}$$

for $\mu = 0, 1, \dots, H-1$, $\nu = 0, 1, \dots, J-1$, where the p_μ are as in Lemma 2.7. Note that H is dependent on R, but that J is fixed. Let $a_{j,k} = g_k(\alpha_j) \in \mathbb{Z}$. We obtain, by Lemma 2.7 and (3.1), the estimate

$$\begin{aligned} |a_{j,k}| = |g_k(\alpha_j)| &= |p_{\mu(k)}(\alpha_j)|\,|f(\alpha_j)|^{\nu(k)} \\ &\leqslant \mathrm{e}^H \left(\frac{R}{H} + 1\right)^H (\mathrm{e}^{AR})^{J-1} \\ &= A(R) \leqslant \lceil A(R) \rceil = B(R) \leqslant 2A(R), \end{aligned}$$

where $\lceil x \rceil$ is the smallest integer not less than x. We apply Lemma 2.8 with $N = 2$ and $n = 2m$ to give the integers $A_1, \dots, A_n$, not all zero, such that

$$\sum_{k=1}^{n} A_k g_k(\alpha_j) = 0$$

for $j = 1, \dots, m$, and

$$|A_k| \leqslant 8nB, \quad \text{where } B = B(R).$$

Now set

$$F(z) = \sum_{k=1}^{n} A_k g_k(z). \tag{3.4}$$

F is meromorphic, takes integer values on E and is 0 at the α_j for $j = 1, \dots, m$. We now estimate $T(r, F)$ for each $r \geqslant R$. Note first that, since the $p_\mu(z)$ are polynomials, all poles of F must come from poles of f, so

$$N(r, F) \leqslant (J-1) N(r, f).$$

Also, for non-negative $x_1, \dots, x_n$,

$$\log^+ \left(\sum_{k=1}^{n} x_k \right) \leqslant \log n + \max_{1 \leqslant k \leqslant n} \log^+ |x_k|.$$

For $r \geqslant R$, we have, by Lemma 2.7, that

$$\begin{aligned} \log |F(z)| &\leqslant \log n + \max_{1 \leqslant k \leqslant n,\, |z| = r} (\log^+ |A_k g_k(z)|) \\ &\leqslant \log n + \log 8nB + H\left(1 + \log\left(\frac{r}{H} + 1\right)\right) + (J-1) \log^+ |f(z)|. \end{aligned}$$

Thus, by integrating we obtain that

$$m(r,F)\leqslant \log n+\log 8nB+H\biggl(1+\log\left(\frac{r}{H}+1\right)\biggr)+(J-1)m(r,f),$$

and therefore

$$\begin{aligned}T(r,F)&\leqslant \log n+\log 8nB+H\biggl(1+\log\left(\frac{r}{H}+1\right)\biggr)+(J-1)T(r,f)\\&\leqslant \log n+\log 16nA+H\biggl(1+\log\left(\frac{r}{H}+1\right)\biggr)+(J-1)T(r,f).\end{aligned}$$

Now, for $r\geqslant R$, since $\Lambda>\lambda$ by (3.1) and $n=2m\leqslant 2r$ and R is large, we have that

$$\begin{aligned}T(r,F)&\leqslant 4\log 2+2\log n+\log\left(\mathrm{e}^{H}\left(\frac{R}{H}+1\right)^{H}\mathrm{e}^{(J-1)\Lambda R}\right)\\&\quad+H\biggl(1+\log\left(\frac{r}{H}+1\right)\biggr)+(J-1)T(r,f)\\&\leqslant 4\log 2+2\log 2r+H\biggl(1+\log\left(\frac{R}{H}+1\right)\biggr)+(J-1)\Lambda R\\&\quad+H\biggl(1+\log\left(\frac{r}{H}+1\right)\biggr)+(J-1)\lambda r\\&\leqslant 2H\biggl(1+\log\left(\frac{r}{H}+1\right)\biggr)+2(J-1)\Lambda r.\end{aligned}$$

It is plain to see by differentiating that $x^{-1}(1+\log(x+1))$ is decreasing for $x>0$. So, for $n=2m\leqslant 2R\leqslant 2r$, this gives that

$$\frac{r}{H}\geqslant\frac{R}{H}=\frac{RJ}{n}=\frac{RJ}{2m}\geqslant\frac{J}{2}$$

and

$$\begin{aligned}2H\biggl(1+\log\left(\frac{r}{H}+1\right)\biggr)&=2r\frac{H}{r}\biggl(1+\log\left(\frac{r}{H}+1\right)\biggr)\\&\leqslant 2r\frac{2}{J}\biggl(1+\log\left(\frac{J}{2}+1\right)\biggr)\\&=\frac{4r}{J}\biggl(1+\log\left(\frac{J}{2}+1\right)\biggr).\end{aligned}$$

Thus,

$$T(r,F)\leqslant\frac{4r}{J}\biggl(1+\log\left(\frac{J}{2}+1\right)\biggr)+2(J-1)\Lambda r, \tag{3.5}$$

and we can say that, for large enough R,

$$T(r,F)<\vartheta r \tag{3.6}$$

for $r \geqslant R$, where $\vartheta > 0$ can be arbitrarily small provided that λ is small enough and J is large enough. We also have that $F(\alpha_j) = 0$ for $j = 1, \dots, m$, where $m \geqslant dR/2$. We apply Lemma 2.6 to give at least dR zeros of F in $E \cap [1, 2R]$. We apply this repeatedly to give an infinite sequence of zeros of F on the real line. Assume that $F(z) \not\equiv 0$. We have that $n(2^t R, 1/F) \geqslant 2^{t-1} dR$, so $n(r, 1/F) \geqslant dr/4$ for all $r \geqslant R$. By application of Lemma 2.1 we find that $N(\mathrm{e}r, 1/F) \geqslant dr/4$; thus, $T(r, 1/F) \geqslant dr/4\mathrm{e}$ and, by the first fundamental theorem,

$$T(r, F) \geqslant \frac{dr}{4\mathrm{e}} - O(1). \tag{3.7}$$

However, if ϑ is small enough, this is incompatible with (3.6). Hence, $F(z) \equiv 0$.

Now, recall from (3.3) and (3.4) that

$$F(z) = \sum_{\nu=0}^{J-1} \left(\sum_{\mu=0}^{H-1} A_{\mu,\nu} p_\mu(z) \right) f(z)^\nu,$$

where at least one $A_{\mu,\nu}$ is non-zero, and where $p_\mu(z)$ has degree μ. Thus, these polynomials cannot cancel each other out; hence, f is algebraic and must have only finitely many poles. Therefore, there exists some $x \in \mathbb{N}$ such that there are no poles in the half-plane $\mathrm{Re}(z) \geqslant x$, so f is analytic in this region. We apply Lemma 2.9 to $f(z - x)$, giving that $f(z - x)$ is a polynomial here. From this, we conclude that $f(z)$ is polynomial in the half-plane $\mathrm{Re}(z) \geqslant x$ and, thus, by the identity theorem $f(z)$ must be a polynomial on the whole plane. The proof is complete.

Appendix A. How small is $\lambda(d)$?

An obvious question to ask about this theorem is, how small must λ be? We will now calculate this. We make no claim as to how sharp these values are, but have sought to present a positive result in a reasonably accessible fashion.

We begin by calculating ϑ in Lemma 2.6. We use (2.3), substituting in $d/96$ for ε, and noting that, since we want $|F(z)| < 1$ in order to force $F(\alpha) = 0$ for $\alpha \in E \setminus U$, we require that $\log|F(z)| < 0$. Hence,

$$\vartheta < \frac{d \log(7/6)}{16(6 - \log(d/96)/\log 2)} = \gamma(d). \tag{A 1}$$

We also require that U encloses at most dR integers, so by (2.4) we need

$$\left(\frac{8\vartheta}{\log 2} + \frac{d}{2} \right) R \leqslant dR,$$

which simplifies to

$$\vartheta \leqslant \frac{d \log 2}{16}. \tag{A 2}$$

We further require, by (3.6) and (3.7), that

$$\vartheta < \frac{d}{4\mathrm{e}}. \tag{A 3}$$

https://doi.org/10.1017/S0013091513000436 Published online by Cambridge University Press

However,

$$\frac{d\log(7/6)}{16(6-\log(d/96)/\log 2)} < \frac{d\log(7/6)}{96} < \frac{d}{48} < \frac{d\log 2}{16} < \frac{d}{4\mathrm{e}};$$

hence, both (A 2) and (A 3) are much looser bounds than (A 1) and may be ignored.

We now move on to λ. The proof of Theorem 1.4 by (3.2) requires that

$$\lambda < \frac{d\log 2}{48}. \tag{A 4}$$

It also requires, by (3.1) and (3.5), that

$$\vartheta = \frac{4}{J}\left(1+\log\left(\frac{J}{2}+1\right)\right) + 2(J-1)\left(6-\frac{\log(d/96)}{\log 2}\right)4\lambda.$$

Suppose that we choose J so large that

$$\frac{4}{J}\left(1+\log\left(\frac{J}{2}+1\right)\right) < \frac{\gamma(d)}{2},$$

and, given this J, choose λ such that

$$2(J-1)\left(6-\frac{\log(d/96)}{\log 2}\right)4\lambda < \frac{\gamma(d)}{2}.$$

The pair (J,λ) will then satisfy (A 1). Furthermore, we have that

$$\lambda < \frac{\gamma(d)}{96} < \frac{d\log(7/6)}{96^2} < \frac{d\log 2}{48},$$

and (A 4) holds. Solving these inequalities using Mathematica for J in terms of d produces the new inequality

$$J > \frac{128\log(d/6144)}{d\log(7/6)\log 2} W\left(\frac{d\log(7/6)\log 2\exp(d\log(7/6)\log 2/64\log(d/6144)-1)}{64\log(d/6144)}\right) - 2,$$

where W is the Lambert W-function. Again using Mathematica, solving for specific values of d gives the following results for J and λ:

$$\begin{array}{lll}
d = 1, & J \gtrapprox 130\,000, & \lambda \lessapprox 2.9\times 10^{-11},\\
d = 0.5, & J \gtrapprox 290\,000, & \lambda \lessapprox 5.6\times 10^{-12},\\
d = 0.1, & J \gtrapprox 2\,000\,000, & \lambda \lessapprox 1.2\times 10^{-13},\\
d = 0.01, & J \gtrapprox 28\,000\,000, & \lambda \lessapprox 5.8\times 10^{-16}.
\end{array}$$

Note that $d = 1$ is essentially meaningless here, as we require our set E to have lower density greater than d, but it provides a useful upper bound.

By comparison, using a similar process on the Fletcher–Langley result (Proposition 1.3) yields a maximal value of λ of roughly 3.6×10^{-4} for d close to 1.

We conclude by asking a question about a topic that does not appear to have been the subject of any research: can any results be obtained by restricting what integer values may be taken? Specifically, for $n \in \{1, 2, 4\}$, is 2^{nz} the slowest-growing transcendental meromorphic function taking only nth powers of integers on the natural numbers? Pólya's result (the corollary to Proposition 1.1) proves this for $n = 1$, but beyond this the way forward is unclear. The restriction to only three integer values of n is due to the sine function: for odd $n \geqslant 3$, $\sin(\pi z/2)$ has the required properties and is smaller than 2^{3z}, and for even $n \geqslant 6$, $\sin(\pi z)$ is sufficient.

Acknowledgements. This research was conducted as part of the author's PhD thesis studies, and was supported by a grant from the Engineering and Physical Sciences Research Council. The author thanks the referee for their careful reading and helpful suggestions.

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