PROPER LEFT TYPE-*A* MONOIDS REVISITED

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(Received 23 September, 1991; revised 10 November, 1992)

Introduction. The relation $\mathcal{R}^*$ is defined on a semigroup S by the rule that $a\mathcal{R}^*b$ if and only if the elements a, b of S are related by the Green's relation $\mathcal{R}$ in some oversemigroup of S. A semigroup S is an E-semigroup if its set $E(S)$ of idempotents is a subsemilattice of S. A left adequate semigroup is an E-semigroup in which every $\mathcal{R}^*$-class contains an idempotent. It is easy to see that, in fact, each $\mathcal{R}^*$-class of a left adequate semigroup contains a unique idempotent [**2**]. We denote the idempotent in the $\mathcal{R}^*$-class of a by a^+.

In this paper we are concerned with left type-A semigroups. These are semigroups S which are left adequate and in which $ae = (ae)^+a$ for each $a \in S$ and $e \in E(S)$. Any inverse semigroup S is left type-A; for an element a of S we have $a^+ = aa^{-1}$ and certainly, for any $e \in E(S)$, $(ae)(ae)^{-1}a = ae$. The class of left type-A semigroups, however, is much larger than the class of inverse semigroups. For example, every right cancellative monoid is a left type-A semigroup.

On any left type-A semigroup there is a minimum right cancellative congruence which we denote by σ. We say that a left type-A semigroup S is *proper* if $\sigma \cap \mathcal{R}^* = \iota$. For an inverse semigroup, being proper is the same as being E-unitary. In the general case, however, a proper left type-A semigroup is E-unitary but the converse is not true [**1**]. A famous result of inverse semigroup theory due to McAlister is that every inverse semigroup has an E-unitary cover [**6, 7**]. The corresponding result for left type-A monoids is that every left type-A monoid has a proper left type-A cover. This is the dual of a theorem in [**1**]. McAlister also gave a structure theorem for E-unitary inverse semigroups in terms of P-semigroups. There is an analogue of this result for left type-A monoids—the dual of Theorem 4.3 of [**1**].

In [**4**] and [**5**] Margolis and Pin develop the theory of a class of E-semigroups called E-dense semigroups. In particular, they describe E-unitary E-dense monoids in terms of groups acting on categories. The class of E-dense semigroups contains the class of inverse semigroups and the techniques introduced by Margolis and Pin can be specialised to obtain the P-theorem of McAlister. These methods also yield another proof of an alternative characterisation of E-unitary inverse semigroups originally due to O'Carroll [**8**], as the inverse subsemigroups of semidirect products of semilattices by groups.

In an earlier paper [**3**], the present authors used the Margolis, Pin techniques to investigate left proper E-dense monoids. In the present paper our objective is to extend their methods so that they can be applied to proper left type-A monoids. To do this we have to make use of actions on certain small categories by right cancellative monoids rather than groups. We introduce the appropriate ideas in Section 1 including the notion of left derived category. We use these ideas in Section 2 to obtain a new characterisation of proper left type-A monoids and to give for the first time a characterisation of E-unitary left type-A monoids.

We introduce two further new descriptions of proper left type-A monoids in

[1] This research was supported by "Plano de Reforço da Capacidade Científica do Departamento de Matemática", Fundação Calouste Gulbenkian.

Sections 3 and 4. First we introduce the class of $\mathscr{R}$-monoids and use the results of Section 2 to show that this class is precisely the class of proper left type-A monoids. In the final section we obtain an analogue of O'Carroll's theorem by showing that a left type-A monoid is proper if and only if it can be embedded by an $\mathscr{R}^*$-preserving embedding in a special submonoid of a semidirect product of a semilattice by a right cancellative monoid.

1. Preliminaries. We start by recalling some definitions and results from [**3**]. We caution the reader that the definition of left type-A category used in this paper differs from that in [**3**].

In what follows $\mathscr{C}$ is always a small category with set of *objects* Obj $\mathscr{C}$ and set of *morphisms* Mor $\mathscr{C}$. For all $v \in \text{Obj}\,\mathscr{C}$, the set of all morphisms with *domain* [*codomain*] v is denoted by $\text{Mor}(v, -)$ [$\text{Mor}(-, v)$]. We use additive notation for the *composition* of morphisms and represent the *identity at an object u* by O_u.

Definition 1.1 [**3**]. On Mor $\mathscr{C}$, we define the relation $\mathscr{R}^*$ as follows, for all $p, q \in \text{Mor}\,\mathscr{C}$,

$$(p, q) \in \mathscr{R}^* \Leftrightarrow [(\forall s, t \in \text{Mor}\,\mathscr{C}) s + p = t + p \Leftrightarrow s + q = t + q],$$

whenever any of these identities exist.

It is easy to check that

Lemma 1.2. *Let $\mathscr{C}$ be a category, $u \in \text{Obj}\,\mathscr{C}$ and $p, q \in \text{Mor}\,\mathscr{C}$. Then*

(a) *if $p \in \text{Mor}(u, -)$ and $(p, q) \in \mathscr{R}^*$, then $q \in \text{Mor}(u, -)$;*

(b) *if p is an idempotent, that is, $p = p + p$, and $p \in \text{Mor}(u, v)$ then $u = v$;*

(c) *if p is an idempotent then*

$$(p, q) \in \mathscr{R}^* \Leftrightarrow \begin{cases} q = p + q, \quad \text{and} \\ (\forall s, t \in \text{Mor}\,\mathscr{C}) s + q = t + q \Rightarrow s + p = t + p; \end{cases}$$

(d) *$\mathscr{R}^*$ is a left congruence on the partial semigroup* Mor $\mathscr{C}$.

Definition 1.3 [**3**]. A category $\mathscr{C}$ is said to be *left abundant* if each $\mathscr{R}^*$-class contains an idempotent.

Definition 1.4. A left abundant category $\mathscr{C}$ in which for all objects u, the idempotents of $\text{Mor}(u, u)$ form a subsemilattice of $\text{Mor}(u, u)$ is said to be *left adequate.*

In a left adequate category each $\mathscr{R}^*$-class, $\mathscr{R}^*_p$, contains exactly one idempotent denoted by p^+.

Definition 1.5. A left adequate category is *left type-A* if for all morphisms p and q,

$$q + p^+ = (q + p^+)^+ + q,$$

whenever either of these elements exists.

Lemma 1.6 [**3**]. *Let $\mathscr{C}$ be a left type-A category and $p, q \in \text{Mor}\,\mathscr{C}$ be such that $p + q$* exists. Then

(a) $(p + q)^+ = (p + q^+)^+$;

(b) $(p + q)^+ + p^+ = (p + q)^+$.

The above definitions and results can all be applied to monoids by regarding a monoid as a category with a single object.

https://doi.org/10.1017/S0017089500009873 Published online by Cambridge University Press

LEMMA 1.7 [1]. *Let M be a left type-A monoid and define the relation σ on S thus: for all $a, b \in S$,*

$$a\sigma b \Leftrightarrow (\exists e \in E(S))ea = eb.$$

Then σ is the least right cancellative monid congruence on S and $E(S)$ is contained in a σ-class.

DEFINITION 1.8. A left type-A monoid is *proper* if $\mathcal{R}^* \cap \sigma = \iota$.

DEFINITION 1.9. A monoid M is E-unitary if $E(M)$ is a unitary subset of M.

Every proper left type A monoid is E-unitary but the converse is false as shown by the dual of Example 3 in [1].

LEMMA 1.10. *For a left type A monoid M, the following conditions are equivalent:*
(1) *M is E-unitary,*
(2) *For $e, a \in M$, $e, ea \in E(M)$ implies $a \in E(M)$,*
(3) *For $e, a \in M$, $e, ae \in E(M)$ implies $a \in E(M)$,*
(4) *$E(M)$ is a σ-class.*

Proof. Suppose that (2) holds and that $e, a \in M$ are such that $e, ae \in E(M)$. Since M is left type-A, $ae = (ae)^+a$ so that $(ae)^+$, $(ae)^+a$ are both in $E(M)$ and by (2), $a \in E(M)$. Thus (2) implies (3).

Suppose that (3) holds and that $e, a \in M$ with $e, ea \in E(M)$. Then $ea = (ea)^+ = ea^+$ so that

$$\begin{aligned}(aea)(aea) &= aea^+aea^+ = aeaea^+ = ae(ea)a^+\\ &= a(ea)a^+ = aa^+ea = aea^+a = aea.\end{aligned}$$

Thus $aea, ea \in E(M)$ so that $a \in E(M)$ by (3) and hence (2) holds.

It now follows that (1), (2) and (3) are equivalent and it is easy to see that (4) is equivalent to (2).

For the restricted setting of idempotent categories we can also define proper categories. A category $\mathscr{C}$ is *idempotent* if for each $u \in \operatorname{Obj} \mathscr{C}$, the monoid $\operatorname{Mor}(u, u)$ is a band. We remark that in an idempotent left type-A category, each monoid $\operatorname{Mor}(u, u)$ is a semilattice.

DEFINITION 1.11. An idempotent, left type A category $\mathscr{C}$ is *proper* if, for all $u, v \in \operatorname{Obj} \mathscr{C}$ and $p, q \in \operatorname{Mor}(u, v)$ we have $p = q$ whenever $(p, q) \in \mathcal{R}^*$.

LEMMA 1.12 [3]. *Let $\mathscr{C}$ be an idempotent left type-A category. The following conditions are equivalent:*
(a) *$\mathscr{C}$ is proper;*
(b) *for all $u, v \in \operatorname{Obj} \mathscr{C}$ and $p, q \in \operatorname{Mor}(u, v)$,*

$$p^+ = q^+ \Rightarrow p = q.$$

Let $\mathscr{C}$ be a category. A *left ideal* I of $\mathscr{C}$ is a subset of $\operatorname{Mor} \mathscr{C}$ such that for all objects u, v, w of $\mathscr{C}$ and for all $x \in \operatorname{Mor}(u, v)$ and $p \in \operatorname{Mor}(v, w)$, $p \in I$ implies $x + p \in I$.

A *right ideal* of $\mathscr{C}$ is defined similarly and a subset of $\operatorname{Mor} \mathscr{C}$ is an *ideal* of $\mathscr{C}$ if it is both a left and a right ideal.

Let $F:\mathscr{C}\to\mathscr{C}$ be an endofunctor. Recall that F is a *full embedding* if F is injective on Obj $\mathscr{C}$ and for each pair u, v of objects of $\mathscr{C}$, F carries $\text{Mor}(u, v)$ bijectively onto $\text{Mor}(uF, vF)$.

We say that a full embedding $F:\mathscr{C}\to\mathscr{C}$ is a *left ideal full embedding* if (Mor $\mathscr{C}$)F is a left ideal of $\mathscr{C}$.

We denote by $E(\mathscr{C})$ the set of all left ideal full embeddings of $\mathscr{C}$ into itself. It is easy to verify that $E(\mathscr{C})$ is a right cancellative submonoid of the monoid of all endofunctors of $\mathscr{C}$.

Definition 1.13. A right cancellative monoid T *acts* on a category $\mathscr{C}$ if there is a monoid morphism $T\to E(\mathscr{C})$. In this case we write ut (resp. pt) for the result of the action of t on the object u (resp. on the morphism p). We then have the following identities and implications where $u, v, w, z \in \text{Obj}\,\mathscr{C}$, $p, q \in \text{Mor}(u, v)$, $r \in \text{Mor}(v, w)$ and $t, t_1, t_2 \in T$:

(1) $(p + r)t = pt + rt$;
(2) $(pt_1)t_2 = p(t_1t_2)$;
(3) $p1 = p$;
(4) $O_ut = O_{ut}$;
(5) $pt = qt \Rightarrow p = q$;
(6) $v = zt \Rightarrow u = yt$ for some $y \in \text{Obj}\,\mathscr{C}$ and $p = st$ for some $s \in \text{Mor}(y, z)$.

By regarding a partially ordered set $\mathscr{X}$ as a category in the usual way we obtain the definition of an action of a right cancellative monoid T on $\mathscr{X}$. We observe that each element t of T induces an order-isomorphism from $\mathscr{X}$ onto the order-ideal $\mathscr{X}t$ of $\mathscr{X}$. It follows from this that if $a, b \in \mathscr{X}$ have a greatest lower bound $a \wedge b$ in $\mathscr{X}$, then

$$(a \wedge b)t = at \wedge bt.$$

In particular, this holds for all a, b when $\mathscr{X}$ is a semilattice.

Let M be an E-unitary left type-A monoid and $T = M/\sigma$. Let $\varphi: M \twoheadrightarrow T$ be the canonical epimorphism associated with σ.

We define the *left-derived category* $\mathscr{C}$ of φ as follows:

$$\text{Obj}\,\mathscr{C} = T,$$

and, for $u, v \in \text{Obj}\,\mathscr{C}$,

$$\text{Mor}(u, v) = \{(u, m, v) : m \in M, u = m\varphi \,.\, v\};$$

composition is given by

$$(u, m, v) + (u, n, w) = (u, mn, w).$$

It is easy to prove that $\mathscr{C}$ is indeed a category where for $u \in \text{Obj}\,\mathscr{C}$, $O_u = (u, 1, u)$. Since M is E-unitary,

$$\text{Mor}(u, u) = \{(u, e, u) : e \in E(M)\}.$$

Thus $\mathscr{C}$ is idempotent. Further, $\mathscr{C}$ is left type-A with $(u, m, v)^+ = (u, m^+, u)$.

Next, we define an action (on the right) of T on $\mathscr{C}$. First, T acts on Obj $\mathscr{C}$ by multiplication and for $(u, m, v) \in \text{Mor}(u, v)$ and $t \in T$ we define $(u, m, v)t = (ut, m, vt)$. It is straightforward to verify that this is an action in the sense of the above definition.

2. E-unitary and proper left type-A monoids. We give structure theorems for the monoids of the title using actions of right cancellative monoids on categories. We start by associating a monoid C_u to each object of a category $\mathscr{C}$ on which a right cancellative monoid T acts.

DEFINITION 2.1. For an object u of a category $\mathscr{C}$,

$$C_u = \{(t,p) : t \in T, p \in \mathrm{Mor}(ut, u)$$

and for $(t,p),(h,q) \in C_u$,

$$(t,p)(h,q) = (th, ph + q).$$

It is routine to verify that C_u actually is a monoid.

THEOREM 2.2. *A monoid M is an E-unitary (resp. a proper) left type-A monoid if and only if M is isomorphic to C_u for some object u of an (resp. a proper) idempotent left type-A category $\mathscr{C}$ acted upon by a right cancellative monoid T.*

Proof. First let T be a right cancellative monoid acting on an idempotent left type-A category $\mathscr{C}$. Then it is easy to see that

$$E(C_u) = \{(1,p) : p \in \mathrm{Mor}(u,u)\}$$

which is clearly isomorphic to the semilattice $\mathrm{Mor}(u,u)$.

If $(t,p) \in C_u$, then $p \in \mathrm{Mor}(ut,u)$ and as $\mathscr{C}$ is left type-A, there is a morphism p^+ in $\mathrm{Mor}(ut,ut)$ such that $p^+\mathscr{R}^*p$. Now $p^+ = p_0t$ for some $p_0 \in \mathrm{Mor}(u,u)$ and it is not difficult to verify that $(1,p_0)\mathscr{R}^*(t,p)$ so that C_u is left adequate.

Now let $(1,q) \in E(C_u)$ and let $(p+q)^+ = rt$ where $r \in \mathrm{Mor}(u,u)$. Then

$$\begin{aligned}((t,p)(1,q))^+(t,p) &= (t,p+q)^+(t,p) = (1,r)(t,p) = (t, rt+p)\\ &= (t,(p+q)^+ + p) = (t,p+q) = (t,p)(1,q),\end{aligned}$$

since $q \in \mathrm{Mor}(u,u)$ is idempotent and $\mathscr{C}$ is left type-A. Thus C_u is left type-A. Further, it is clear that C_u is E-unitary.

Now, suppose that $\mathscr{C}$ is proper. Let $(t,p),(h,q) \in C_u$ be related by $\mathscr{R}^* \cap \sigma$. Then $(t,p)^+ = (h,q)^+$ so that $(1,p_0) = (1,q_0)$, where $p_0, q_0 \in \mathrm{Mor}(u,u)$ are such that $p_0t = p^+$, $q_0t = q^+$. As $(t,p)\sigma(h,q)$ we have

$$(1,r)(t,p) = (1,r)(h,q)$$

for some $r \in \mathrm{Mor}(u,u)$ from which it follows that $t = h$. Thus $p,q \in \mathrm{Mor}(ut,u)$ and $p^+ = q^+$ so that by Lemma 1.12, $p = q$. Therefore $(t,p) = (h,q)$ and C_u is proper.

For the converse we take T to be M/σ and $\mathscr{C}$ to be the left derived category of the canonical epimorphism $M \twoheadrightarrow T$ as defined in Section 1. We observed there that $\mathscr{C}$ is idempotent and left type-A.

If M is proper, let $u,v \in T$ and suppose that (u,m,v), (u,n,v) are $\mathscr{R}^*$-related morphisms in $\mathrm{Mor}(u,v)$. Then $u = m\varphi \,.\, v = n\varphi \,.\, v$ and $m^+ = n^+$. Hence $m\varphi = n\varphi$ since T is right cancellative. Thus $(m,n) \in \sigma \cap \mathscr{R}^*$ and so $m = n$ since M is proper. Thus $\mathscr{C}$ is proper in this case.

We now show that M is isomorphic to C_1, where 1 is the identity of T, by defining $\psi : M \to C_1$ by putting

$$m\psi = (m\sigma, (m\sigma, m, 1)).$$

Clearly, ψ is injective. Let $(t, (t, n, 1)) \in C_1$. Then $(t, n, 1) \in \mathrm{Mor}(t, 1)$ and so $t = n\varphi \,.\, 1 = n\varphi$. Thus $n\psi = (t, (t, n, 1))$ and ψ is onto. To see that ψ is a morphism, let $m, n \in M$. Then

$$\begin{aligned}(mn)\psi &= ((mn)\sigma, ((mn)\sigma, mn, 1)) \\ &= (m\sigma n\sigma, (m\sigma n\sigma, mn, 1)) \\ &= (m\sigma, (m\sigma, m, 1))(n\sigma, (n\sigma, n, 1)) \\ &= m\psi n\psi\end{aligned}$$

and so ψ is an isomorphism.

3. $\mathcal{R}$-monoids. Here we present a new characterisation of a proper left type-A monoid M as an $\mathcal{R}$-monoid $\mathcal{R}(T, \mathcal{X}, \mathcal{Y})$, obtained by means of the left-derived category of the canonical epimorphism of M onto $T = M/\sigma$.

DEFINITION 3.1. Let $\mathcal{X}$ be a partially ordered set and let $\mathcal{Y}$ be a subsemilattice of $\mathcal{X}$. Let T be a right cancellative monoid with identity 1, which acts on $\mathcal{X}$. Then

$$\mathcal{S}(T, \mathcal{X}, \mathcal{Y}) = \{(t, at) : a \in \mathcal{Y} \text{ and } (\forall b\, \mathcal{Y}) at \wedge b \in \mathcal{Y}t\},$$

and for $(t, at), (h, bh) \in \mathcal{S}(T, \mathcal{X}, \mathcal{Y})$ we define

$$(t, at)(h, bh) = (th, ath \wedge bh).$$

If $\mathcal{Y}$ has a greatest element f, we put

$$\mathcal{R}(T, \mathcal{X}, \mathcal{Y}) = \{(t, at) \in \mathcal{S}(T, \mathcal{X}, \mathcal{Y}) : at \leq f\},$$

LEMMA 3.2. *$\mathcal{S}(T, \mathcal{X}, \mathcal{Y})$ is a left type-A semigroup with semilattice of idempotents isomorphic to $\mathcal{Y}$.*

Proof. Let $(t, at), (h, bh) \in \mathcal{S}(T, \mathcal{X}, \mathcal{Y})$. Then $at \wedge b \in \mathcal{Y}t$ so that $at \wedge b = ct$ for some $c \in \mathcal{Y}$ and hence $ath \wedge bh = (at \wedge b)h = cth$ giving

$$(t, at)(h, bh) = (th, cth).$$

Let $d \in \mathcal{Y}$. We now prove that $cth \wedge d$ exists and belongs to $\mathcal{Y}th$. As $(h, bh) \in \mathcal{S}(T, \mathcal{X}, \mathcal{Y})$, $bh \wedge d \in \mathcal{Y}h$ and so $bh \wedge d = b_0 h$ for some $b_0 \in \mathcal{Y}$. Also, as $(t, at) \in \mathcal{S}(T, \mathcal{X}, \mathcal{Y})$, $at \wedge b_0$ and $at \wedge b$ exist. Hence $ath \wedge b_0 h$ and $ath \wedge bh$ exist and

$$ath \wedge b_0 h = ath \wedge (bh \wedge d) = (ath \wedge bh) \wedge d = cth \wedge d.$$

Thus $cth \wedge d$ exists. Further $ath \wedge b_0 h = (at \wedge b_0)h$ and since $(t, at) \in \mathcal{S}(T, \mathcal{X}, \mathcal{Y})$, we have $at \wedge b_0 = xt$ for some $x \in \mathcal{Y}$. Therefore $cth \wedge d \in \mathcal{Y}th$. We conclude that $(th, cth) \in \mathcal{S}(T, \mathcal{X}, \mathcal{Y})$ and $\mathcal{S}(T, \mathcal{X}, \mathcal{Y})$ is closed.

It is now a routine matter to verify that $\mathcal{S}(T, \mathcal{X}, \mathcal{Y})$ is a semigroup with set of idempotents

$$\{(1, a) : a \in \mathcal{Y}\},$$

which is clearly isomorphic to the semilattice $\mathcal{Y}$.

https://doi.org/10.1017/S0017089500009873 Published online by Cambridge University Press

Next, let $(t, at) \in \mathcal{S}(T, \mathcal{X}, \mathcal{Y})$. Then $(1, a) \in E(\mathcal{S}(T, \mathcal{X}, \mathcal{Y}))$ and

$$(1, a)(t, at) = (t, at).$$

Let $(h, bh), (h_1, b_1h_1) \in \mathcal{S}(T, \mathcal{X}, \mathcal{Y})$ be such that

$$(h, bh)(t, at) = (h_1, b_1h_1)(t, at).$$

Then $ht = h_1t$ and $(bh \wedge a)t = (b_1h_1 \wedge a)t$. As T is right cancellative $h = h_1$ and, by the definition of action, $bh \wedge a = b_1h_1 \wedge a$. Thus

$$(h, bh)(1, a) = (h_1, b_1h_1)(1, a).$$

It follows that

$$(t, at)^+ = (1, a).$$

Finally, we prove that $\mathcal{S}(T, \mathcal{X}, \mathcal{Y})$ satisfies the type-A condition. Let $(t, at) \in \mathcal{S}(T, \mathcal{X}, \mathcal{Y})$ and $(1, b) \in E(\mathcal{S}(T, \mathcal{X}, \mathcal{Y}))$. Then

$$\begin{aligned}((t, at)(1, b))^+(t, at) &= (t, at \wedge b)^+(t, at)\\ &= (1, a_0)(t, at),\end{aligned}$$

where $a_0t = at \wedge b$. Thus

$$\begin{aligned}(1, a_0)(t, at) = (t, at \wedge b \wedge at) &= (t, at \wedge b)\\ &= (t, at)(1, b).\end{aligned}$$

The semigroup $\mathcal{S}(T, \mathcal{X}, \mathcal{Y})$ is therefore left type-A.

We now assume that $\mathcal{Y}$ has a greatest element f so that $\mathcal{R}(T, \mathcal{X}, \mathcal{Y})$ is defined.

LEMMA 3.3. *$\mathcal{R}(T, \mathcal{X}, \mathcal{Y})$ is a proper left type-A monoid with semilattice of idempotents isomorphic to $\mathcal{Y}$.*

Proof. We first prove that $\mathcal{R}(T, \mathcal{X}, \mathcal{Y})$ is a subsemigroup of $\mathcal{S}(T, \mathcal{X}, \mathcal{Y})$. Let $(t, at), (h, bh) \in \mathcal{R}(T, \mathcal{X}, \mathcal{Y})$. Then $bh \leq f$ and so $ath \wedge bh \leq f$. Thus $(t, at)(h, bh) \in \mathcal{R}(T, \mathcal{X}, \mathcal{Y})$. Clearly, $E(\mathcal{S}(T, \mathcal{X}, \mathcal{Y})) = E(\mathcal{R}(T, \mathcal{X}, \mathcal{Y}))$ and so, for all $(t, at) \in \mathcal{R}(T, \mathcal{X}, \mathcal{Y})$, we have $(1, a) \in \mathcal{R}(T, \mathcal{X}, \mathcal{Y})$ and so $\mathcal{R}(T, \mathcal{X}, \mathcal{Y})$ is a left type-A subsemigroup of $\mathcal{S}(T, \mathcal{X}, \mathcal{Y})$. Both $\mathcal{S}(T, \mathcal{X}, \mathcal{Y})$ and $\mathcal{R}(T, \mathcal{X}, \mathcal{Y})$ have identity $(1, f)$.

To prove that $\mathcal{R}(T, \mathcal{X}, \mathcal{Y})$ is proper, let (t, at) and (h, bh) be elements of $\mathcal{R}(T, \mathcal{X}, \mathcal{Y})$ such that

$$((t, at), (h, bh)) \in \mathcal{R}^* \cap \sigma.$$

Then $(t, at)^+ = (h, bh)^+$, that is $(1, a) = (1, b)$. Also, for some idempotent $(1, e)$,

$$(1, e)(t, at) = (1, e)(h, bh).$$

Hence $t = h$ and so $(t, at) = (h, bh)$.

Notice that the over semigroup $\mathcal{S}(T, \mathcal{X}, \mathcal{Y})$ is also proper

DEFINITION 3.4. The monoid $\mathcal{R}(T, \mathcal{X}, \mathcal{Y})$ is called an *$\mathcal{R}$-monoid.*

Our objective is to show that for any proper left type-A monoid M there is a choice of T, $\mathcal{X}$ and $\mathcal{Y}$ such that M is isomorphic to $\mathcal{R}(T, \mathcal{X}, \mathcal{Y})$. We do this by showing that all monoids C_u arising from actions of right cancellative monoids on proper, idempotent left type-A categories are isomorphic to $\mathcal{R}$-monoids. We start by associating a partially ordered set $\mathcal{X}$ with any idempotent left type-A category $\mathcal{C}$.

DEFINITION 3.5 [3]. Let $\mathcal{C}$ be an idempotent, left type-A category. On Mor $\mathcal{C}$, we define a relation $\preccurlyeq$ as follows: for all $p, q \in \operatorname{Mor} \mathcal{C}$,

$$p \preccurlyeq q \Leftrightarrow (\exists a \in \operatorname{Mor} \mathcal{C}) p^+ = a^+, a + q^+ = a.$$

Also, we define on Mor $\mathcal{C}$ a relation $\sim$ by the rule: for all $p, q \in \operatorname{Mor} \mathcal{C}$

$$p \sim q \Leftrightarrow (p \preccurlyeq q \text{ and } q \preccurlyeq p).$$

The relation $\sim$ is an equivalence on Mor $\mathcal{C}$.

Let $\mathcal{X}$ be the set of all $\sim$-classes on Mor $\mathcal{C}$. On $\mathcal{X}$ we define a relation $\leq$ as follows, for all $A_p, A_q \in \mathcal{X}$,

$$A_p \leq A_q \Leftrightarrow p \preccurlyeq q.$$

The relation $\leq$ is well-defined and is a partial order on $\mathcal{X}$.

From [3] we have the following lemma.

LEMMA 3.6. *Let $\mathcal{C}$ be an idempotent left type-A category. Then*

(a) $\mathcal{R}^* \subseteq \sim$;

(b) *if $\mathcal{C}$ is proper, and $p, q \in \operatorname{Mor}(u, v)$ for some $u, v \in \operatorname{Obj} \mathcal{C}$, then*

$$p \sim q \Leftrightarrow p = q.$$

Let T be a right cancellative monoid acting on a proper idempotent, left type-A category $\mathcal{C}$. For an object u of $\mathcal{C}$ we shall describe the monoid C_u as an $\mathcal{R}$-monoid $\mathcal{R}(T, \mathcal{X}, \mathcal{Y})$ where $\mathcal{X}$ is the partially ordered set defined above and

$$\mathcal{Y} = \{A_p \in \mathcal{X} : A_p \cap \operatorname{Mor}(u, u) \neq \varnothing\}.$$

In order to obtain the properties required to show that C_u is isomorphic to $\mathcal{R}(T, \mathcal{X}, \mathcal{Y})$ we need several lemmas.

LEMMA 3.7. *If $p \in \operatorname{Mor} \mathcal{C}$ and $t \in T$, then $(pt)^+ = p^+t$.*

Proof. Let $p \in \operatorname{Mor}(u, v)$ so that $p^+ \in \operatorname{Mor}(u, u)$, $pt \in \operatorname{Mor}(ut, uv)$ and $p^+t \in \operatorname{Mor}(ut, ut)$. We note first that $p^+t + pt = (p^+ + p)t = pt$.

Now suppose that $r, s \in \operatorname{Mor} \mathcal{C}$ are such that $r_1 + pt = s_1 + pt$. Then $r_1, s_1 \in \operatorname{Mor}(v, ut)$ for some object v and by the definition of action we have $v = wt$ for some w and $v_1 = rt$, $s_1 = st$ for some $r, s \in \operatorname{Mor}(w, u)$. Thus

$$(r + p)t = rt + pt = st + pt = (s + p)t$$

and hence $r + p = s + p$ from which we obtain $r + p^+ = s + p^+$ and consequently,

$$rt + p^+t = (r + p^+)t = (s + p^+)t = st + p^+t.$$

Thus $p^+t\mathcal{R}^*pt$ and as p^+t is idempotent we have $(pt)^+ = p^+t$.

LEMMA 3.8. *Let $p, q \in \operatorname{Mor} \mathcal{C}$ and $t \in T$. Then*

$$p \preccurlyeq q \Leftrightarrow pt \preccurlyeq qt.$$

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Proof. If $p \leqslant q$, then $p^+ = a^+$ and $a + q^+ = a$ for some morphism a. Using the previous lemma we obtain $(pt)^+ = (at)^+$ and $at + (qt)^+ = at$ so that $pt \leqslant qt$.

Conversely, if $b \in \operatorname{Mor} \mathscr{C}$ is such that $(pt)^+ = b$ and $b + (qt)^+ = b$, then since $(qt)^+ \in \operatorname{Mor}(vt, vt)$ for some object v we have $b \in \operatorname{Mor}(-, vt)$.

From the definition of action, it follows that $b = ct$ for some $c \in \operatorname{Mor}(-, v)$. Using the previous lemma we now obtain $p^+ = c^+$ and $c + q^+ = c$ so that $p \leqslant q$ as required.

LEMMA 3.9. *Let $p, q \in \operatorname{Mor} \mathscr{C}$ and $t \in T$. If p is an idempotent, then*

$$p \leqslant qt \Rightarrow (\exists a \in \operatorname{Mor} \mathscr{C}) p = at.$$

Proof. Suppose that $p \leqslant qt$. There exists $b \in \operatorname{Mor} \mathscr{C}$ such that $p = p^+ = b^+$ and $b + (qt)^+ = b$. For some $u \in \operatorname{Obj} \mathscr{C}$, we have $(qt)^+ \in \operatorname{Mor}(ut, ut)$. So, $b \in \operatorname{Mor}(-, ut)$ and there exists $a \in \operatorname{Mor}(-, u)$ such that $b = at$. Now, by Lemma 3.7,

$$p = b^+ = (at)^+ = a^+ t,$$

as required.

LEMMA 3.10. *If $t, h \in T$ and $p \in \operatorname{Mor}(ut, u)$, $q \in \operatorname{Mor}(uh, u)$, then*

$$A_{ph} \wedge A_q = A_{ph+q}.$$

Proof. Clearly, $ph + q$ is defined and belongs to $\operatorname{Mor}(uth, u)$. By Lemmas 3.6 and 3.7, $A_{ph} = A_{p^+h}$ and $A_q = A_{q^+}$. Let $a = ph + q^+$. Then, by Lemma 1.6,

$$a^+ = (ph + q^+)^+ = (ph + q)^+ \quad \text{and} \quad a + q^+ = a.$$

Thus $A_{ph+q} \leqslant A_{q^+}$. Also, let $b = (ph + q)^+ + p^+h$. Then, as $\mathscr{C}$ is left type-A,

$$\begin{aligned} b^+ &= ((ph + q)^+ + p^+h)^+ = ((ph + q^+)^+ + ph)^+ = (ph + q^+)^+ \\ &= (ph + q)^+ \end{aligned}$$

and

$$b + p^+h = b.$$

It follows that

$$A_{ph+q} \leqslant A_{p^+h}.$$

Next, let $r \in \operatorname{Mor}(v, w)$, for some $v, w \in \operatorname{Obj} \mathscr{C}$, and suppose that $A_r \leq A_{ph}, A_q$. There exist morphisms x, y such that

$$r^+ = x^+ = y^+ \quad \text{and} \quad x + p^+h = x, y + q^+ = y.$$

Thus, $x + ph$, $y \in \operatorname{Mor}(v, uh)$ and

$$(x + ph)^+ = (x + p^+h)^+ = x^+ = r^+ = y^+.$$

As $\mathscr{C}$ is proper

$$x + ph = y.$$

Let $c = x + (ph + q)^+$. Then $c + (ph + q)^+ = c$ and

$$\begin{aligned} c^+ &= (x + (ph + q)^+)^+ = (x + ph + q)^+ \\ &= (y + q)^+ = (y + q^+)^+ = y^+ = r^+, \end{aligned}$$

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and so $A_r \le A_{ph+q}$. Thus

$$A_{ph} \wedge A_q = A_{ph+q}$$

as required.

THEOREM 3.11. *A monoid M is proper left type-A if and only if M is isomorphic to some $\mathcal{R}$-monoid $\mathcal{R}(T, \mathcal{X}, \mathcal{Y})$.*

Proof. If $M \cong \mathcal{R}(T, \mathcal{X}, \mathcal{Y})$, then by Lemma 3.3, M is proper left type-A.

To prove the converse it suffices, by Theorem 2.2, to show that if T is a right cancellative monoid which acts on a proper idempotent left type-A category $\mathcal{C}$ and if u is an object of $\mathcal{C}$, then $C_u \cong \mathcal{R}(T, \mathcal{X}, \mathcal{Y})$. We use the partially ordered set $\mathcal{X}$ and subset $\mathcal{Y}$ already defined. It is immediate from Lemma 3.10 that if $p, q \in \mathrm{Mor}(u, u)$, then $A_p \wedge A_q = A_{p+q}$, from which it follows that $\mathcal{Y}$ is a semilattice which by (b) of Lemma 3.6 is isomorphic to $\mathrm{Mor}(u, u)$. Thus $\mathcal{Y}$ has a greatest element namely, A_{O_u}.

By Lemma 3.8, the rule $A_p t = A_{pt}$ gives a well-defined mapping from $\mathcal{X}$ to $\mathcal{X}$ for all $t \in T$. By (a) of Lemma 3.6, $A_p = A_{p^+}$ so that if $A_p \le A_q t$, then $p^+ \preccurlyeq qt$ so that by Lemma 3.9, $p^+ = at$ for some $a \in \mathrm{Mor}\,\mathcal{C}$. Hence $A_p = A_{at} = A_a t$ and $\mathcal{X}t$ is an order-ideal of $\mathcal{X}$. It is now easy to verify (using Lemma 3.8) that we have defined an action of T on $\mathcal{X}$.

We can therefore define the monoids

$$\mathcal{S}(T, \mathcal{X}, \mathcal{Y}) = \{(t, A_p t) \in T \times \mathcal{X} : A_p \in \mathcal{Y} \text{ and } (\forall A_q \in \mathcal{Y}) A_p t \wedge A_q \in \mathcal{Y}\}$$

and

$$\mathcal{R}(T, \mathcal{X}, \mathcal{Y}) = \{(t, A_p t) \in \mathcal{S}(T, \mathcal{X}, \mathcal{Y}) : A_p t \le A_{O_u}\}.$$

If $(t, p) \in C_u$, then $p \in \mathrm{Mor}(ut, u)$ and $p^+ = rt$ for some $r \in \mathrm{Mor}(u, u)$. Also, $A_p = A_{p^+} = A_r t$ and $A_r \in \mathcal{Y}$. Let $A_q \in \mathcal{Y}$ with $q \in \mathrm{Mor}(u, u)$. Then $p + q \in \mathrm{Mor}(ut, u)$ so that $(p+q)^+ \in \mathrm{Mor}(ut, ut)$ and hence $(p+q)^+ = xt$ for some $x \in \mathrm{Mor}(u, u)$. Hence, using Lemma 3.10,

$$A_r t \wedge A_q = A_p \wedge A_q = A_{p+q} = A_x t \in \mathcal{Y}t.$$

Further, $p \preccurlyeq O_u$ since $p^+ = p^+$ and $p + O_u = p$, so that $A_r t = A_p \le A_{O_u}$. Thus (t, A_p) is in $\mathcal{R}(T, \mathcal{X}, \mathcal{Y})$ and we can define a mapping

$$\psi : C_u \to \mathcal{R}(T, \mathcal{X}, \mathcal{Y})$$

by

$$(t, p)\psi = (t, A_p).$$

Using Lemma 3.10, it is routine to verify that ψ is a morphism. Now suppose that $(t, p)\psi = (h, q)\psi$. Then $t = h$ and $A_p = A_q$ so that $p \sim q$. Hence $p, q \in \mathrm{Mor}(ut, u)$ and, by (b) of Lemma 3.6, $p = q$. Thus ψ is injective.

If $(T, A_p t) \in \mathcal{R}(T, \mathcal{X}, \mathcal{Y})$, then $A_p \in \mathcal{Y}$ and we can assume that $p \in \mathrm{Mor}(u, u)$. Also, $A_p t \le A_{O_u}$ so that $pt \preccurlyeq O_u$ and hence $pt = a^+$ and $a + O_u = a$ for some $a \in \mathrm{Mor}\,\mathcal{C}$. Thus $a \in \mathrm{Mor}(ut, u)$ so that $(t, a) \in C_u$ and we have

$$(t, a)\psi = (t, A_a) = (t, A_{a^+}) = (t, A_{pt}) = (t, A_p t).$$

Hence ψ is an isomorphism and the proof is complete.

4. The semidirect product of a right cancellative monoid by a semilattice. In this section we prove that any proper left type-A monoid can be embedded in a special subsemigroup of a semidirect product $T*Y$ of a right cancellative monoid T by a semilattice Y.

Let T be a right cancellative monoid acting (according to the definition of Section 1) on a semilattice Y. On the product set $T \times Y$ we define an operation in the usual way:

$$(t, a)(h, b) = (th, ah \wedge b).$$

Then we obtain a semigroup $T*Y$ called the *semidirect product of T by Y*. Also,

$$E(T*Y) = \{(1, a) : a \in Y\}$$

and $T*Y$ is a monoid if and only if Y has a greatest element. In general, $T*Y$ is neither left nor right abundant. Inside it, however, sits a proper left type-A subsemigroup,

$$W = w(T, Y) = \{(t, at) : a \in Y, t \in T\},$$

which coincides with $T*Y$ when T is a group.

LEMMA 4.1. *The subset W of $T*Y$ is a full proper left type-A subsemigroup of $T*Y$. Moreover, W is a monoid if and only if Y has a greatest element.*

Proof. Let $(t, at), (h, bh) \in W$. Then

$$(t, at)(h, bh) = (th, ath \wedge bh)$$

and as $ath \wedge bh \leq ath$ we have $ath \wedge bh = cth$ for some $c \in Y$ by the definition of action. Hence W is a subsemigroup. Clearly,

$$E(W) = E(T*Y) = \{(1, a) : a \in Y\}$$

so that W is full and $E(W) \cong Y$. Also, W is a monoid if and only if $T*Y$ is a monoid.

To prove that W is left abundant, let $(t, at) \in W$. Then $(1, a) \in E(W)$ and

$$(1, a)(t, at) = (t, at \wedge at) = (t, at).$$

Now, for all $(h, bh), (h_1, b_1h_1) \in W$, if

$$(h, bh)(t, at) = (h_1, b_1h_1)(t, at)$$

then

$$(ht, bht \wedge at) = (h_1t, b_1h_1t \wedge at).$$

It follows that $h = h_1$ and $bh \wedge a = b_1h_1 \wedge a$, so that

$$(h, bh)(1, a) = (h_1, b_1h_1)(1, a)$$

and so $(t, at)\mathcal{R}^*(1, a)$.

Next, we show that W satisfies the type-A condition. Let $(t, at) \in W$ and $(1, e) \in E(W)$. Then

$$((t, at)(1, e))^+(t, at) = (t, at \wedge e)^+(t, at) = (1, c)(t, at),$$

where $ct = at \wedge e$. Also,

$$(t, at)(1, e) = (t, at \wedge e) = (t, ct) = (1, c)(t, at).$$

Thus, W is left type-A.

https://doi.org/10.1017/S0017089500009873 Published online by Cambridge University Press

Finally, we prove that W is proper. Let $(t, at), (h, bh) \in W$ be such that

$$((t, at), (h, bh)) \in \mathcal{R}^* \cap \sigma.$$

Then $(t, at)^+ = (h, bh)^+$, that is $(1, a) = (1, b)$. Furthermore, for some idempotent $(1, e)$,

$$(1, e)(t, at) = (1, e)(h, bh)$$

and so $t = h$. Whence, $(t, at) = (h, bh)$ and W is proper, as required.

We can now state the main theorem of the section.

THEOREM 4.2. *For a proper left type-A monoid M, the following conditions are equivalent*:

(a) *M is proper*;

(b) *M is isomorphic to a submonoid V of the submonoid $W = W(T, Y)$ of a semidirect product $T * Y$ of a right cancellative monoid T by a semilattice Y, where Y has a greatest element, and $\mathcal{R}^*_V \subseteq \mathcal{R}^*_W$.*

We use $\mathcal{R}^*_V$ (resp. $\mathcal{R}^*_W$) to denote the relation $\mathcal{R}^*$ on the monoid V (resp. W).

Proof. In view of Lemma 4.1, it is easy to show that (b) implies (a). That (a) implies (b) follows from the next proposition by Theorem 2.2. First we need a definition.

DEFINITION 4.3. Let $\mathcal{C}$ be a category. A subset I of Mor $\mathcal{C}$ is an $\mathcal{R}^*$-ideal of $\mathcal{C}$ if I is an ideal and I is a union of $\mathcal{R}^*$-classes.

It is easy to check that both the intersection and the union of a family of $\mathcal{R}^*$-ideals of $\mathcal{C}$ are again $\mathcal{R}^*$-ideals. In particular, the set $\mathcal{I}^*(\mathcal{C})$ of all $\mathcal{R}^*$-ideals of $\mathcal{C}$ is a semilattice under intersection.

PROPOSITION 4.4. *Let $\mathcal{C}$ be a proper, idempotent, left type-A category and let T be a right cancellative monoid acting on $\mathcal{C}$. For $u \in$ Obj $\mathcal{C}$, the monoid C_u is isomorphic to a submonoid V of the submonoid $W = W(T, \mathcal{I}^*(\mathcal{C}))$ of a semidirect product $T * \mathcal{I}^*(\mathcal{C})$ with $\mathcal{R}^*_V \subseteq \mathcal{R}^*_W$.*

To establish this result we must first define an action of T on $\mathcal{I}^*(\mathcal{C})$. From Lemma 4.2, 4.3 and 4.5 of [**3**] we have

LEMMA 4.5. *Let $p, q \in$ Mor $\mathcal{C}$. Then*

(a) *$A^*(p) = \{r \in \text{Mor } \mathcal{C} : (\exists x \in \text{Mor } \mathcal{C}) x^+ = (x + p)^+\}$ is the least $\mathcal{R}^*$-ideal which contains p*;

(b) *$A_p \le A_q \Leftrightarrow A^*(p) \subseteq A^*(q)$.*

DEFINITION 4.6. Let I be an $\mathcal{R}^*$-ideal of $\mathcal{C}$ and let $t \in T$. Then

$$I \,.\, t = \bigcup_{p \in I} A^*(pt).$$

It is easy to see that $I \,.\, t$ is an $\mathcal{R}^*$-ideal and that it is the least $\mathcal{R}^*$-ideal which contains It.

LEMMA 4.7. *Let $I, I_1, I_2 \in \mathcal{I}^*(\mathcal{C})$, $t, t_1, t_2 \in T$ and $p \in$ Mor $\mathcal{C}$. Then*

(a) *$I \,.\, 1 = I$*;

(b) *$\left(\bigcup_\lambda I_\lambda\right) . \, t = \bigcup_\lambda (I_\lambda \,.\, t)$, for any collection $\{I_\lambda : \lambda \in \Lambda\}$ of $\mathcal{R}^*$-ideals*;

(c) $A^*(p)=A^*(pt)$;
(d) $I.(t_1t_2)=(I.t_1).t_2$;
(e) $I_1\subseteq I_2 \Leftrightarrow I_1.t\subseteq I_2.t$;
(f) $I_1=I_2\Leftrightarrow I_1.t=I_2.t$;
(g) $I\subseteq I_2.t\Rightarrow(\exists K\in\mathscr{I}^*(\mathscr{C}))\ I=K.t$;
(h) $(I_1\cap I_2).t=I_1.t\cap I_2.t$.

Proof. (a) and (b) are clear from the definition.

(c) Certainly $A^*(pt)\subseteq A^*(p).t$. On the other hand, if $q\in A^*(p)$, then by Lemma 4.5(b), $q\leq p$ so that $qt\leq pt$, giving $A^*(qt)\subseteq A^*(pt)$. It now follows from the definition that $A^*(p).t\subseteq A^*(pt)$.

(d) This is an easy consequence of the definition and (c).

(e) If $I_1\subseteq I_2$, then $I_1.t\subseteq I_2.t$ follows from (b).

If $I_1.t\subseteq I_2.t$, let $r\in I_1$. Then $rt\in I_1.t$ and so $rt\in A^*(qt)$ for some $q\in I_2$. By Lemma 4.5(b), $rt\leq qt$ and so by Lemma 3.8, $t\leq q$. Hence $A^*(r)\subseteq A^*(q)\subseteq I_2$ and $r\in I_2$.

(f) follows immediately from (e).

(g) Let $K=\{b\in I_2: bt\in I\}$. If $b\in K$ and $x,y\in\operatorname{Mor}\mathscr{C}$ are such that $x+b+y$ exists, then $x+b+y\in I_2$ since I_2 is an ideal and $(x+b+y)t=xt+bt+yt$ is in I since $bt\in I$ and I is an ideal. Hence $x+b+y\in K$ and K is an ideal.

If $c\in\operatorname{Mor}\mathscr{C}$ and $c\mathscr{R}^*b$, then $c\in I_2$ since I_2 is an $\mathscr{R}^*$-ideal. Also $ct\mathscr{R}^*bt$ and I is an $\mathscr{R}^*$-ideal, so $ct\in I$. Thus $c\in K$ and K is an $\mathscr{R}^*$-ideal.

It is easy to see that $K.t\subseteq I$. If $q\in I$, then $q\in I_2.t$ so $q\in A^*(pt)$ for some $p\in I_2$. Hence $q\leq pt$ and so $q^+\leq pt$. By Lemma 3.9, $q^+=at$ for some $a\in\operatorname{Mor}\mathscr{C}$. Now $at\leq pt$ and so by Lemma 3.8, $a\leq p$. Thus $a\in I_2$ and $at=q^+$ is in I so that $a\in K$. Hence $q^+=at\in K.t$ and as $K.t$ is an $\mathscr{R}^*$-ideal we also have $q\in K.t$. Consequently, $K.t\subseteq I$ and the proof is complete.

(h) We have now shown that we have an action of T on the semilattice $\mathscr{I}^*(\mathscr{C})$ and as remarked in Section 1, it follows that

$$(I_1\cap I_2).t=I_1.t\cap I_2.t.$$

LEMMA 4.8. *If $t,h\in T$, $u\in\operatorname{Obj}\mathscr{C}$ and $p\in\operatorname{Mor}(ut,u)$, $q\in\operatorname{Mor}(uh,u)$, then*

$$A^*(ph+q)=A^*(p).h\cap A^*(q).$$

Proof. By Lemma 4.7(c) $A^*(p).h=A^*(ph)$ and as both $A^*(ph)$ and $A^*(q)$ are ideals we have $ph+q\in A^*(ph)\cap A^*(q)$, so that

$$A^*(ph+q)\subseteq A^*(p).h\cap A^*(q).$$

On the other hand, if $r\in A^*(ph)\cap A^*(q)$, then by Lemma 4.5(b), $A_r\leq A_{ph}$ and $A_r\leq A_q$ and so by Lemma 3.10, $A_r\leq A_{ph+q}$ and hence $r\in A^*(ph+q)$. The result follows.

We can now prove Proposition 4.4. Since we have an action of T on $\mathscr{I}^*(\mathscr{C})$ we have a semidirect product $T*\mathscr{I}^*(\mathscr{C})$ and the proper left type-A submonoid W is defined.

If $(t,p)\in C_u$, then $p\in\operatorname{Mor}(ut,u)$ so that $p^+\in\operatorname{Mor}(ut,ut)$ and $p^+=rt$ for some $r\in\operatorname{Mor}(u,u)$. Now by Lemmas 3.6(a), 4.5(b) and 4.7(c) we have $A^*(p)=A^*(rt)=A^*(r).t$ so that $(t,A^*(p))=(t,A^*(r).t)\in W$. Thus we can define a mapping

$$\psi: C_u\to W$$

given by

$$(t,p)\psi = (t, A^*(p)).$$

It follows from Lemma 4.8 that ψ is a morphism.

If $(t,p),(h,q) \in C_u$ and $(t,p)\psi = (h,q)\psi$, then $t = h$ and $A^*(p) = A^*(q)$. Thus $p,q \in \mathrm{Mor}(ut,u)$ and $p \sim q$ so that by Lemma 3.6(b), $p = q$ and so ψ is injective.

We now have $C_u \cong V = \mathrm{Im}\,\psi$. If $v_1, v_2 \in V$ with $v_1 \mathcal{R}^*_V v_2$, then $v_1 = (t,p)\psi$ and $v_2 = (h,q)\psi$ for some $(t,p),(h,q) \in C_u$ and $(t,p)\mathcal{R}^*(h,q)$. Then $(t,p)^+ = (h,q)^+$ that is $r = s$ where $rt = p^+$ and $st = q^+$. From the proof of Lemma 4.1,

$$(t, A^*(p))^+ = (1, A^*(r)) \quad \text{and} \quad (h, A^*(q))^+ = (1, A^*(s))$$

so that $(t,A^*(p))^+ = (h, A^*(q))^+$ and so $v_1 \mathcal{R}^*_W v_2$. Thus $\mathcal{R}^*_V \subseteq \mathcal{R}^*_W$ and the proof is complete.

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