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COMPATIBLE LOCALLY CONVEX TOPOLOGIES ON NORMED SPACES: CARDINALITY ASPECTS

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Dedicated to the centenary of G. W. Mackey (1916–2006)

Abstract

For a normed infinite-dimensional space, we prove that the family of all locally convex topologies which are compatible with the original norm topology has cardinality greater or equal to $\mathfrak{c}$.

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1. Introduction

Let X be a vector space over $\mathbb{R}$ and let τ be a topology on X. Denote by $(X,\tau)^*$ the set of all τ-continuous linear forms $l : X \to \mathbb{R}$. A topology η on X is said to be compatible with τ if $(X,\eta)^* = (X,\tau)^*$. Write $\mathrm{LCT}(X,\tau)$ for the set of all locally convex vector space topologies η on X, which are compatible with τ.

Let $w(\tau)$ be the coarsest topology on X with respect to which all elements $l \in (X,\tau)^*$ are continuous. The following known statement implies in particular that $w(\tau)$ is the least element of the partially ordered set $\mathrm{LCT}(X,\tau)$ and therefore *this set is nonempty*.

Proposition 1.1. *Let (X,τ) be a topological vector space. Then $w(\tau)$ is a locally convex vector space topology on X, $w(\tau) \le \tau$ and $w(\tau) \in \mathrm{LCT}(X,\tau)$.*

We next formulate the Mackey–Arens theorem, one of the most relevant results of linear functional analysis, which asserts that the set $\mathrm{LCT}(X,\tau)$ also contains a top element.

Proposition 1.2 (Mackey–Arens theorem). *Let (X,τ) be a topological vector space. Then there exists a topology $m(\tau)$ on X such that $m(\tau) \in \mathrm{LCT}(X,\tau)$ and*

$$w(\tau) \le \eta \le m(\tau), \quad \textit{for all } \eta \in \mathrm{LCT}(X,\tau).$$

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For a topological vector space (X, τ) the topology $m(\tau)$ is called *the Mackey topology* of (X, τ), while (X, τ) is called *a Mackey space* if $m(\tau) = \tau$. The following facts are well known:

(ms$_1$) if (X, τ) is a metrisable locally convex topological vector space, then it is a Mackey space;

(ms$_2$) if (X, τ) is an infinite-dimensional metrisable locally convex topological vector space, it may happen that $w(\tau) = \tau = m(\tau)$ and hence $\operatorname{card}(\mathrm{LCT}(X, \tau)) = 1$ (for example, let (X, τ) be $\mathbb{R}^{\mathbb{N}}$ endowed with the usual product topology);

(ns) if (X, τ) is an infinite-dimensional normable topological vector space, then we have $w(\tau) \neq \tau = m(\tau)$ and hence $\operatorname{card}(\mathrm{LCT}(X, \tau)) \geq 2$.

In connection with **(ns)** the following question can be posed:

Question 1.3. Let $(X, \|\cdot\|)$ be an infinite-dimensional normed space and let τ be the norm topology of $(X, \|\cdot\|)$. What is the cardinality of $\mathrm{LCT}(X, \tau)$?

It seems that the first published result in this direction was the following assertion.

Theorem 1.4 [6, Theorem 1.3]. *Let $(X, \|\cdot\|)$ be an infinite-dimensional reflexive Banach space and let τ be the norm topology of $(X, \|\cdot\|)$. Then*

$$\operatorname{card}(\mathrm{LCT}(X, \tau)) \geq \mathfrak{c}.$$

Recall that a subset of a poset (= partially ordered set) in which no two distinct elements are comparable is called an *antichain*. In the next section we will prove a general assertion, from which we derive the following statement.

Theorem 1.5. *Let $(X, \|\cdot\|)$ be an infinite-dimensional normed space and let τ be the norm topology of $(X, \|\cdot\|)$. Then:*

(a) *the poset $\mathrm{LCT}(X, \tau)$ contains an antichain $\mathfrak{A}$ such that $\operatorname{card}(\mathfrak{A}) \geq \mathfrak{c}$;*

(b) $\operatorname{card}(\mathrm{LCT}(X, \tau)) \geq \mathfrak{c}$.

From the proof of Theorem 1.4 contained in [6], it can be concluded that the following assertion holds.

Theorem 1.6. *Let $(X, \|\cdot\|)$ be an infinite-dimensional reflexive Banach space and let τ be the norm topology of $(X, \|\cdot\|)$. Then the poset $\mathrm{LCT}(X, \tau)$ contains an antichain $\mathfrak{A}$ such that:*

(a) $\operatorname{card}(\mathfrak{A}) \geq \mathfrak{c}$, *and if τ_1 and τ_2 are distinct elements of $\mathfrak{A}$, then the topological spaces (X, τ_1) and (X, τ_2) have noncomparable (with respect to $\subset$) sets of convergent sequences;*

(b) $\operatorname{card}(\mathrm{LCT}(X, \tau)) \geq \mathfrak{c}$.

Remark 1.7. Question 1.3 is treated in the realm of locally quasi-convex topological abelian groups in [2]. This class contains in particular the locally convex topological vector spaces, and the notions of dual group and compatible topologies can be

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defined in this broader context. The paper [2] deals with the poset of locally quasi-convex compatible topologies $\mathcal{C}(G, \tau)$ defined on a locally quasi-convex group (G, τ). Estimates of the possible length of chains and antichains in $\mathcal{C}(G, \tau)$ are given for some classes of groups.

2. Almost disjoint sets, equicontinuous bi-orthogonal systems and proof of Theorem 1.5

A pair of (infinite) sets, C and D, are *almost disjoint* [7] if card $(C \cap D) < \aleph_0$.

LEMMA 2.1 ([7, Theorem IV.14.1]; see also [9, Lemma] and [1, Lemma 2.5.3]). *There exists a family $\mathcal{A}$ with cardinality $\mathfrak{c}$ consisting of pairwise almost disjoint infinite subsets of $\mathbb{N}$.*

Two proofs of this statement can be found in [6]. It is also a consequence of the following theorem.

THEOREM 2.2 (Tarski, [8, Theorem 5.2, page 120]). *Let $\mathfrak{m}$ and $\mathfrak{n}$ be cardinal numbers with $\mathfrak{n}$ infinite and let T be a set having cardinality $\mathfrak{n}$. The following assertions are equivalent:*

(i) $\mathfrak{m} \leq \mathfrak{n}^{\aleph_0}$;

(ii) *there exists a family $\mathcal{A}$ with cardinality $\mathfrak{m}$ consisting of pairwise almost disjoint infinite subsets of T.*

Let (X, τ) be an infinite-dimensional topological vector space. If T is a set containing at least two elements, we say that a family $(e_t, e_t^*)_{t \in T}$ of elements of $(X, \tau) \times (X, \tau)^*$ is

- *bi-orthogonal* if $e_t^*(e_t) = 1$ for $t \in T$ and $e_s^*(e_t) = 0$ for $s, t \in T$ such that $s \neq t$.

A bi-orthogonal family $(e_t, e_t^*)_{t \in T}$ of elements of $(X, \tau) \times (X, \tau)^*$ is called

- *equicontinuous* if $(e_t^*)_{t \in T}$ is a τ-equicontinuous family;
- *total* if $(e_t^*)_{t \in T}$ separates points of X;
- *fundamental* if the closed vector subspace of (X, τ) generated by $(e_t)_{t \in T}$ is the whole of X.

The following assertion will be used to prove the main theorem of this paper.

PROPOSITION 2.3. *Let $(X, \|\cdot\|)$ be an infinite-dimensional normed space and let τ be the norm topology of $(X, \|\cdot\|)$. Then there exists an equicontinuous bi-orthogonal sequence (e_n, e_n^*), $n = 1, 2, \ldots,$ of elements of $(X, \tau) \times (X, \tau)^*$.*

PROOF. Denote again by $\|\cdot\|$ the dual norm on $(X, \tau)^*$.

Suppose first that (X, τ) is separable. Then by [4, Theorem 14.1.5, page 290] we can find and fix a bi-orthogonal sequence (x_n, x_n^*), $n = 1, 2, \ldots,$ of elements of $(X, \tau) \times (X, \tau)^*$. Write

$$e_n = x_n \|x_n^*\|, \quad l_n = x_n^* / \|x_n^*\|, \quad n = 1, 2, \ldots.$$

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Then (e_n, l_n), $n = 1, 2, \ldots$, is a bi-orthogonal sequence of elements of $(X, \tau) \times (X, \tau)^*$ such that $\|l_n\| = 1$, $n = 1, 2, \ldots$.

Suppose now that (X, τ) is nonseparable. Fix an infinite-dimensional separable vector subspace X_0 of (X, τ) and let (e_n, l_n), $n = 1, 2, \ldots$, be a bi-orthogonal sequence of elements of $(X_0, \tau|_{X_0}) \times (X_0, \tau|_{X_0})^*$ such that $\|l_n\| = 1$, $n = 1, 2, \ldots$. By the Hahn–Banach extension theorem, there is a sequence $e_n^* \in (X, \tau)^*$, $n = 1, 2, \ldots$, such that

$$e_n^*|_{X_0} = l_n \quad \text{and} \quad \|e_n^*\| = 1, \quad n = 1, 2, \ldots.$$

Since $\|e_n^*\| = 1$, $n = 1, 2, \ldots$, the sequence (e_n, e_n^*), $n = 1, 2, \ldots$, is an *equicontinuous* bi-orthogonal sequence of elements of $(X, \tau) \times (X, \tau)^*$. □

Remark 2.4. Proposition 2.3 is best possible in the following sense: under the additional set-theoretical axiom ♣, the existence of a nonseparable Banach space which does not admit any uncountably infinite bi-orthogonal system can be established (see [3, Theorem 4.41, page 151]).

Theorem 2.5 (cf. [3, Theorem 4.12, page 135]). *Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space and let τ be the norm topology of $(X, \|\cdot\|)$. Denote by $\mathfrak{n}$ the w^*-density character of $(X, \tau)^*$ and let T be a set with* $\operatorname{card}(T) = \mathfrak{n}$. *Then there exists a total bi-orthogonal system $(e_t, e_t^*)_{t \in T}$ of elements of $(X, \tau) \times (X, \tau)^*$.*

Notation 2.6. For a topological vector space (X, τ), a set T, a nonempty subset $C \subset T$ and a bi-orthogonal system $(\mathbf{e}, \mathbf{e}^*) := (e_t, e_t^*)_{t \in T}$ of elements of $(X, \tau) \times (X, \tau)^*$ such that

$$\sup_{t \in T} |e_t^*(x)| < \infty, \quad \text{for all } x \in X,$$

denote by

- $p_{\mathbf{e}^*, C}$ the semi-norm on X defined by $p_{\mathbf{e}^*, C}(x) = \sup_{t \in C} |e_t^*(x)|$, $x \in X$;
- X_C the vector subspace of X generated by the set $\{e_t : t \in C\}$;
- $\tau'_{\mathbf{e}^*, C}$ the locally convex vector space topology on X generated by $p_{\mathbf{e}^*, C}$;
- $\tau_{\mathbf{e}^*, C}$ the least upper bound (in the set of all topologies on X) of $w(\tau)$ and $\tau'_{\mathbf{e}^*, C}$.

The following statement may be of independent interest.

Proposition 2.7. *Let (X, τ) be an infinite-dimensional topological vector space, T an infinite set and $(e_t, e_t^*)_{t \in T}$ a bi-orthogonal equicontinuous system of elements of $(X, \tau) \times (X, \tau)^*$.*

(a) *If $C \subset T$ is a nonempty set, then $\tau_{\mathbf{e}^*, C}$ is a locally convex vector space topology on X compatible with τ.*

(a′) *If $C \subset T$ is an infinite set, then $\tau_{\mathbf{e}^*, C}|_{X_C}$ is strictly finer than $w(\tau)|_{X_C}$; in particular, $\tau_{\mathbf{e}^*, C}$ is strictly finer than $w(\tau)$.*

(b) *If $B, D \subset T$ are almost disjoint infinite subsets, then the topologies $\tau_{\mathbf{e}^*, B}$ and $\tau_{\mathbf{e}^*, D}$ are incomparable.*

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PROOF. (a) From the τ-equicontinuity of $(e_t^*)_{t\in T}$, the semi-norm $p_{\mathbf{e}^*,C}$ is τ-continuous, that is, $\tau'_{\mathbf{e}^*,C} \le \tau$. From this and from $w(\tau) \le \tau$, we have $w(\tau) \le \tau_{\mathbf{e}^*,C} \le \tau$. This implies (a).

(a′) Clearly, $\tau_{\mathbf{e}^*,C}|_{X_C} \ge w(\tau)|_{X_C}$. Suppose that $\tau_{\mathbf{e}^*,C}|_{X_C} = w(\tau)|_{X_C}$. This implies

$$\tau'_{\mathbf{e}^*,C} \le w(\tau)|_{X_C}.$$

From this inequality, there are $m \in \mathbb{N}$ and $x_i^* \in (X,\tau)^*$, $i = 1,\dots,m$, such that

$$p_{\mathbf{e}^*,C}(x) \le \max_{1\le i\le m} |x_i^*(x)|, \quad \text{for all } x \in X_C. \tag{2.1}$$

Using the bi-orthogonality, it is easy to see that $p_{\mathbf{e}^*,C}|_{X_C}$ *is a norm*. From this and from (2.1) we conclude that the finite sequence x_i^*, $i = 1,\dots,m$, separates points of X_C. However, this contradicts the fact that X_C *is infinite-dimensional* (just note that the set C is infinite and the family $(e_t)_{t\in T}$ is linearly independent).

(b) Let $B, D \subset T$ be almost disjoint infinite subsets. Suppose that $\tau_{\mathbf{e}^*,B} \le \tau_{\mathbf{e}^*,D}$. This implies

$$\tau_{\mathbf{e}^*,B}|_{X_B} \le \tau_{\mathbf{e}^*,D}|_{X_B}.$$

Since $B \cap D$ is finite,

$$\tau_{\mathbf{e}^*,D}|_{X_B} = w(\tau)|_{X_B}.$$

From the last two relations,

$$\tau_{\mathbf{e}^*,B}|_{X_B} \le w(\tau)|_{X_B}$$

in contradiction to (a′), according to which $\tau_{\mathbf{e}^*,B}|_{X_B}$ is strictly finer than $w(\tau)|_{X_B}$. Consequently, the inequality $\tau_{\mathbf{e}^*,B} \le \tau_{\mathbf{e}^*,D}$ is not true. One can prove similarly that the inequality $\tau_{\mathbf{e}^*,D} \le \tau_{\mathbf{e}^*,B}$ is not true either. □

The following observation was prompted by a question posed by the referee (see Question 2.11 below).

REMARK 2.8. The incomparable topologies $\tau_{\mathbf{e}^*,B}$ and $\tau_{\mathbf{e}^*,B}$ obtained in Proposition 2.7(b) might be isomorphic as we prove next.

Let X be an infinite-dimensional real separable Hilbert space, $(e_n)_{n\in\mathbb{N}}$ an orthonormal basis of X and $B, D \subset \mathbb{N}$ almost disjoint infinite subsets. Then the topological vector spaces $(X,\tau_{\mathbf{e}^*,B})$ and $(X,\tau_{\mathbf{e}^*,D})$ are isomorphic.

In fact, let $\varphi : \mathbb{N} \to \mathbb{N}$ be a bijection such that $\varphi(B) = D$ and $\varphi(\mathbb{N}\backslash B) = \mathbb{N}\backslash D$. Then the linear isometry $u_\varphi : X \to X$ defined by $u_\varphi e_n = e_{\varphi(n)}$, $n = 1,2,\dots$, establishes an isomorphism between the topological vector spaces $(X,\tau_{\mathbf{e}^*,B})$ and $(X,\tau_{\mathbf{e}^*,D})$.

THEOREM 2.9. *Let (X,τ) be an infinite-dimensional topological vector space for which there exists an infinite equicontinuous bi-orthogonal system $(e_t, e_t^*)_{t\in T}$ of elements of $(X,\tau)\times(X,\tau)^*$. Then the poset* LCT(X,τ) *contains an antichain $\mathfrak{A}$ such that*

$$\operatorname{card}(\mathfrak{A}) \ge (\operatorname{card}(T))^{\aleph_0}.$$

In particular,

$$\operatorname{card}(\mathrm{LCT}(X,\tau)) \ge \operatorname{card}(\mathfrak{A}) \ge (\operatorname{card}(T))^{\aleph_0}.$$

Proof. By Theorem 2.2, we can find and fix a family $\mathcal{A}$ with cardinality $\operatorname{card}(T)^{\aleph_0}$ consisting of pairwise almost disjoint infinite subsets of T. By Proposition 2.7:

(a) if $A \in \mathcal{A}$, then $\tau_{\mathbf{e}^*,A} \in \mathrm{LCT}(X, \tau)$;
(b) if $B, D \in \mathcal{A}$ and $B \neq D$, then the topologies $\tau_{\mathbf{e}^*,B}$ and $\tau_{\mathbf{e}^*,D}$ are not comparable.

Consequently, the collection

$$\mathfrak{A} = \{\tau_{\mathbf{e}^*,A} : A \in \mathcal{A}\}$$

is an antichain in the poset $\mathrm{LCT}(X, \tau)$ and

$$\operatorname{card}(\mathfrak{A}) = \operatorname{card}(\mathcal{A}).$$

Hence,

$$\operatorname{card}(\mathrm{LCT}(X, \tau)) \geq \operatorname{card}(\mathfrak{A}) = (\operatorname{card}(T))^{\aleph_0}$$

and Theorem 2.9 is proved. □

Proof of Theorem 1.5. By Proposition 2.3, we can apply Theorem 2.9 for $T = \mathbb{N}$ to see that the poset $\mathrm{LCT}(X, \tau)$ contains an antichain $\mathfrak{A}$ such that

$$\operatorname{card}(\mathfrak{A}) \geq (\operatorname{card}(\mathbb{N}))^{\aleph_0} = \mathfrak{c}.$$

This implies $\operatorname{card}(\mathrm{LCT}(X, \tau)) \geq \operatorname{card}(\mathfrak{A}) \geq \mathfrak{c}$. □

Remark 2.10. With the notation of Theorem 1.5, let us call a subset $\mathfrak{R}$ of $\mathrm{LCT}(X, \tau)$ a *tvs-antichain* if from $\tau_1, \tau_2 \in \mathfrak{R}$, $\tau_1 \neq \tau_2$, it follows that (X, τ_1) and (X, τ_2) are nonisomorphic as topological vector spaces. The following question was posed to us by the referee.

Question 2.11. Let $(X, \|\cdot\|)$ be an infinite-dimensional normed space and let τ be the norm topology of $(X, \|\cdot\|)$. Does the poset $\mathrm{LCT}(X, \tau)$ contain a tvs-antichain $\mathfrak{R}$ such that $\operatorname{card}(\mathfrak{R}) \geq \mathfrak{c}$?

Remark 2.8 shows that the arguments used for the proof of Theorem 1.5 do not produce a tvs-antichain of the cardinality required in Question 2.11. However, using a different approach, it can be shown that the answer to the referee's question is positive. The complete proof will appear elsewhere.

Note added in proof

Professor Alexander Gouberman has just pointed out to us that the existence of a family of power $\mathfrak{c}$ of locally convex compatible vector space topologies for an infinite-dimensional normed space can also be derived from the paper by Kiran [5].

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